System of Linear Equations:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
    &\vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
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**New perspective**: think of the LHS as a “function/map/transformation”, \( T(\vec{x}) = A\vec{x} \). \( T \) maps/transforms a vector \( \vec{x} \) to another vector \( A\vec{x} \).
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**New perspective:** think of the LHS as a “function/map/ transformation”, \( T(\vec{x}) = A\vec{x} \). \( T \) maps/transforms a vector \( \vec{x} \) to another vector \( A\vec{x} \).

Two very nice properties it enjoys are

\[ T(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v}) \]
\[ T(c\vec{u}) = A(c\vec{u}) = cA\vec{u} = cT(\vec{u}) \]
A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$. 

So $T$ is a linear transformation.
Definition

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Example: Let $T : \mathbb{R}^1 \to \mathbb{R}^1$ be defined by $T(x) = 5x$. 
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**Example:** Let $T : \mathbb{R}^1 \to \mathbb{R}^1$ be defined by $T(x) = 5x$.

For any $u, v \in \mathbb{R}^1$,

- $T(u + v) = 5(u + v) = 5u + 5v = T(u) + T(v)$ and

- for any $c \in \mathbb{R}$, $T(cu) = 5cu = c5u = cT(u)$. 

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Linear Transformations

**Definition**

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- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
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Some notes:
  - Most functions are **not** linear transformations.
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  \[ \cos(x + y) \neq \cos(x) + \cos(y) \]. Or \((2x)^2 \neq 2(x^2)\).
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  \[
  T(\vec{0}) = T(0 \cdot \vec{0}) = 0 T(\vec{0}) = \vec{0}.
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- The two conditions could be written as one: For any vectors \( \vec{u}, \vec{v} \in \mathbb{R}^n \) and real numbers \( a, b \in \mathbb{R} \),
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Let $A$ be any $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$. We have already seen that $T$ has what it takes:
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A linear transformation defined by a matrix is called a **matrix transformation**.

Important Fact: Conversely, any linear transformation is associated to a matrix transformation (by using bases).
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Important Fact Conversely any linear transformation is associated to a matrix transformation (by using bases).
Mona Lisa transformed
Matrix transformations are **important** and are also **cool**!
Matrix transformations are important and are also cool!

Example 1, a shear: Consider the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the $2 \times 2$ matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$
Matrix transformations are important and are also cool!

**Example 1, a shear:** Consider the matrix transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the $2 \times 2$ matrix

$$A = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

For any horizontal vector $\vec{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 3/2 \cdot 0 \\ 0 \cdot x_1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$
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**Example 1, a shear:** Consider the matrix transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by the 2 \( \times \) 2 matrix

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So \( T \) is the identity on horizontal vectors.
For any vertical vector \( \vec{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \)
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For any **vertical** vector $\vec{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

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So a vertical vector is pushed perfectly horizontally, a distance \( \frac{3}{2} \) times its length:
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So a vertical vector is pushed perfectly horizontally, a distance \( \frac{3}{2} \) times its length:

\((0, 1)\) \(\to\) \((3/2, 1)\) \(\to\) \((2, 1)\) \(\to\) \((2+3/2, 1)\)
Example 2, scaling:

Use

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$
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So $$T$$ stretches horizontally and contracts vertically:
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So \( T \) stretches horizontally and contracts vertically:
Example 3, reflection through a line:
Use

\[ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
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So \( T \) exchanges the two coordinates.
Example 3, reflection through a line:

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A = \begin{bmatrix}
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So horizontal unit vector is rotated \( \theta \) clockwise.

Similarly, for the vertical unit vector \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), so all of plane rotates:
Example 4, rotation:

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So horizontal unit vector is rotated c-clockwise an angle \( \theta \).
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\end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}
\]

So horizontal unit vector is rotated \( c \)-clockwise an angle \( \theta \).

Similarly, for the vertical unit vector \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), so all of plane rotates:
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**Question:** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix linear transformation. Suppose \( A \) is the matrix of \( T \) and \( \vec{u} \in \mathbb{R}^n \) is given. What is \( T(\vec{u}) \)?
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**Answer:** Solve the system of equations given by $A\vec{x} = \vec{b}$. Any solution is such a vector $\vec{u}$. 

Reminder: There may be no solution or exactly one solution or a parameterized family of solutions.
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- no solution or
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Suppose $T$ is a matrix linear transformation with matrix $A$ below, and we are seeking all vectors $\vec{u}$ so that $T(\vec{u}) = \vec{b}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

How many solutions are there?

