# Introduction to Linear Transformation

Math 4A – Xianzhe Dai

UCSB

April 14 2014

Based on the 2013 Millett and Scharlemann Lectures

System of Linear Equations:

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Two very nice properties it enjoys are

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# Linear Transformations

## Definition

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

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So $T$ is a linear transformation.

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$$T(\vec{0}) = T(0 \cdot \vec{0}) = 0\,T(\vec{0}) = \vec{0}.$$

- The two conditions could be written as one: For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and real numbers $a, b \in \mathbb{R}$,

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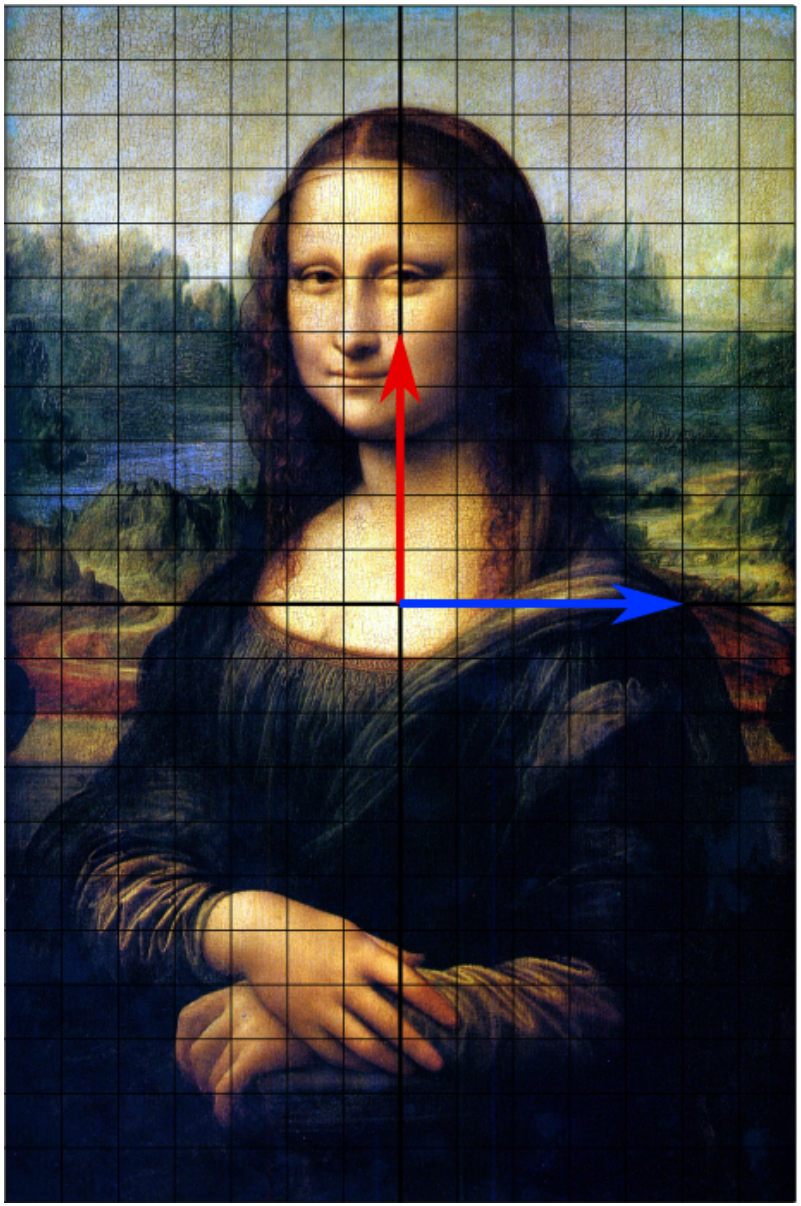
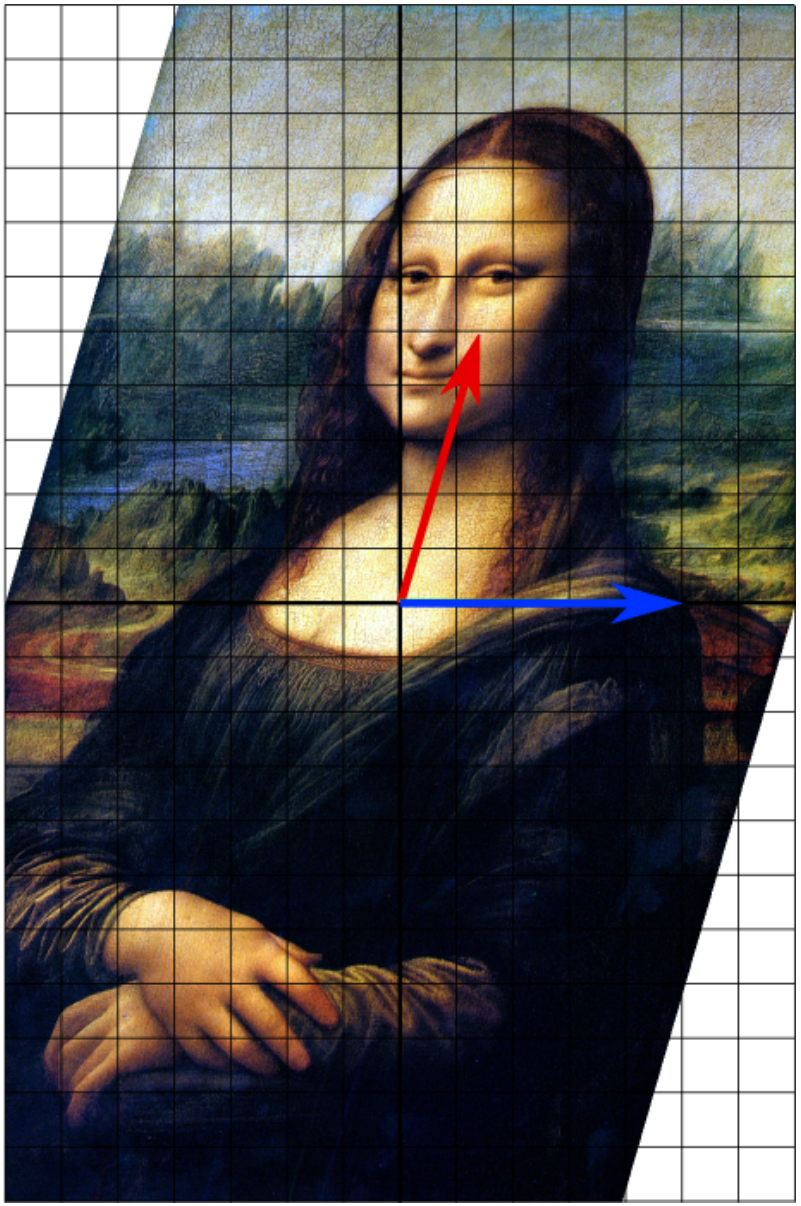
Important Fact Conversely any linear transformation is associated to a matrix transformation (by using bases).





