Last time 1/13/12

$$m\ddot{x} + b\dot{x} + kx = 0$$

$x = e^{rt}$

General sol$^{\underline{n}}$s:

1). characteristic eq$^{\underline{n}}$

$$mr^2 + br + k = 0$$

$\Rightarrow$ two roots r_1, r_2.

2) two ~~real~~ distinct roots r_1, r_2, $r_1 \neq r_2$

general sol$^{\underline{n}}$

$$x(t) = c_1 \boxed{e^{r_1 t}} + c_2 \boxed{e^{r_2 t}}$$

overdamped

3). two identical (real) roots $r_1 = r_2$

no oscillation

general sol$^{\underline{n}}$:

$x = V(t)e^{r_1 t}$

$$x(t) = c_1 \boxed{e^{r_1 t}} + c_2 \boxed{t\, e^{r_1 t}}$$

fudge factor

critically damped

4). complex roots —

to be continued!

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Sol$^{\underline{n}}$s for linear homogeneous diff. eq$^{\underline{n}}$s

linear

$$\boxed{y'' + p(t)\, y' + q(t)\, y = 0}$$

homogeneous

need not be const. !

— Uniqueness and Existence Theorem: assuming $p(t)$, $q(t)$ continuous on (a,b) and $t_0 \in (a,b)$, then there is a unique sol$^{\underline{n}}$ of the diff. eq$^{\underline{n}}$ satisfying the initial cond$^{\underline{n}}$s:

$$y(t_0) = A, \qquad y'(t_0) = B$$

for any $A, B \in \mathbb{R}$.

$\Longleftrightarrow$

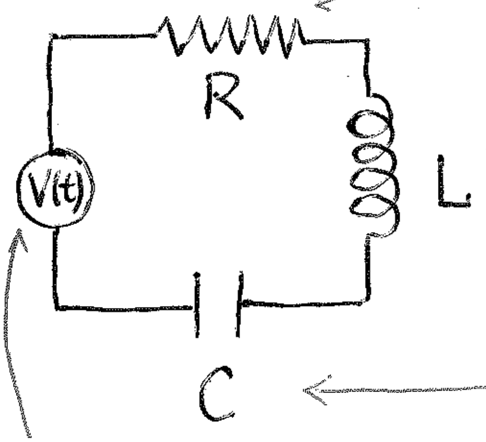
— Sol$^{\underline{n}}$ Space Theorem: the space of sol$^{\underline{n}}$s of the diff. eq$^{\underline{n}}$ is a vector space of dimension 2. i.e. the general sol$^{\underline{n}}$ can be written as

$$y(t) = C_1\, y_1(t) + C_2\, y_2(t)$$

for two linearly independent sol$^{\underline{n}}$s $y_1(t)$, $y_2(t)$.

Electric Circuit

1/11/12.



Resistor: $V = RI$ (Ohm's law)
current
resistance (const.)

inductor: $V = L\dfrac{dI}{dt}$ (Faraday's law)
inductance (const.)

capacitor: $V = \dfrac{1}{C} Q$
charge
capacitance (const.)

applied voltage

— $I = \dfrac{dQ}{dt}$

— Kirchoff's law: input voltage = sum of voltage drop around circuit

$\Longrightarrow$

$$RI + L\frac{dI}{dt} + \frac{1}{C}\int I\,dt = V(t)$$

or, in terms of Q:

$$\boxed{L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)}$$

Compare w/ mass-spring system:

$$\boxed{m\ddot{x} + b\dot{x} + kx = f(t)}$$

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2^{nd} order linear homog. diff. eq$^{\underline{n}}$.
w/. const. coeff's:

$$\boxed{ay'' + by' + cy = 0}$$

a, b, c real numbers.

Char. eq$^{\underline{n}}$.

$$ar^2 + br + c = 0.$$

3. <u>complex roots</u>: $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ (This happens when $b^2 - 4ac < 0$)

$\Rightarrow$ general sol$^{\underline{n}}$.

$$y(t) = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

underdamped $\longrightarrow$ oscillation!

4. Phase-Amplitude form

$$y(t) = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

$$= e^{\alpha t}[A\cos(\beta t - \delta)]$$

$$A = \sqrt{C_1^2 + C_2^2}, \quad \tan\delta = C_2/C_1$$

$$= \underbrace{Ae^{\alpha t}}_{\text{time-varying amplitude}} \cos\Big[\underset{\substack{\uparrow \\ \text{quasi-frequency}}}{\beta}\big(t - \underbrace{\tfrac{\delta}{\beta}}_{\text{phase shift}}\big)\Big]$$

Euler's Formula De-mystified:

1/13/12

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad !?$$

Taylor expansion

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots$$

$$= 1 + i\theta + \frac{1}{2!} i^2\theta^2 + \frac{1}{3!} i^3\theta^3 + \cdots$$

$$= \underbrace{1}_{\text{no } i\text{'s}} + \underline{\underline{i}}\,\theta \underbrace{- \frac{1}{2!}\theta^2}_{\text{no } i\text{'s}} - \frac{1}{3!}\underline{\underline{i}}\,\theta^3 + \underbrace{\cdots}_{\text{no } i\text{'s}}$$

$$= \underbrace{\left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right)}_{= \cos\theta} + i\underbrace{\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)}_{= \sin\theta}$$

$$= \cos\theta + i\sin\theta \quad !!$$