Rank of a linear transf$^{\underline{n}}$ surjective? 02/01/12

$$T : V \longrightarrow W$$

- $\mathrm{Im}(T) = \{ w \in W \mid w = T(v) \text{ for some } v \in V \}$

  $\subset W$ is a <u>subspace</u>.

- $\mathrm{rank}(T) = \dim(\mathrm{Im}(T))$.

- For $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T(x) = Ax$ for an $m \times n$ matrix A,

  $$\mathrm{rank}(T) = \mathrm{rank}(A)$$

T is surjective
$\iff \mathrm{Im}(T) = W$
$\iff \mathrm{rank}(T) = \dim W$

Kernel of a linear transf$^{\underline{n}}$ injective ?

- $\ker(T) = \{ v \in V \mid T(v) = 0 \}$

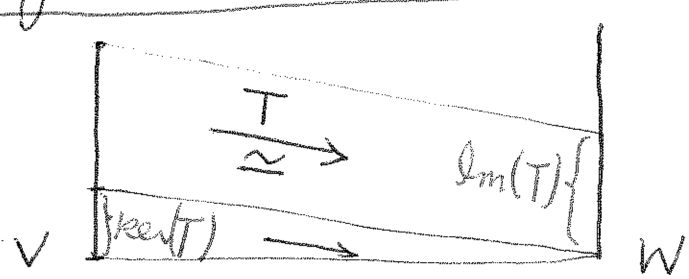
  $\subset V$ is a <u>subspace</u>.

- $\mathrm{nullity}(T) = \dim(\ker(T))$

T is injective
$\iff \ker(T) = \{0\}$
$\iff \mathrm{nullity}(T) = 0$

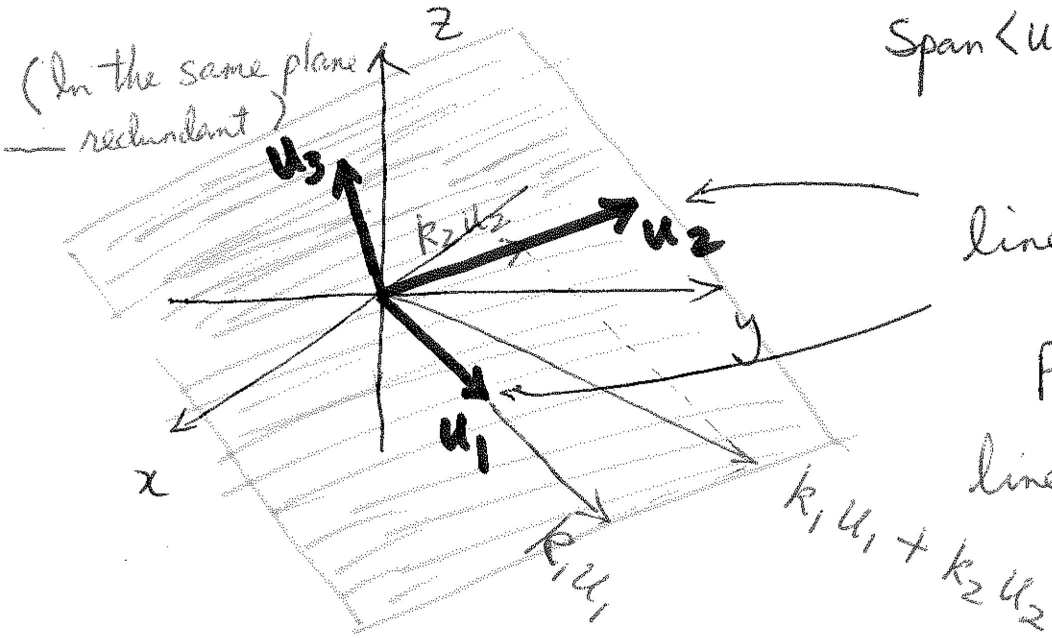
Dimension Theorem :

$$\boxed{\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim V}$$



Linear Combination & Span (Review)

02/01/12



$\text{Span}\langle u_1, u_2, u_3 \rangle$
$= \text{Span}\langle u_1, u_2 \rangle$

linearly independent.

