Sol^{n}s HW 03/09/12

1. $A = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}$ Find eigenvalues:

$$0 = \begin{vmatrix} 3-\lambda & 2 \\ 7 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 20 = (\lambda - 5)(\lambda + 4)$$

$\Rightarrow \lambda_1 = 5, \quad \lambda_2 = -4.$

Find eigenvectors: $\lambda_1 = 5$: $A - 5I = \begin{bmatrix} -2 & 2 \\ 7 & -7 \end{bmatrix} \to$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$\lambda_2 = -4$: $A - (-4)I = \begin{bmatrix} 7 & 2 \\ 7 & 2 \end{bmatrix} \longrightarrow$

$\begin{bmatrix} 7 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{V}_2 = \begin{bmatrix} 1 \\ -7/2 \end{bmatrix}.$

$\Rightarrow P = [\vec{V}_1 | \vec{V}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -7/2 \end{bmatrix}$ diagonalizes A:

$P^{-1}AP = D = \begin{bmatrix} 5 & 0 \\ 0 & -4 \end{bmatrix}$ i.e. $A = PDP^{-1}$

$$\Rightarrow e^{tA} = e^{t(PDP^{-1})} = P e^{tD} P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -7/2 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} \frac{7}{9} & \frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} \end{bmatrix} = \begin{bmatrix} \frac{7}{9}e^{5t} + \frac{2}{9}e^{-4t} & \frac{2}{9}e^{5t} - \frac{2}{9}e^{-4t} \\ \frac{7}{9}e^{5t} - \frac{7}{9}e^{-4t} & \frac{2}{9}e^{5t} + \frac{7}{9}e^{-4t} \end{bmatrix}$$

2. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda_1 = i$, $\lambda_2 = -i$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Set

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \Rightarrow P^{-1}AP = D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

or $A = PDP^{-1} \Longrightarrow$

$$e^{tA} = e^{t(PDP^{-1})} = P e^{tD} P^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{it}+e^{-it}}{2} & \frac{e^{it}-e^{-it}}{2i} \\ \frac{ie^{it}-ie^{-it}}{2} & \frac{e^{it}+e^{-it}}{2} \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

by plugging in Euler's formula $e^{it} = \cos t + i\sin t$

$(\Rightarrow e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i\sin t)$.

3. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = I + N$, $\quad N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Compute

$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = N^2 N = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow e^{tN} = I + tN + \frac{t^2}{2!}N^2 + \cancel{\frac{t^3}{3!}N^3}^{\,0} + \cancel{\cdots}^{\,0}$$

$$= I + tN + \frac{t^2}{2}N^2 = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$e^{tA} = e^{t(I+N)} = e^{tI} \cdot e^{tN} \quad \left(\text{Since } \underline{IN = NI}\right)$$

$$= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t & \frac{t^2}{2}e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}.$$