Matrix Algebra

Math 4A – Xianzhe Dai

UCSB

April 16, 2014

Based on the 2013 Millett and Scharlemann Lectures

Last Time:Matrix of Linear Transformation

Definition

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$.

Last Time:Matrix of Linear Transformation

Definition

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$.

For any linear transformation $T(\vec{0}) = \vec{0}$

Last Time:Matrix of Linear Transformation

Definition

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$.

For any linear transformation $T(\vec{0}) = \vec{0}$

$$T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$$

Last Time:Matrix of Linear Transformation

Definition

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$.

For any linear transformation $T(\vec{0}) = \vec{0}$

$$T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$$

This has important implications: if you know $T(\vec{u})$ and $T(\vec{v})$, then you know the values of T on all the linear combinations of $\vec{u}$ and $\vec{v}$.

Last Time:Matrix of Linear Transformation

Definition

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ with these properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- For any vector $\vec{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$.

For any linear transformation $T(\vec{0}) = \vec{0}$

$$T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$$

This has important implications: if you know $T(\vec{u})$ and $T(\vec{v})$, then you know the values of T on all the linear combinations of $\vec{u}$ and $\vec{v}$.

By using standard basis, all linear transformations are matrix transformation

Standard basis of $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Standard basis of $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$$

is a linear combination of the standard basis.

Standard basis of $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$$

is a linear combination of the standard basis.
Since T is a linear transformation,

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots x_n\vec{e}_n) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \ldots x_n T(\vec{e}_n)$$

Standard basis of $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$$

is a linear combination of the standard basis.
Since T is a linear transformation,

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots x_n\vec{e}_n) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \ldots x_n T(\vec{e}_n)$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots x_n\vec{a}_n = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x}.$$

And

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

And

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is 1 *to* 1 if $\vec{u} \neq \vec{v}$ implies that $T(\vec{u}) \neq T(\vec{v})$.

And

$$A = \left[\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{array} \right]$$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is 1 *to* 1 if $\vec{u} \neq \vec{v}$ implies that $T(\vec{u}) \neq T(\vec{v})$.

Put another words, $T(\vec{u}) = T(\vec{v})$ implies $\vec{u} = \vec{v}$.

And

$$A = \left[\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{array} \right]$$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is 1 *to* 1 if $\vec{u} \neq \vec{v}$ implies that $T(\vec{u}) \neq T(\vec{v})$.

Put another words, $T(\vec{u}) = T(\vec{v})$ implies $\vec{u} = \vec{v}$.

But $T(\vec{u}) = T(\vec{v})$ means that $T(\vec{u} - \vec{v}) = \vec{0}$ (T linear transformation).

And

$$A = \left[\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{array} \right]$$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is 1 *to* 1 if $\vec{u} \neq \vec{v}$ implies that $T(\vec{u}) \neq T(\vec{v})$.

Put another words, $T(\vec{u}) = T(\vec{v})$ implies $\vec{u} = \vec{v}$.

But $T(\vec{u}) = T(\vec{v})$ means that $T(\vec{u} - \vec{v}) = \vec{0}$ (T linear transformation).

Thus, saying that T is 1 *to* 1 is the same as saying that $T(\vec{w}) = \vec{0}$ exactly when $\vec{w} = \vec{0}$ (only trivial solution).

And

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is 1 *to* 1 if $\vec{u} \neq \vec{v}$ implies that $T(\vec{u}) \neq T(\vec{v})$.

Put another words, $T(\vec{u}) = T(\vec{v})$ implies $\vec{u} = \vec{v}$.

But $T(\vec{u}) = T(\vec{v})$ means that $T(\vec{u} - \vec{v}) = \vec{0}$ (T linear transformation).

Thus, saying that T is 1 *to* 1 is the same as saying that $T(\vec{w}) = \vec{0}$ exactly when $\vec{w} = \vec{0}$ (only trivial solution).

