Consider the right circular helix \( \gamma(t) = (a \sin t, a \cos t, b) \). By Examples 2.1 and 2.4, we know that the signed curvature \( \kappa = \frac{a}{\sqrt{a^2 + b^2}} \) and the torsion is \( \tau = \frac{\dot{b}}{\sqrt{a^2 + b^2}} \), respectively. Solving for \( a \) and \( b \), we find that \( a = \frac{\dot{b}}{\sqrt{a^2 + b^2}} \) and \( b = \frac{\dot{a}}{\sqrt{a^2 + b^2}} \). Using these values of \( a \) and \( b \), we see that \( \gamma \) then has curvature \( \kappa \) and torsion \( \tau \). By Theorem 2.3, we know that all such curves can be obtained from \( \gamma \) by a rigid motion.

First we note that \( \delta(t) = \mathbf{t}(t) \), where \( \mathbf{t} \) is the unit speed normal to \( \gamma(t) \). Thus \( \delta'(t) = \kappa \mathbf{n}(t) \). Since \( \kappa \) is never zero, this shows that \( \delta \) is regular. Moreover, since \( \mathbf{n}(t) \) is a unit vector, then \( \|\delta\| = \kappa \), thus showing that if \( s \) is a unit speed parameter for \( \gamma \), then \( \frac{d}{dt} = \kappa \).

To find the curvature \( \kappa^\delta \) of \( \delta \), let \( \mathbf{t}^\delta = \mathbf{n} \) be the unit tangent of \( \delta \). Then
\[
\frac{dt^\delta}{ds} = \frac{d\mathbf{n}}{dt} \cdot \frac{dt}{ds} = (-\kappa \mathbf{t} + \tau \mathbf{b}) \frac{1}{\kappa}.
\]

Thus \( \kappa^\delta = \left\|\frac{dt^\delta}{ds}\right\| = \left(1 + \frac{\tau^2}{\kappa^2}\right)^{1/2} \). This formula also gives us the unit normal \( \mathbf{n}^\delta \) for \( \delta \) is \( \mathbf{n}^\delta = \frac{1}{\kappa}(-\mathbf{t} + \frac{\tau}{\kappa} \mathbf{b}) \). This means that the binormal \( \mathbf{b}^\delta \) of \( \delta \) is
\[
\mathbf{b}^\delta = \mathbf{t}^\delta \times \mathbf{n}^\delta = \frac{1}{\kappa^\delta}(\mathbf{b}^\delta - \frac{\tau}{\kappa} \mathbf{t}).
\]

Taking the derivative and using that \( (\mathbf{b}^\delta)' = -\tau \mathbf{n} \), we find that the torsion \( \tau^\delta \) of \( \delta \) is
\[
\tau^\delta = \frac{\kappa \tau' - \tau \kappa'}{\kappa(\kappa^2 + \tau^2)}.
\]

Without loss of generality, we may assume that \( \gamma \) has unit speed. By assumption, \( \mathbf{t} \cdot \mathbf{a} = \cos \theta \) for \( \mathbf{a} \) a fixed unit vector and \( \theta \) a fixed angle. Thus, taking a derivative and using \( \mathbf{t}' = \kappa \mathbf{n} \), we see that
\[
\kappa \mathbf{n} \cdot \mathbf{a} = 0.
\]

As \( \kappa > 0 \), \( \mathbf{n} \cdot \mathbf{a} = 0 \). Since \( \{\mathbf{t}, \mathbf{n}, \mathbf{b}\} \) form an orthonormal basis, this means that \( \mathbf{a} = c \mathbf{t} + d \mathbf{b} \). That \( \mathbf{t} \cdot \mathbf{a} = \cos \theta \) forces \( c = \cos \theta \), and thus, since \( \mathbf{a} \) is a unit vector, \( d = \pm \sin \theta \). Thus \( \mathbf{a} = \mathbf{t} \cos \theta \pm \mathbf{b} \sin \theta \). Now, taking the derivative and using the Frenet-Serret equations, we see that
\[
0 = \kappa \mathbf{n} \cos \theta \mp \tau \mathbf{n} \sin \theta.
\]

Thus \( \tau = \pm \kappa \cot \theta \).

Conversely, suppose \( \tau = \lambda \kappa \) for some fixed constant \( \lambda \). Let \( \theta \in [0, \pi) \) be such that \( \lambda = \cot \theta \), and define \( \mathbf{a} = \mathbf{t} \cos \theta + \mathbf{b} \sin \theta \). Taking a derivative as above, we see that \( \mathbf{a}' = 0 \). Hence \( \mathbf{a} \) is a fixed unit vector. Moreover, \( \mathbf{t} \cdot \mathbf{a} = \cos \theta \), and so \( \mathbf{t} \) makes a fixed angle \( \theta \) with \( \mathbf{a} \).

