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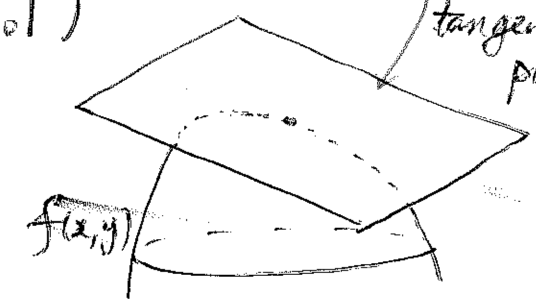
① Partial derivatives and linear approximation

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + o(|x-x_0|+|y-y_0|)$$



② Linearization of a nonlinear system

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \qquad (x_e, y_e) \text{ equilibrium} \implies f(x_e,y_e) = g(x_e,y_e) = 0$$

Set $u = (x - x_e)$, $v = (y - y_e)$ then

error terms

$$\begin{cases} u' = \frac{\partial f}{\partial x}(x_e,y_e)\, u + \frac{\partial f}{\partial y}(x_e,y_e)\, v + o(|u|+|v|) \\ v' = \frac{\partial g}{\partial x}(x_e,y_e)\, u + \frac{\partial g}{\partial y}(x_e,y_e)\, v + o(|u|+|v|) \end{cases}$$

Throwing away the error terms we obtain a linear system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = J(x_e,y_e)\begin{bmatrix} u \\ v \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

When u, v are small (i.e. (x,y) close to (x_e, y_e)), the behavior of u, v approximate that of (x,y).

Stability via linearization

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$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

$x = x_e$, $y = y_e$ equilibrium

Consider Jacobian

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

① If $J(x_e, y_e) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_e, y_e) & \frac{\partial f}{\partial y}(x_e, y_e) \\ \frac{\partial g}{\partial x}(x_e, y_e) & \frac{\partial g}{\partial y}(x_e, y_e) \end{bmatrix}$

has all its eigenvalues negative or with negative real parts, then (x_e, y_e) is stable.

② If one of the eigenvalues of $J(x_e, y_e)$ is positive or has positive real part, then (x_e, y_e) is unstable.

③ Borderline cases can not be decided by eigenvalues of the Jacobian!

i.e. none of the eigenvalues is positive or has positive real part, And one is zero or purely imaginary.