A) Zero.
B) One
C) Infinity
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iClicker question

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What if $\vec{b} = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$?
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Recall the idea: row reduce the augmented matrix \([A : \vec{b}]\) to merely echelon form.
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- Augmentation column is pivot column $\iff$ no solutions.
- Augmentation column is only non-pivot column $\iff$ unique solution.
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Definition

A **linear transformation** is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$. 
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This has important implications: if you know $T(\vec{u})$ and $T(\vec{v})$, then you know the values of $T$ on all the linear combinations of $\vec{u}$ and $\vec{v}$.
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**Matrix transformation**: Let $A$ be any $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$. 
Example: Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation so that

$$T\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \quad T\left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
Matrix Is Everywhere

**Example:** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation so that

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What is $T\left(\begin{bmatrix} -1 \\ 7 \end{bmatrix}\right)$?

\[
T\left(\begin{bmatrix} -1 \\ 7 \end{bmatrix}\right) = T\left(-1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 7 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)
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\[
= -1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \end{bmatrix}
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$$= -1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \end{bmatrix}$$

In fact, nothing can stop us from using the same idea to compute $T\left( \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right)$ or $T(\vec{x})$ for any vector $\vec{x} \in \mathbb{R}^2$:
We can carry this much further: All linear transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are matrix linear transformations.
We can carry this much further: All linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ are matrix linear transformations.

Why?

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

(standard basis of $\mathbb{R}^n$)
So by the property of linear transformation

\[ T(\vec{x}) = x_1 T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 T \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + x_n T \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \]

only need to know each \( T(\vec{e}_j) \) where

\[ \vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th} \text{entry} \]
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Denote \( \vec{a}_j = T(\vec{e}_j) \)

\[ T(\vec{x}) = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots x_n \vec{a}_n \]
Ahha: In matrix notation this is written:

\[
T(\vec{x}) = \begin{bmatrix}
\vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} = A\vec{x}
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Ahha: In matrix notation this is written:

\[ T(\vec{x}) = [\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x} \]

That is, the matrix

\[ A = [\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n]\]

is the matrix of \( T \)!
Recap:
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$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \ldots + x_n \vec{e}_n) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \ldots + x_n T(\vec{e}_n) =$$
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Example
Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a (matrix) linear transformation.

**Definition**

\( T \) is 1 to 1 if \( \vec{u} \neq \vec{v} \) implies that \( T(\vec{u}) \neq T(\vec{v}) \).
Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a (matrix) linear transformation.

**Definition**

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If \( T(\vec{u}) = T(\vec{v}) \) we see that \( T(\vec{u} - \vec{v}) = \vec{0} \) (\( T \) linear transformation).
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Saying that \( T \) is 1 to 1 is the same as saying that \( T(\vec{w}) = \vec{0} \) exactly when \( \vec{w} = \vec{0} \) (only trivial solution).
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The only solution to the homogeneous equation is the zero solution. And, as a consequence, $n \leq m$. 
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Examples: shears, contractions and expansions, rotations, reflections.
Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

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$T$ is **onto** if, for any $\vec{v} \in \mathbb{R}^m$, is there a $\vec{u} \in \mathbb{R}^n$ such that $T(\vec{u}) = \vec{v}$.

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$T$ is an **isomorphism** if $T$ is both 1 to 1 and onto.

Saying that $T$ is **isomorphism** is the same as saying that $T$ is a bijection that respects the vector space structure.
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The non-homogeneous equation must always have exactly one solution.
Suppose \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a (matrix) linear transformation.

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\( T \) is an **isomorphism** if \( T \) is both 1 to 1 and onto.

Saying that \( T \) is isomorphism is the same as saying that \( T \) is a bijection that respects the vector space structure.

This means that the reduced echelon form of the matrix of \( T \) must have exactly \( n \) non-zero rows, the same as the number of columns.

The non-homogeneous equation must always have exactly one solution.

Examples: shears, contractions and expansions, rotations, reflections
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$T$ is 1 to 1 if there is a pivot 1 in every column of the reduced echelon form, i.e. there are no free variables. Said differently, the column vectors of the matrix of $T$ are linearly independent.

$T$ is onto if there is a pivot 1 in every row of the reduced echelon form. Said differently, the column vectors of the matrix of $T$ span the whole space $\mathbb{R}^m$.

$T$ is an isomorphism if there is a pivot 1 in every row and column, i.e. the reduced echelon matrix is the identity matrix. Said differently, the column vectors of the matrix of $T$ are linearly independent and span the whole space.
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