Suppose \( \gamma \) lies on the surface of a sphere. By translation, we may assume that \( \gamma \) lies on the surface of a sphere centered at the origin with radius \( r \). Thus
\( \gamma \cdot \gamma = r^2 \). Differentiating, we find

\[
\begin{align*}
\gamma \cdot t &= 0 \\
\kappa \gamma \cdot n + 1 &= 0 \\
\kappa' \gamma \cdot n + \kappa \tau \gamma \cdot b &= 0
\end{align*}
\]

where we have used \( \gamma \cdot t = 0 \) to simplify the later equations. The second equation implies that \( \gamma \cdot n = -1/\kappa \) and the third equation implies that \( \gamma \cdot b = -\frac{\kappa'}{\tau} \). Differentiating this last equation yields

\[
-\tau \gamma \cdot n = \frac{d}{ds} \left( \frac{\kappa'}{\kappa^2 \tau} \right),
\]

and hence we see that

\[
\tau = \frac{d}{ds} \left( \frac{\kappa'}{\kappa^2 \tau} \right).
\]

Conversely, suppose (0.1) holds, and consider \( f(s) = \rho^2 + (\rho/\sigma)^2 \), where \( \rho = 1/\kappa \) and \( \sigma = 1/\tau \). Then (0.1) reads as \( \rho/\sigma = -\frac{d}{ds} (\rho/\sigma) \). Taking a derivative of \( f \), we see that

\[
f' = 2\rho' + 2\rho' \sigma \frac{d}{ds} (\rho/\sigma)
= 2\rho' (1 + \frac{\sigma}{\rho} \frac{d}{ds} (\rho/\sigma))
= 0.
\]

Thus \( f(t) \) is constant. Since \( f \geq 0 \), we can write \( f(t) = r^2 \) for some constant \( r \).

This then shows that \( \gamma + \rho n + \rho' \sigma b \) has constant length \( r^2 \). Setting \( a = \rho n + \rho' \sigma b \), the Frenet-Serret equations imply that \( a' \) is constant. Since \( a' = 0 \), and \( \gamma \) lies on the circle of radius \( r \) with center \( -a \).

Finally, one could compute \( \kappa \) and \( \tau \) for Viviani’s curve to find that (0.1) holds.

\(\text{(2.22)}\) Consider the functions \( \lambda_{ij} = v_i \cdot v_j \). The initial conditions give us \( \lambda_{ij}(s_0) = \delta_{ij} \), where \( \delta_{ij} \) is 0 if \( i \neq j \) and 1 if \( i = j \). We compute

\[
\lambda'_{ij} = \sum_k a_{ik} \lambda_{kj} + a_{jk} \lambda_{ki}.
\]

Since \( a_{ij} = -a_{ji} \), we have that

\[
\sum_k a_{ik} \delta_{kj} + a_{jk} \lambda_{ki} = a_{ij} + a_{ji} = 0,
\]

and so the functions \( s \mapsto \delta_{ij} \) are a solution of the ODE. By Picard’s Theorem, since \( \lambda_{ij}(s_0) = \delta_{ij}(s_0) \), we must have \( \lambda_{ij} = \delta_{ij} \). Thus \( \{v_i\} \) form an orthonormal basis for all \( s \).

\(\text{(4.1)}\) Let \( U = \{(x, y, 0): d((x, y), (x_0, y_0)) < r\} \) be an open disc in the \( xy \)-plane, where \( d \) is the standard Euclidean distance. Then the inclusion \( i: \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( i(x, y) = (x, y, z) \) is a homeomorphism from \( B_r(x_0, y_0) \), the open disc of radius \( r \) and center \( (x_0, y_0) \) in \( \mathbb{R}^2 \), to \( U \).
We have $S = \{(x, y, z) : x^2 + y^2 = 1\}$. Define $U = \{(r, \theta) : \pi < r < 2\pi, \theta \in [0, 2\pi]\}$. In other words, $U$ is the annulus $\{(x, y) : \pi < x^2 + y^2 < 4\pi^2\}$. Define the map $f : U \to S$ by $f(r, \theta) = (\cos \theta, \sin \theta, \tan(r - 3\pi/2))$. Because $\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$ is a homeomorphism, we see that $f$ is a homeomorphism onto $S$.

First we note that since each map $\sigma^x_h, \sigma^y_h, \sigma^z_h$ is formed from combinations of simple functions and the square root is always taken with its argument nonnegative, each map is a homeomorphism. To show that the maps in fact form an atlas, we need to show that for any $P \in S^2$, the unit sphere, $P$ lies in the image of at least one of the maps $\sigma^x_h, \sigma^y_h, \sigma^z_h$. The image of the map $\sigma^x_h$ is the set of points on the unit sphere with $\pm x > 0$, and similarly for $\sigma^y_h, \sigma^z_h$. Thus if $P$ were not in the image of any of the maps, then we would have $P = (0, 0, 0)$, which is impossible.