Mona Lisa transformed

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Example 1, a shear: Consider the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the $2 \times 2$ matrix

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For any horizontal vector $\vec{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

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So $T$ is the identity on horizontal vectors.

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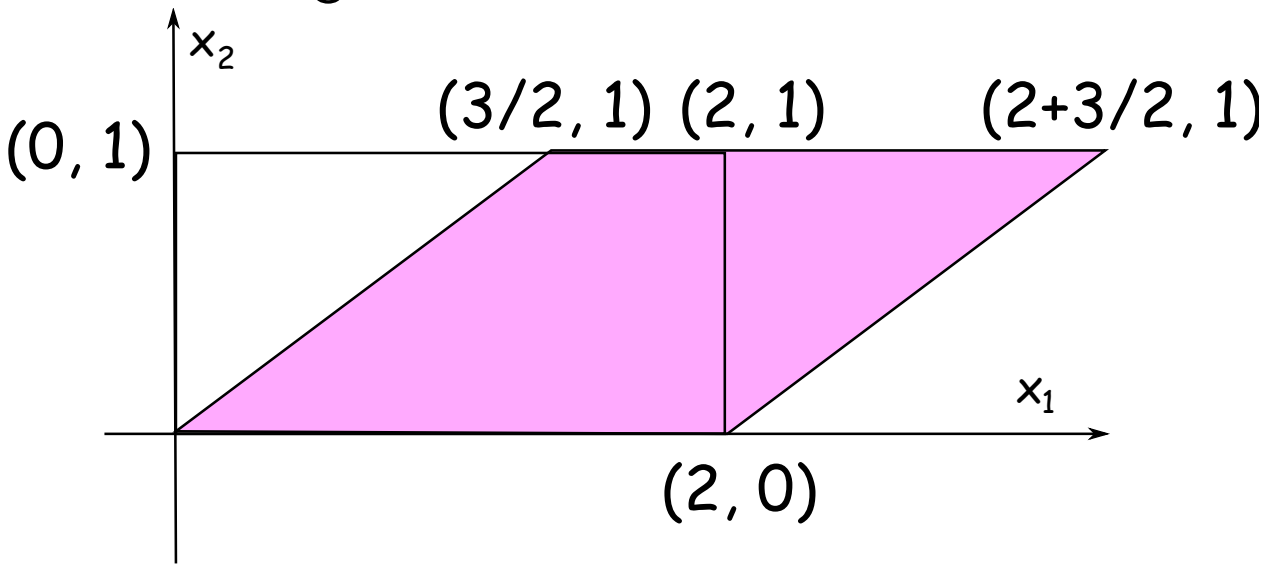