This means that the reduced echelon form of the matrix of T must have a pivot in each of its n columns (no free variables), i.e. exactly n non-zero rows.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is onto if, for any $\vec{v} \in \mathbb{R}^m$, there is a $\vec{u} \in \mathbb{R}^n such\ that T(\vec{u}) = \vec{v}$.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is onto if, for any $\vec{v} \in \mathbb{R}^m$, there is a $\vec{u} \in \mathbb{R}^n such\ that T(\vec{u}) = \vec{v}$.

Saying that T is onto is the same as saying that $T(\vec{u}) = \vec{v}$ always has a solution.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is onto if, for any $\vec{v} \in \mathbb{R}^m$, there is a $\vec{u} \in \mathbb{R}^n such\ that T(\vec{u}) = \vec{v}$.

Saying that T is onto is the same as saying that $T(\vec{u}) = \vec{v}$ always has a solution.

This means that the reduced echelon form of the matrix of T must have a pivot in each of its m rows (so that the augumentation column can never be pivot), i.e. exactly m non-zero rows.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is onto if, for any $\vec{v} \in \mathbb{R}^m$, there is a $\vec{u} \in \mathbb{R}^n$ *such that* $T(\vec{u}) = \vec{v}$.

Saying that T is onto is the same as saying that $T(\vec{u}) = \vec{v}$ always has a solution.

This means that the reduced echelon form of the matrix of T must have a pivot in each of its m rows (so that the augumentation column can never be pivot), i.e. exactly m non-zero rows.

Definition

T is an isomorphism if T is both 1 *to* 1 and onto.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is onto if, for any $\vec{v} \in \mathbb{R}^m$, there is a $\vec{u} \in \mathbb{R}^n$*such that*$T(\vec{u}) = \vec{v}$.

Saying that T is onto is the same as saying that $T(\vec{u}) = \vec{v}$ always has a solution.

This means that the reduced echelon form of the matrix of T must have a pivot in each of its m rows (so that the augumentation column can never be pivot), i.e. exactly m non-zero rows.

Definition

T is an isomorphism if T is both 1 *to* 1 and onto.

This means that the reduced echelon form of the matrix of T must have exactly n non-zero rows, the same as the number of columns.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Definition

T is onto if, for any $\vec{v} \in \mathbb{R}^m$, there is a $\vec{u} \in \mathbb{R}^n$ *such that* $T(\vec{u}) = \vec{v}$.

Saying that T is onto is the same as saying that
$T(\vec{u}) = \vec{v}$ always has a solution.

This means that the reduced echelon form of the matrix of T must have a pivot in each of its m rows (so that the augumentation column can never be pivot), i.e. exactly m non-zero rows.

Definition

T is an isomorphism if T is both 1 *to* 1 and onto.

This means that the reduced echelon form of the matrix of T must have exactly n non-zero rows, the same as the number of columns.

Examples: shears, contractions and expansions, rotations,

Summarizing Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

Summarizing Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

T is 1 *to* 1 if there is a pivot 1 in every column of the reduced echelon form, i.e. there are no free variables.

Summarizing Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

T is 1 *to* 1 if there is a pivot 1 in every column of the reduced echelon form, i.e. there are no free variables.
Said differently, the column vectors of the matrix of T are linearly independent.

Summarizing Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

T is 1 *to* 1 if there is a pivot 1 in every column of the reduced echelon form, i.e. there are no free variables.
Said differently, the column vectors of the matrix of T are linearly independent.

T is onto if there is a pivot 1 in every row of the reduced echelon form.

Summarizing Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

T is 1 *to* 1 if there is a pivot 1 in every column of the reduced echelon form, i.e. there are no free variables.
Said differently, the column vectors of the matrix of T are linearly independent.

T is onto if there is a pivot 1 in every row of the reduced echelon form.
Said differently, the column vectors of the matrix of T span the whole space $\mathbb{R}^m$.

Summarizing Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

T is 1 *to* 1 if there is a pivot 1 in every column of the reduced echelon form, i.e. there are no free variables.
Said differently, the column vectors of the matrix of T are linearly independent.

T is onto if there is a pivot 1 in every row of the reduced echelon form.
Said differently, the column vectors of the matrix of T span the whole space $\mathbb{R}^m$.

