Last time:

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System of DE's:

$$\vec{x}' = A\vec{x} \qquad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

General sol^n

$$\vec{x}(t) = c_1 \vec{x}_1(t) + \cdots + c_n \vec{x}_n(t)$$

for n linearly indep. sol^n's $\vec{x}_1(t), \cdots, \vec{x}_n(t)$.

How to find $\vec{x}_1(t), \cdots, \vec{x}_n(t)$?

$\vec{x}(t) = e^{\lambda t}\vec{v}$ is a sol^n. $\Longleftrightarrow$ $A\vec{v} = \lambda\vec{v}$. i.e. λ is an eigenvalue and $\vec{v}$ a corresponding eigenvector

$\Longrightarrow$

If $\lambda_1, \cdots, \lambda_n$ eigenvalues of A

$\vec{v}_1, \cdots, \vec{v}_n$ linearly indep. eigenvectors

then the general sol^n

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n.$$

Short of (linearly indep) eigenvectors? 02/15/12

Repeated eigenvalues: $\lambda_1 = \lambda_2$ but only one eigenvector $\vec{v}_1$

$\Rightarrow$ a linearly indep new sol$^{\underline{n}}$ is given by

$$\boxed{t e^{\lambda_1 t} \vec{v}_1 + e^{\lambda_1 t} \vec{u}}$$

where $\vec{u}$ is a sol$^{\underline{n}}$ of

$$\boxed{(A - \lambda_1 I)\vec{u} = \vec{v}_1}$$

$\vec{u}$ is called a generalized eigenvector since $(A - \lambda_1 I)^2 \vec{u} = 0$

Complex eigenvalues 02/15/12

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$ $n \times n$ matrix w. real entries, i.e. $a_{ij} \in \mathbb{R}$.

$\Rightarrow$ complex eigenvalues appear in pairs, with each member of the pair complex conjugate of the other.

For simplicity, assume A 2×2 matrix.

If $\lambda_1 = \alpha + i\beta$ $(\beta \neq 0)$ $\Longrightarrow$ $\boxed{\lambda_2 = \bar{\lambda}_1} = \alpha - i\beta$.

$\vec{v}_1$ eigenvector $\Longrightarrow$ $\boxed{\vec{v}_2 = \overline{\vec{v}_1}}$

$\Rightarrow$ general sol$^{\underline{n}}$ of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = \underbrace{C_1 e^{(\alpha+i\beta)t}\vec{v}_1 + C_2 e^{(\alpha-i\beta)t}\overline{\vec{v}_1}}_{\text{still complex.}}$$

Real form: Euler $\Rightarrow$ $e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$.

Write $\vec{v}_1 = \vec{p} + i\vec{q}$ (with $\vec{p}, \vec{q}$ real vectors)

$$\Rightarrow e^{(\alpha+i\beta)t}\vec{v}_1 = \underbrace{e^{\alpha t}\left[\cos\beta t \cdot \vec{p} - \sin\beta t \cdot \vec{q}\right]}_{\text{real part}} + i\underbrace{e^{\alpha t}\left[\sin\beta t \cdot \vec{p} + \cos\beta t \cdot \vec{q}\right]}_{\text{imaginary part}}$$

$$\Rightarrow \boxed{\vec{x}(t) = C_1 e^{\alpha t}\left[\cos\beta t \cdot \vec{p} - \sin\beta t \cdot \vec{q}\right] + C_2 e^{\alpha t}\left[\sin\beta t \cdot \vec{p} + \cos\beta t \cdot \vec{q}\right]}$$