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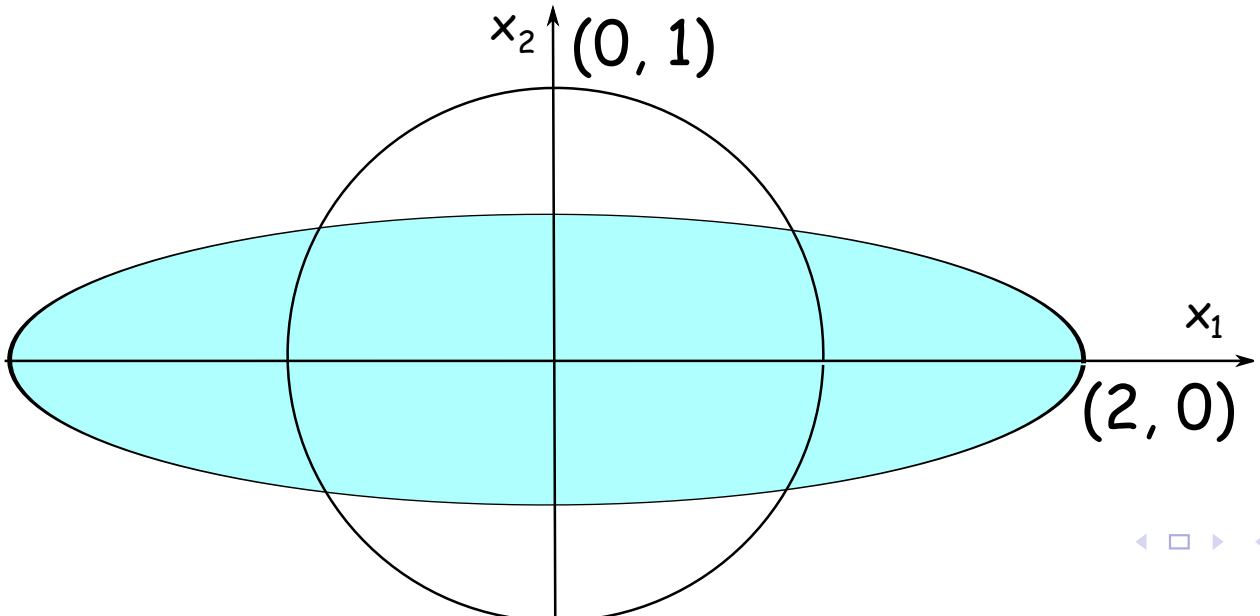

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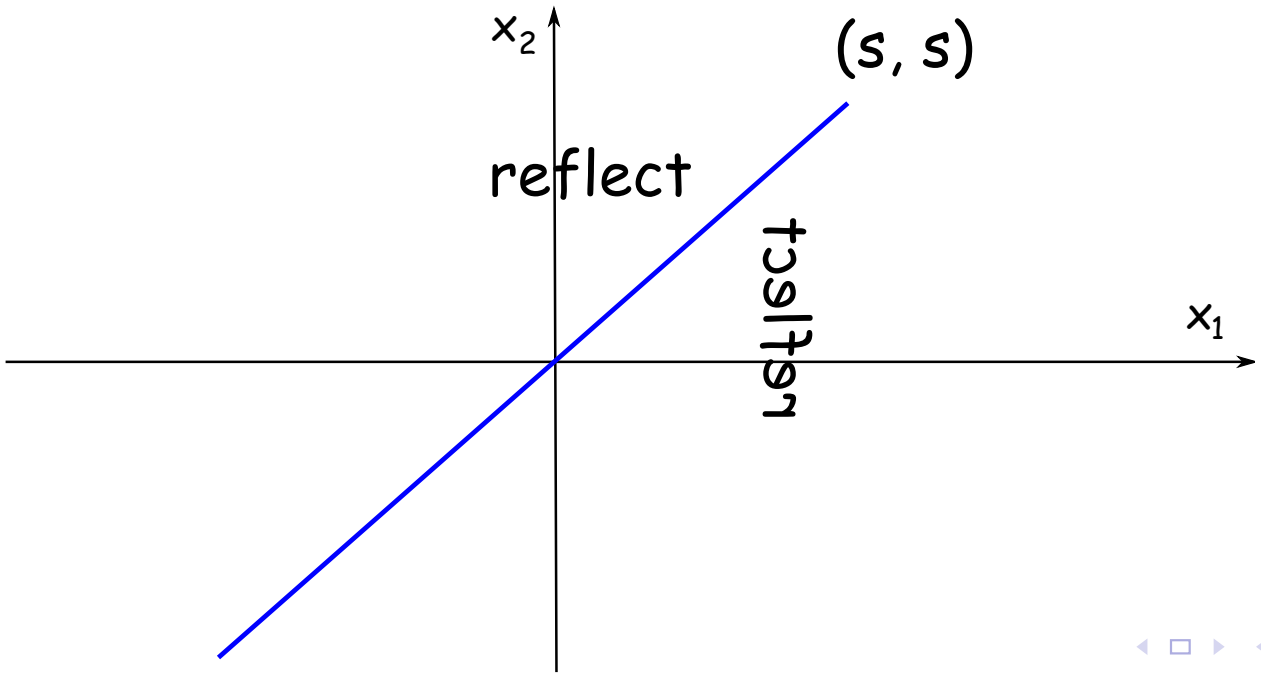

