1.1 i. Recall that \( \dot{\gamma} = (\frac{1}{2}(1+t)^{\frac{3}{2}}, \frac{1}{2}(1-t)^{\frac{3}{2}}, \frac{1}{\sqrt{2}}) \), and \( \|\dot{\gamma}\| = 1 \). Hence \( \kappa(t) = \|\dot{\gamma}(t)\| \), and since
\[
\ddot{\gamma}(t) = \left( \frac{1}{4}(1+t)^{-\frac{3}{2}}, \frac{1}{2}(1-t)^{-\frac{3}{2}}, 0 \right),
\]
it follows that
\[
\kappa(t) = \frac{1}{\sqrt{8(1-t^2)}}.
\]
ii. Since \( \dot{\gamma} = (-\frac{4}{5}\sin(t), \cos(t), \frac{3}{5}\cos(t)) \), and \( \|\dot{\gamma}\| = 1 \), it is again the case that \( \kappa(t) = \|\dot{\gamma}(t)\| \).

The equation
\[
\ddot{\gamma} = (-\frac{4}{5}\cos(t), \sin(t), \frac{3}{5}\cos(t)),
\]
then implies that
\[
\kappa(t) = \|\ddot{\gamma}(t)\| = 1.
\]
iii. Since \( \|\dot{\gamma}\| \neq 1 \) the curvature is given by \( \kappa(t) = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} \). Now \( \dot{\gamma}(t) = (1, \sinh(t)) \) and \( \ddot{\gamma}(t) = (0, \cosh(t)) \). Therefore
\[
\|\dot{\gamma} \times \ddot{\gamma}\| = \begin{vmatrix} 0 & \cosh(t) \\ 1 & \sinh(t) \end{vmatrix} = \cosh(t),
\]
and
\[
\kappa(t) = \text{sech}^2(t).
\]
iv.
\[
\kappa(t) = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{1}{3|\cos(t)\sin(t)|}.
\]

This goes to \( \infty \) when \( t \) is an integer multiple of \( \pi/2 \), at these times \( (\cos^3 t, \sin^3 t) = (\pm 1, 0), (0, \pm 1) \).

1.2 Let \( s(t) \) be the arclength from a fixed point \( p \) on the curve \( \gamma \). Since \( \gamma \) is regular, by proposition 1.3.5 \( s \) is a smooth function of \( t \). Since \( \kappa(t) = \|\frac{d^2}{ds^2}(s(t))\| \) and \( \|\cdot\| : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R} \) is smooth \( \kappa(t) \) can be given as a composition of smooth functions as long as \( \frac{d^2}{ds^2}(s(t)) \neq (0,0) \). Of course \( \frac{d^2}{ds^2} = (0,0) \) contradicts the assumption \( \kappa(t) > 0 \), showing that \( \kappa(t) \) is indeed smooth.

For a counter example when \( \kappa = 0 \) one could use \( \gamma(t) = (t, t^3) \), then \( \dot{\gamma}(t) = (1, 3t^2), \ddot{\gamma}(t) = (0, 6t), \)
\[
\|\dot{\gamma} \times \ddot{\gamma}\| = \begin{vmatrix} 0 & 6t \\ 1 & 3t^2 \end{vmatrix} = 6|t|, \quad \|\dot{\gamma}\| = \frac{6|t|}{(1+9t^4)^{\frac{1}{2}}},
\]
and
\[
\kappa(t) = \frac{6|t|}{(1+9t^4)^{\frac{1}{2}}} \text{ is not differentiable at } t = 0.
\]

2.2 Assume that \( \gamma \) is unit speed parameterized then \( t = \dot{\gamma} \) is smooth. Since the counter clockwise rotation \( r : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( r(x,y) = (-y,x) \) is smooth, \( \mathbf{n}_s = r(t) \) is also smooth. Finally, the dot product \( \cdot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) is smooth so \( \kappa_s = \dot{\gamma} \cdot \mathbf{n}_s \) is smooth. Since any unit speed parameterization is smooth for regular curves this also proves the non-unit speed case by composing with such a reparameterization.

Alternatively use propositions 2.2.1 and 2.2.3 to show that \( \kappa_s \) is the derivative of a smooth function and is therefore smooth.

2.5 Let \( s(t) \) be a unit speed parameterization of \( \gamma \) then
\[
\frac{d\gamma^\lambda}{ds} = t - \lambda \kappa_s t = (1 - \lambda \kappa_s) t.
\]

