Complex Tricks That Makes Life Simpler (and Easier)

02/17/12

- $z = a + ib \rightsquigarrow \bar{z} = a - ib$

$\Rightarrow \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

- If $\lambda = a + ib$ solves

$$a_n \lambda^n + \cdots + a_1 \lambda + a_0 = 0 \qquad (\underline{a_i \in \mathbb{R}}!)$$

then $\bar{\lambda} = a - ib$ also solves the same eq$^{\underline{n}}$:

$$a_n \bar{\lambda}^n + \cdots + a_1 \bar{\lambda} + a_0$$

$$= \overline{a_n \lambda^n + \cdots + a_1 \lambda + a_0} = \bar{0} = 0.$$

$\Rightarrow$ complex sol$^{\underline{n}}$s come in pairs!

- If A is an $n \times n$ matrix w/. real entries, and $\lambda = a + ib$ is an eigenvalue whose eigenvector is $\vec{v}$, then $\overline{\vec{v}}$ is an eigenvector of A w/. eigenvalue $\bar{\lambda}$:

$$A\vec{v} = \lambda \vec{v} \quad \Rightarrow \quad \overline{A\vec{v}} = \overline{\lambda \vec{v}}$$

$$\Rightarrow \quad A\overline{\vec{v}} = \bar{\lambda}\,\overline{\vec{v}}$$

Complex eigenvalues 02/15/12

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$ $n \times n$ matrix w. real entries, i.e. $a_{ij} \in \mathbb{R}$.

$\Rightarrow$ complex eigenvalues appear in pairs, with each member of the pair complex conjugate of the other.

For simplicity, assume A 2×2 matrix.

If $\lambda_1 = \alpha + i\beta \quad (\beta \neq 0) \Longrightarrow \boxed{\lambda_2 = \bar{\lambda}_1} = \alpha - i\beta$.

$\vec{v}_1$ eigenvector $\Longrightarrow \boxed{\vec{v}_2 = \overline{\vec{v}_1}}$

$\Rightarrow$ general sol$^{\underline{n}}$ of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = \underbrace{C_1 e^{(\alpha+i\beta)t}\vec{v}_1}_{} + \underbrace{C_2 e^{(\alpha-i\beta)t}\overline{\vec{v}_1}}_{}$$

still complex.

Real form: Euler $\Rightarrow$ $e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$.

Write $\vec{v}_1 = \vec{p} + i\vec{q}$ (with $\vec{p}, \vec{q}$ real vectors)

$$\Rightarrow e^{(\alpha+i\beta)t}\vec{v}_1 = \underbrace{e^{\alpha t}\left[\cos\beta t\cdot\vec{p} - \sin\beta t\cdot\vec{q}\right]}_{\text{real part}} + i\,\underbrace{e^{\alpha t}\left[\sin\beta t\cdot\vec{p} + \cos\beta t\cdot\vec{q}\right]}_{\text{imaginary part}}$$

$$\Rightarrow \boxed{\vec{x}(t) = C_1 e^{\alpha t}\left[\cos\beta t\cdot\vec{p} - \sin\beta t\cdot\vec{q}\right] + C_2 e^{\alpha t}\left[\sin\beta t\cdot\vec{p} + \cos\beta t\cdot\vec{q}\right]}$$

Decoupling and Nonhomog System

(02/17/12)

$$\boxed{\vec{x}\,' = A\vec{x} + \vec{f}(t)}$$

① Eigenvalues & eigenvectors $\lambda_1, \cdots, \lambda_n$

$\vec{v}_1, \cdots, \vec{v}_n$ linearly indep

$\Rightarrow$ $P = [\vec{v}_1 | \cdots | \vec{v}_n]$ diagonizes A:

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

② $\vec{x} = P\vec{u}$ decouples the system:

$$\vec{u}\,' = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \vec{u} + \underbrace{P^{-1}\vec{f}(t)}_{= \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}}$$

i.e.
$$\begin{cases} u_1' = \lambda_1 u_1 + g_1(t) \\ \quad \vdots \\ u_n' = \lambda_n u_n + g_n(t) \end{cases}$$

③ Solve each u_i (using integrating factors)

Solve $\vec{x}$ by
$$\vec{x} = P\vec{u}.$$