Multiplying the equations $(x - z)\cos \theta = (1 - y)\sin \theta$ and $(x + z)\sin \theta = (1 + y)\cos \theta$ yields

$$(x^2 - z^2)\sin \theta \cos \theta = (1 - y^2)\sin \theta \cos \theta.$$

Thus $x^2 + y^2 - z^2 = 1$ unless $\cos \theta = 0$ or $\sin \theta = 0$. If $\sin \theta = 0$, then $x = z$ and $y = -1$, which is on the hyperboloid. Similarly, if $\cos \theta = 0$, then $x = -z$ and $y = 1$, which is on the hyperboloid. Thus all the lines lie within the hyperboloid.

Consider now the line $L_{\theta}$ given by $(x - z)\cos \theta = (1 - y)\sin \theta$ and $(x + z)\sin \theta = (1 + y)\cos \theta$. Setting $z = 0$, we see that $L_{\theta}$ passes through $(\sin 2\theta, -\cos 2\theta, 0)$. Multiplying the two equations for the line by $\sin \theta$ and $\cos \theta$, respectively, and then adding, we then see that $L_{\theta}$ is parallel to $(\cos 2\theta, \sin 2\theta, 1)$, from which it follows that we get all of the lines by taking $0 < \theta < \pi$. To see that the lines cover the hyperboloid, let $(x, y, z)$ be a point in the hyperboloid. If $x \neq z$, then choose $\theta$ so that $\cot \theta = \frac{1 - y}{x - z}$. This shows that $(x, y, z)$ is on $L_{\theta}$. Similarly, if $x \neq -z$, then set $\cot \theta = \frac{1 + y}{x + z}$, so that $(x, y, z)$ is on $L_{\theta}$. The only points we haven’t considered are $(0, \pm 1, 0)$, which lie on the lines $L_{\pi/2}$, $L_0$, respectively. Finally, suppose that $0 \leq \theta, \phi < \pi$ and that $\theta \neq \phi$. We want to show that $L_{\theta}$ does not intersect $L_{\phi}$. Suppose not. Then there is a point $(x, y, z)$ in both. This means that $(1 - y)\tan \theta = (1 - y)\tan \phi$ and $(1 + y)\cot \theta = (1 + y)\cot \phi$. This forces $y = 1$ and $z = -1$, which is impossible.

Because the lines cover the hyperboloid and do not intersect, they give us our map. Explicitly, let $U = \{(r, \theta) : 1 < r < 2, \theta \in [0, 2\pi]\}$ be as in Exercise 4.2. Define the map $\sigma : U \to H$, with $H$ the hyperboloid, by

$$(r, \theta) \mapsto (\sin \theta, -\cos \theta, 0) + \tan(\pi(r - 3/2))(\cos \theta, \sin \theta, 1).$$

Roughly speaking, this map says to start on the circle $x^2 + y^2 = 1$ in $H$ and then move out along the line $L_{\theta/2}$. Using the information above, one easily sees that this is in fact a homeomorphism.

Finally, we can consider also the lines given by

$$(x - z)\cos \phi = (1 + y)\sin \phi, \quad (x + z)\sin \phi = (1 - y)\cos \phi,$$

Checking as above, we see that these lines cover $H$. Finally, one can check that the lines $L_{\theta}, M_{\phi}$ intersect at the point

$$\left(\frac{\cos(\theta - \phi)}{\sin(\theta + \phi)}, \frac{\sin(\theta - \phi)}{\sin(\theta + \phi)}, \frac{\cos(\theta + \phi)}{\sin(\theta + \phi)}\right)$$

so long as $\theta + \phi \neq k\pi$.
for some integer $k$. For each $0 \leq \theta < \pi$, there is exactly one $0 \leq \phi < \pi$ such that
$\theta + \pi = k\pi$, and so each line $L_\theta$ intersects all but one line $M_\phi$.

(4.5) Suppose to the contrary that $S^2$ can be covered by a single surface patch. That is, suppose we have a homeomorphism $\sigma: U \rightarrow S^2$ for $U \subset \mathbb{R}^2$ an open set. Since $S^2$ is closed and bounded in $\mathbb{R}^3$, it is compact. As $\sigma$ is a homeomorphism, this means that $U$ is compact, and in particular, is closed. Since $U$ is both open and closed, it must be either $\mathbb{R}^2$ or the empty set. It clearly isn’t the empty set, and because $U$ is compact, it cannot be $\mathbb{R}^2$. Thus we arrive at a contradiction, and so $S^2$ cannot be covered by a single surface patch.