This shows that \( \gamma^\lambda \) is regular as long as \( \lambda \kappa_s \neq 1 \), and its unit tangent vector and normal vectors coincide with those of \( \gamma \). Now let \( s^\lambda(s) \) be a unit speed paramaterization for \( \gamma^\lambda \), then its inverse \( s(s^\lambda) \) has derivative \( \frac{ds}{ds^\lambda} = \frac{1}{|1-\lambda \kappa_s|} \), and the curvature of \( \gamma^\lambda \) is
\[
\kappa^\lambda = \frac{dt}{ds^\lambda} \cdot \mathbf{n}_s = \frac{dt}{ds} \frac{ds}{ds^\lambda} \cdot \mathbf{n}_s = \frac{\kappa_s}{|1-\lambda \kappa_s|}.
\]
2.6 The three points, \( \gamma(s_0), \gamma(s_0 \pm \delta s) \) are on a circle of radius \( r \) and center \( \epsilon \) provided
\[
r^2 = \|\gamma(s_0) - \epsilon\|^2 = \|\gamma(s_0 + \delta s) - \epsilon\|^2 = \|\gamma(s_0 - \delta s) - \epsilon\|^2.
\]
There will always be a unique \( r \) and \( \epsilon \) solving these equations provided \( \delta s \) is non-zero, and in the case of a closed curve, small enough to guarantee \( \gamma(s_0 + \delta s) \neq \gamma(s_0 - \delta s) \). Let \( a(s, \delta s) \) and \( b(s, \delta s) \) be given by
\[
\epsilon - \gamma(s_0) = a(s, \delta s)t + b(s, \delta s)n_s.
\]
Then it must be that
\[
r^2 = \|\gamma(s_0 + \delta s) - \epsilon\|^2,
\]
\[
= \|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 - \|\epsilon - \gamma(s_0)\|^2,
\]
\[
= \|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 - 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [\epsilon - \gamma(s_0)] + \|\epsilon - \gamma(s_0)\|^2,
\]
\[
= \|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 - 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [\epsilon - \gamma(s_0)] + r^2.
\]
Which, in turn, implies that
\[
\|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 = 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [a(s_0, \delta s)t + b(s_0, \delta s)n_s].
\]
The left hand side of this is on the order of \( \delta s^2 \), so dividing by \( \delta s \) and taking \( \delta s \) to zero yields
\[
\gamma(s_0) \cdot \lim_{\delta s \to 0} (a(s_0, \delta s)t + b(s_0, \delta s)n_s) = \lim_{\delta s \to 0} a(s_0, \delta s) = 0.
\]
Note that
\[
0 = \|\gamma(s_0 + \delta s) - \epsilon\|^2 + \|\epsilon - \gamma(s_0 - \delta s)\|^2 - 2\|\gamma(s_0) - \epsilon\|^2,
\]
\[
= \|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 + \|\gamma(s_0) - \gamma(s_0 - \delta s)\|^2 + \|\epsilon - \gamma(s_0)\|^2
\]
\[
- 2\|\gamma(s_0) - \epsilon\|^2,
\]
\[
= \|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 - 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [\epsilon - \gamma(s_0)]
\]
\[
+ \|\gamma(s_0) - \gamma(s_0 - \delta s)\|^2 + 2[\gamma(s_0) - \gamma(s_0 - \delta s)] \cdot [\epsilon - \gamma(s_0)],
\]
implies that
\[
\|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 + \|\gamma(s_0) - \gamma(s_0 - \delta s)\|^2
\]
and
\[
2[\gamma(s_0 + \delta s) - 2\gamma(s_0) + \gamma(s_0 - \delta s)] \cdot [a(s_0, \delta s)t + b(s_0, \delta s)n_s]
\]
are equal. Dividing by \( \delta s^2 \) and taking \( \delta s \) to zero yields
\[
2\|\gamma(s_0)\|^2 = 2\gamma(s_0) \cdot \lim_{\delta s \to 0} b(s, \delta s)n_s.
\]
Since \( \gamma \) is unit speed, this is just
\[
2 = 2\kappa_s \lim_{\delta s \to 0} b(s, \delta s).
\]
Finally, this shows that if \( \kappa_s \neq 0 \) then \( \epsilon \) converges as \( \delta s \) goes to zero, and it converges to
\[
\epsilon(s_0) = \gamma(s_0) + \frac{1}{\kappa_s}n_s.
\]
2.7 The tangent to \( \epsilon \) is
\[
\dot{\epsilon} = t - \frac{\dot{\kappa}_s}{\kappa_s^2}n_s - t = -\frac{\dot{\kappa}_s}{\kappa_s^2}n_s.
\]
Therefore since \( \dot{\kappa}_s > 0 \) the arclength is
\[
s'(s) = \int_{s_0}^{s} \frac{\dot{\kappa}_s}{\kappa_s^2}ds' = -\frac{1}{\kappa_s} + C.
\]
Since the unit tangent to \( \epsilon \) is \( t' = -n_s \), the signed unit normal is \( n'_s = t \). Thus the signed curvature of \( \epsilon \) is
\[
\kappa'_s = -\frac{dn_s}{ds} \frac{ds}{ds'} \cdot t = \frac{\kappa_s^3}{\kappa_s}. 
\]
For the cycloid, $\gamma(t) = a(t - \sin t, 1 - \cos t)$, the first and second derivative are

$$\dot{\gamma}(t) = a(1 - \cos t, \sin t) \quad \text{and} \quad \ddot{\gamma}(t) = a(\sin t, \cos t).$$

Thus the signed unit normal is

$$n_s = \frac{1}{\|\dot{\gamma}\|} a(-\sin t, 1 - \cos t),$$

and the curvature is

$$\kappa = \frac{a^2}{\|\dot{\gamma}\|^3} \begin{vmatrix} \sin t & \cos t \\ 1 - \cos t & \sin t \end{vmatrix} = \frac{a^2}{\|\dot{\gamma}\|^3} (1 - \cos(t)).$$

The signed curvature is always negative so

$$\frac{1}{\kappa_s} n_s = \frac{\|\dot{\gamma}\|^2}{a(1 - \cos(t))} (\sin t, -1 + \cos t).$$

Since $\|\dot{\gamma}\|^2 = a^2(2 - 2 \cos t)$ this is just

$$\frac{1}{\kappa_s} n_s = 2a(\sin t, -1 + \cos t),$$

and

$$\epsilon(t) = a(t - \sin t, 1 - \cos t) + 2a(\sin t, -1 + \cos t) = a(t + \sin t, -1 + \cos(t)).$$

Using the reparameterization $t = \tilde{t} - \pi$ this becomes

$$\epsilon(\tilde{t}) = a(\tilde{t} - \sin \tilde{t}, -1 - \cos \tilde{t}),$$

which is just $\gamma(\tilde{t}) - (0, 2)$. 