T is an isomorphism if there is a pivot 1 in every row and column, i.e. the reduced echelon matrix is the identity matrix.

Summarizing Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a (matrix) linear transformation.

T is 1 *to* 1 if there is a pivot 1 in every column of the reduced echelon form, i.e. there are no free variables.
Said differently, the column vectors of the matrix of T are linearly independent.

T is onto if there is a pivot 1 in every row of the reduced echelon form.
Said differently, the column vectors of the matrix of T span the whole space $\mathbb{R}^m$.

T is an isomorphism if there is a pivot 1 in every row and column, i.e. the reduced echelon matrix is the identity matrix.
Said differently, the column vectors of the matrix of T are linearly independent and span the whole space.

Matrix Algebra

Let's look at some of the properties of matrix algebra. Suppose that A is an $m \times n$ matrix,

Matrix Algebra

Let's look at some of the properties of matrix algebra. Suppose that A is an $m \times n$ matrix, that is A has m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix Algebra

Let's look at some of the properties of matrix algebra. Suppose that A is an $m \times n$ matrix, that is A has m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Entry in i^{th} row, j^{th} column of a matrix A denoted $(A)_{ij}$.

Matrix Algebra

Let's look at some of the properties of matrix algebra. Suppose that A is an $m \times n$ matrix, that is A has m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Entry in i^{th} row, j^{th} column of a matrix A denoted $(A)_{ij}$.
So for the matrix above, $(A)_{ij} = a_{ij}$.

Matrix Algebra

Let's look at some of the properties of matrix algebra. Suppose that A is an $m \times n$ matrix, that is A has m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Entry in i^{th} row, j^{th} column of a matrix A denoted $(A)_{ij}$.
So for the matrix above, $(A)_{ij} = a_{ij}$.
The diagonal entries are those in which $i = j$.

Matrix Algebra

Let's look at some of the properties of matrix algebra. Suppose that A is an $m \times n$ matrix, that is A has m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Entry in i^{th} row, j^{th} column of a matrix A denoted $(A)_{ij}$.
So for the matrix above, $(A)_{ij} = a_{ij}$.
The diagonal entries are those in which $i = j$.
So, for the matrix A above,

$$\{a_{11}, a_{22}, ..., a_{mm}\}$$

are the diagonal entries (if $m \leq n$; if $n < m$ must stop at a_{nn}.)

A square matrix has as many rows as columns (i. e. $m = n$.)

A square matrix has as many rows as columns (i. e. $m = n$.)

A diagonal matrix is square and all off-diagonal terms are zero.

A square matrix has as many rows as columns (i. e. $m = n$.)

A diagonal matrix is square and all off-diagonal terms are zero.
So A is diagonal $\iff$ whenever $i \neq j$, the entry $(A)_{ij} = 0$.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix}$$

A square matrix has as many rows as columns (i. e. $m = n$.)

A diagonal matrix is square and all off-diagonal terms are zero.
So A is diagonal $\iff$ whenever $i \neq j$, the entry $(A)_{ij} = 0$.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix}$$

Also: upper triangular and lower triangular:

$$\begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ * & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & a_{mm} \end{bmatrix}$$

Matrix addition: Can add two matrices of the same size! entry by entry.

Matrix addition: Can add two matrices of the same size! entry by entry.

Scalar multiplication: Can multiply any matrix by a real number, sometimes called a scalar.

Matrix addition: Can add two matrices of the same size! entry by entry.

Scalar multiplication: Can multiply any matrix by a real number, sometimes called a scalar.

Matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} =$$

$$\begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}; \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots a_{in}b_{nj}$$

Another interpretation: suppose A is an $m \times n$ matrix and

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$$

is an $n \times p$ matrix, then AB is the $m \times p$ matrix given by:

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

Another interpretation: suppose A is an $m \times n$ matrix and

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$$

is an $n \times p$ matrix, then AB is the $m \times p$ matrix given by:

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

In other words, each column of AB is given by $A\vec{b}_1, A\vec{b}_2, \cdots, A\vec{b}_n$.

