02/22/12

Last time:

Solving Systems of DE

$$\boxed{\vec{x}' = A\vec{x} + \vec{f}(t)}$$

Two methods available:

a) Eigenvalue - Eigenvector + Undetermined Coeff's.

General sol$^{\underline{n}}$ $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

— $\vec{x}_h(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$

provided $\vec{v}_1, \dots, \vec{v}_n$ a full set of linearly indep. eigenvectors

— If short of eigenvectors (this happens only when eigenvalues repeat), need fudge factor AND generalized eigenvectors

Say $\lambda_1 = \lambda_2$ repeat twice but only one eigenvector $\vec{v}_1$

$\rightsquigarrow$ one sol$^{\underline{n}}$ $e^{\lambda_1 t}\vec{v}_1$

second sol$^{\underline{n}}$: $te^{\lambda_1 t}\vec{v}_1 + e^{\lambda_1 t}\vec{u}$

w/. $\vec{u}$ the generalized eigenvector:

$$\boxed{(A - \lambda_1 I)\vec{u} = \vec{v}_1}$$

$\vec{x}' = A\vec{x}$

$\Longrightarrow$ general sol$^{\underline{n}}$

$c_1 e^{\lambda_1 t}\vec{v}_1 + c_2(\quad)$

$+ c_3 \cdots$

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a) "guessing game" for $\vec{x}_p$:

If $\vec{f}(t) = e^{kt}\vec{v}_0$, then $\vec{x}_p(t) = e^{kt}\vec{w}$ w/

$\vec{x}' = A\vec{x} + \vec{f}(t)$

$$\boxed{(A - kI)\vec{w} = -\vec{v}_0}$$

Works only when $\vec{f}(t)$ of special form

b). Diagonalization/Decoupling + Integrating factors.

If $\vec{v}_1, \cdots, \vec{v}_n$ a full set of linearly indep. eigenvectors

$$P = [\vec{v}_1 | \cdots | \vec{v}_n] : \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Set $\vec{x} = P\vec{u} \implies$

System decouples

$$\boxed{\vec{u}' = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\vec{u} + P^{-1}\vec{f}(t)} = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

$$\iff \begin{cases} u_1' = \lambda_1 u_1 + g_1(t) \\ \vdots \\ u_n' = \lambda_n u_n + g_n(t) \end{cases}$$

— Solve each using integrating factors $\Rightarrow \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

Finally general sol$^{\underline{n}}$.

$$\vec{x} = P\vec{u}.$$

Works (for us) only when we have a full set of eigenvectors (but can be generalized to cover all the situations.)

Stability of System of Homog. DE: (02/22/12)

$$\boxed{\vec{x}' = A\vec{x}}$$

- A solⁿ $\vec{x}(t)$ represents a curve in $\mathbb{R}^n$ whose tangent vectors are given by $A\vec{x}$ at $\vec{x}$.

- The zero solⁿ $\vec{x}(t) \equiv 0$ represents a constant curve — <u>equilibrium solⁿ</u>.

- An equilibrium solⁿ is called <u>stable</u> if all nearby solⁿs (i.e. nearby curves) stay bounded.
- Stability here completely determined by eigenvalues (and eigenvectors) of A:

  <u>stable</u> if all eigenvalues are negative or have negative real parts;

  <u>unstable</u> if one eigenvalue ~~has~~ is positive or has positive real part.

<u>Borderline Case</u>: next time.