Pick out u_1, u_2 (the linearly independent set of vectors) by RREF.

If $u_1, \ldots, u_k$ linearly independent, then

$$\dim \text{Span}\langle u_1, \ldots, u_k \rangle = k.$$

Ex: $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ 02/01/12

$$T(v) = \left[\begin{bmatrix}2\\-1\\1\end{bmatrix}\begin{bmatrix}-4\\2\\-2\end{bmatrix}\begin{bmatrix}3\\-2\\1\end{bmatrix}\begin{bmatrix}6\\-3\\3\end{bmatrix}\right]\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}$$

1). $\boxed{\operatorname{Im}(T) = ? \qquad \operatorname{rank}(T) = ?}$

$$\operatorname{Im}(T) = \operatorname{span}\left\langle \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}-4\\2\\-2\end{bmatrix}, \begin{bmatrix}3\\-2\\1\end{bmatrix}, \begin{bmatrix}6\\-3\\3\end{bmatrix} \right\rangle$$

the column vectors

To eliminate <u>redandant</u> vectors:

pivots (first nonzero entry in a row)

$$\begin{bmatrix}2 & -4 & 3 & 6\\-1 & 2 & -2 & -3\\1 & -2 & 1 & 3\end{bmatrix} \xrightarrow[\text{(EROS)}]{\text{RREF}} \begin{bmatrix}2 & -4 & 3 & 6\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

1st 3rd are linearly indep.

The rests are redundant.

$$\Rightarrow \operatorname{Im}(T) = \operatorname{span}\left\langle \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}3\\-2\\1\end{bmatrix} \right\rangle$$

and $\operatorname{rank}(T) = 2$.

2). $\boxed{\ker(T) = ? \qquad \operatorname{nullity}(T) = ?}$

$$\ker(T) = \left\{ \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} \;\middle|\; \begin{cases}2v_1 - 4v_2 + 3v_3 + 6v_4 = 0\\ -v_1 + 2v_2 - 2v_3 - 3v_4 = 0\\ v_1 - 2v_2 + v_3 + 3v_4 = 0\end{cases} \right\} \overset{\text{RREF}}{\Longleftrightarrow} \begin{cases}2v_1 - 4v_2 + \cancel{3v_3} + 6v_4 = 0\\ v_3 = 0\end{cases}$$

$$\Longleftrightarrow \begin{cases}v_1 = 2v_2 - 3v_4\\ v_3 = 0\end{cases}$$

arbitrary const's (free parameters)

$$= \left\{ \begin{bmatrix}2v_2 - 3v_4\\ 1v_2\\ 0\\ 1v_4\end{bmatrix} \;\middle|\; \begin{matrix}v_2, v_4\\ \text{arbitrary}\end{matrix} \right\} = \operatorname{span}\left\langle \underset{v_2}{\begin{bmatrix}2\\1\\0\\0\end{bmatrix}}, \underset{v_4}{\begin{bmatrix}-3\\0\\0\\1\end{bmatrix}} \right\rangle$$

and $\operatorname{nullity}(T) = 2$.

02/10/12

Linear diff. eq$^{\underline{n}}$. $\longrightarrow$ 1st order system

$$x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x = f(t)$$

$$\xrightarrow[\substack{\vdots \\ x_n = x^{(n-1)}}]{\substack{x_1 = x \\ x_2 = x'}} \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = -a_0 x_1 - \cdots - a_{n-1} x_n + f(t) \end{cases}$$

i.e. $$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}' = A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{=\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{bmatrix}}_{=\vec{f}(t)}, \quad A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

i.e. $$\boxed{\vec{x}\,' = A\vec{x} + \vec{f}(t)}$$

Eigenvalues and Eigenvector

$$T : V \longrightarrow V \quad \text{linear transf}^{\underline{n}}.$$

A scalar λ is called an <u>eigenvalue</u> of T if we can solve the eq$^{\underline{n}}$

$$T(v) = \lambda v$$

for some $\boxed{v \neq 0}$; v is called an <u>eigenvector</u>.