Last time

V, W vector spaces

01/30/12

$T: V \longrightarrow W$ is a linear transformation

if it preserves both the vector addition and scalar multiplication, i.e. $\forall\, u, v \in V$, $\forall\, k$ scalar

$$\boxed{\begin{aligned} T(u+v) &= T(u) + T(v) \\ T(ku) &= k\,T(u) \end{aligned}}$$

— preserves vector addition

— preserves scalar multiplication

$$\Longleftrightarrow \quad T(k_1 u_1 + k_2 u_2) = k_1 T(u_1) + k_2 T(u_2).$$

$$\Longleftrightarrow \quad \boxed{T(k_1 u_1 + \cdots + k_n u_n) = k_1 T(u_1) + \cdots + k_n T(u_n)}$$

vectors remain

scalars pull out.

<u>Linear transf^{n}s and matrices</u>:

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transf^n $\longleftrightarrow$ $m \times n$ matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

$$T\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

general form of a linear transf^n from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Standard matrix of T:

$$A = \big[\, T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n) \,\big]$$

$e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, \cdots, 0)$, $e_n = (0, 0, \cdots, 1)$

standard basis of $\mathbb{R}^n$.

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Rank of a linear transf$^{\underline{n}}$ surjective?

$$T : V \longrightarrow W$$

- $Im(T) = \{ w \in W \mid w = T(v) \text{ for some } v \in V \}$
  $\subset W$ is a <u>subspace</u>.

- $\text{rank}(T) = \dim(Im(T))$.

- For $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T(x) = Ax$ for an $m \times n$ matrix A,
  $$\text{rank}(T) = \text{rank}(A)$$

$$\boxed{\begin{array}{l} T \text{ is surjective} \\ \iff Im(T) = W \\ \iff \text{rank}(T) = \dim W \end{array}}$$

Kernel of a linear transf$^{\underline{n}}$ injective?

- $\ker(T) = \{ v \in V \mid T(v) = 0 \}$
  $\subset V$ is a <u>subspace</u>.

- $\text{nullity}(T) = \dim(\ker(T))$.

$$\boxed{\begin{array}{l} T \text{ is injective} \\ \iff \ker(T) = \{0\} \\ \iff \text{nullity}(T) = 0 \end{array}}$$

Dimension Theorem:

$$\boxed{\text{nullity}(T) + \text{rank}(T) = \dim V}$$