System (15) has the same linearization (14) as system (13); by reasoning similar to that above, you should predict that the solutions to (15) will spiral outward, making the equilibrium at the origin unstable. ■

Summary

An equilibrium solution of a nonlinear autonomous system can be analyzed by studying a closely related linear system called the linearization. In most cases the classification of this linearization according to the scheme of Chapter 6 predicts the stability of the solution of the nonlinear system, and often its geometry as well.

7.2 Problems

Original Equilibrium *Show that the system in each of Problems 1–6 has an equilibrium point at the origin. Compute the Jacobian, then discuss the type and stability of the equilibrium point.*

1. $x' = -2x + 3y + xy$, $y' = -x + y - 2xy^2$
2. $x' = -y - x^3$, $y' = x - y^3$
3. $x' = x + y + 2xy$, $y' = -2x + y + y^3$
4. $x' = y$, $y' = -\sin x - y$
5. $x' = x + y^2$, $y' = x^2 + y^2$
6. $x' = \sin y$, $y' = -\sin x + y$

Almost Linear *For each system in Problems 7–9, determine the type and stability of each real equilibrium point by calculating the Jacobian matrix at each equilibrium.*

7. $x' = 1 - xy$, $y' = x - y^3$
8. $x' = x - 3y + 2xy$, $y' = 4x - 6y - xy$
9. $x' = 4x - x^3 - xy^2$, $y' = 4y - x^2y - y^3$

10. **Linearization Completion** Complete the analysis of Example 1 by providing the details of the linearization about the point $(-1, 0)$ for $x' = y$, $y' = -y + x - x^3$.

11. **Strong Spring** Determine the stability of the equilibrium solutions of the strong spring $\ddot{x} + \dot{x} + x + x^3 = 0$.

12. **Weak Spring** Determine the stability of the equilibrium solutions of the weak spring $\ddot{x} + \dot{x} + x - x^3 = 0$.

13. **Liénard Equation** A generalized damped mass-spring equation, the Liénard[4] equation, is $\ddot{x} + p(x)\dot{x} + q(x) = 0$. If $q(0) = 0$, $q'(0) > 0$ and $p(0) > 0$, show that the origin is a stable equilibrium point.

14. **Conservative Equation** A second-order differential equation of the form $\ddot{x} + F(x) = 0$ is called a **conservative differential equation**. (See Sec. 4.6.) Find the equilibrium points of the conservative equation

$$\ddot{x} + x - x^2 - 2x^3 = 0$$

and determine their type and stability.

15. **Predator-Prey Equations** In Sec. 2.5 we introduced the Lotka-Volterra predator-prey system

$$x' = (a - by)x,$$
$$y' = (cx - d)y$$

and determined its equilibrium points $(0, 0)$ and $(d/c, a/b)$. Use the Jacobian matrix to analyze the stability around the equilibrium point $(d/c, a/b)$. Interpret the trajectories of this system as plotted in Fig. 2.5.3.

IDE **Lotka-Volterra**
This tool lets you experiment on screen.

Damped Mass-Spring Systems *The second-order linear differential equation $m\ddot{x} + b\dot{x} + kx = 0$ models vibrations of a mass m attached to a spring with spring constant k and with damping constant b. For the nonlinear variations in Problems 16–19, use your intuition to decide whether the zero solution ($x = \dot{x} \equiv 0$) is stable or unstable. Check your intuition by transforming to a first-order system and linearizing.*

[4]Alfred Liénard (1869–1958) was a French mathematician and applied physicist.