Another interpretation: suppose A is an $m \times n$ matrix and

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$$

is an $n \times p$ matrix, then AB is the $m \times p$ matrix given by:

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

In other words, each column of AB is given by $A\vec{b}_1, A\vec{b}_2, \cdots, A\vec{b}_n$.

Major warning: Matrix multiplication is not commutative. That is, in general, $AB \neq BA$

Other strange properties:

- Can't cancel: If $AB = AC$ it does not follow that $B = C$, even when $A \neq 0$
- Can multiply to zero: If $AB = 0$ it does not follow that $A = 0$ or $B = 0$

Other strange properties:

- Can't cancel: If $AB = AC$ it does not follow that $B = C$, even when $A \neq 0$
- Can multiply to zero: If $AB = 0$ it does not follow that $A = 0$ or $B = 0$

Example of both: Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$$

But mostly the properties are nice:

Theorem

If A, B, and C are sized so that the products are defined, and $t \in \mathbb{R}$, then

1. $A(BC) = (AB)C$ *(associative),*
2. $A(B + C) = AB + AC$ *(distributes),*
3. $(B + C)A = BA + CA$ *(distributes),*
4. $t(AB) = (tA)B = A(tB)$ *(associative)*

Only the first rule is not easy to see (it takes a bit of computation).

Definition

The $n \times n$ diagonal matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is called the identity matrix.

Easy to check:

- For A any $m \times n$ matrix, $AI_n = A$.
- For B any $n \times p$ matrix, $I_n B = B$.
- So if A is a square $n \times n$ matrix then $I_n A = A = AI_n$.

The last matrix operation is the matrix transpose.

The last matrix operation is the matrix transpose.

The matrix transpose A^T is obtained from A by swapping the rows for columns (and columns for rows). I.e. $(A^T)_{ij} = (A)_{ji}$.

The last matrix operation is the matrix transpose.

The matrix transpose A^T is obtained from A by swapping the rows for columns (and columns for rows). I.e. $(A^T)_{ij} = (A)_{ji}$.

It enjoys the following properties.

Theorem

For any matrices A and B (of the same size) and $s \in \mathbb{R}$, we have

1. $(A^T)^T = A$,

The last matrix operation is the matrix transpose.

The matrix transpose A^T is obtained from A by swapping the rows for columns (and columns for rows). I.e. $(A^T)_{ij} = (A)_{ji}$.

It enjoys the following properties.

Theorem

For any matrices A and B (of the same size) and $s \in \mathbb{R}$, we have

1. $(A^T)^T = A$,
2. $(A + B)^T = A^T + B^T$,

The last matrix operation is the matrix transpose.

The matrix transpose A^T is obtained from A by swapping the rows for columns (and columns for rows). I.e. $(A^T)_{ij} = (A)_{ji}$.

It enjoys the following properties.

Theorem

For any matrices A and B (of the same size) and $s \in \mathbb{R}$, we have

1. $(A^T)^T = A$,
2. $(A + B)^T = A^T + B^T$,
3. $(sA)^T = sA^T$.

The last matrix operation is the matrix transpose.

The matrix transpose A^T is obtained from A by swapping the rows for columns (and columns for rows). I.e. $(A^T)_{ij} = (A)_{ji}$.

It enjoys the following properties.

Theorem

For any matrices A and B (of the same size) and $s \in \mathbb{R}$, we have

1. $(A^T)^T = A$,
2. $(A + B)^T = A^T + B^T$,
3. $(sA)^T = sA^T$.
4. $(AB)^T = B^T A^T$. *(Order of multiplication reversed!)*

The last matrix operation is the matrix transpose.

The matrix transpose A^T is obtained from A by swapping the rows for columns (and columns for rows). I.e. $(A^T)_{ij} = (A)_{ji}$.

It enjoys the following properties.

Theorem

For any matrices A and B (of the same size) and $s \in \mathbb{R}$, we have

1. $(A^T)^T = A$,
2. $(A + B)^T = A^T + B^T$,
3. $(sA)^T = sA^T$.
4. $(AB)^T = B^T A^T$. *(Order of multiplication reversed!)*

Check by computations