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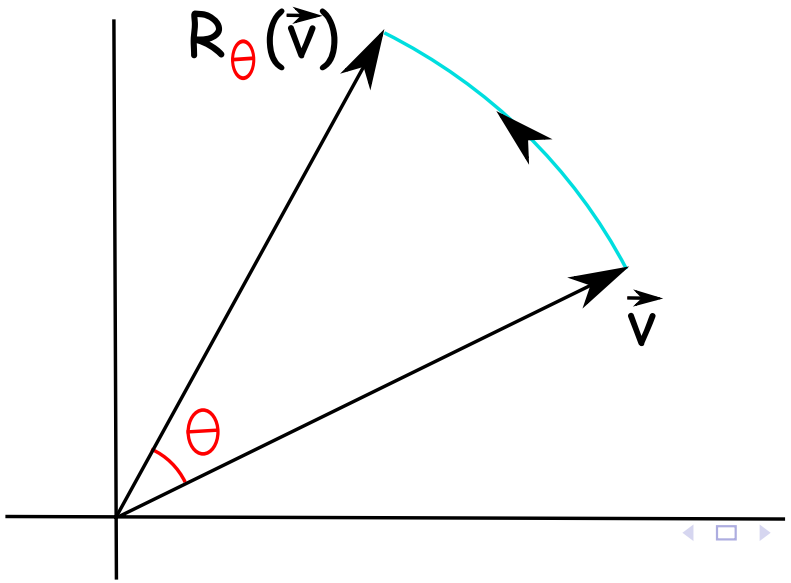

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Answer: Solve the system of equations given by $A\vec{x} = \vec{b}$. Any solution is such a vector $\vec{u}$.

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Question: Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix linear transformation. Suppose $A$ is the matrix of $T$ and $\vec{b} \in \mathbb{R}^m$ is given.
Find a vector $\vec{u}$ so that

$$T(\vec{u}) = \vec{b}.$$

Answer: Solve the system of equations given by $A\vec{x} = \vec{b}$. Any

solution is such a vector $\vec{u}$.

Reminder: There may be

- no solution or
- exactly one solution or
- a parameterized family of solutions

# iClicker question

Suppose $T$ is a matrix linear transformation with matrix $A$ below, and we are seeking all vectors $\vec{u}$ so that $T(\vec{u}) = \vec{b}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

How many solutions are there?

A) Zero.
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What if $\vec{b} = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$?

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If there are solutions, reduced echelon form makes it easy to describe them.

## Definition

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
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Matrix transformation: Let $A$ be any $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$.

# Matrix Is Everywhere

Example: Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation so that

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \qquad T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

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$$T(\begin{bmatrix} -1 \\ 7 \end{bmatrix}) = T(-1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = -1\,T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + 7\,T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

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In fact, nothing can stop us from using the same idea to compute $T(\begin{bmatrix} 2 \\ -4 \end{bmatrix})$ or $T(\vec{x})$ for any vector $\vec{x} \in \mathbb{R}^2$:

Example

We can carry this much further: All linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ are matrix linear transformations

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Why?

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

(standard basis of $\mathbb{R}^n$)

So by the property of linear transformation

$$T(\vec{x}) = x_1 T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right) + x_2 T\left(\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\right) + \ldots x_n T\left(\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right)$$

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Ahha: In matrix notation this is written:

$$T(\vec{x}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x}$$

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That is, the matrix

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

is the matrix of $T$!

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Example

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

**Definition**

$T$ is *1 to 1* if $\vec{u} \neq \vec{v}$ implies that $T(\vec{u}) \neq T(\vec{v})$.

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