ADIABATIC LIMIT, HEAT KERNEL AND ANALYTIC TORSION

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Introduction

The adiabatic limit refers to the geometric degeneration in which the metric is been blown up along certain directions. The study of the adiabatic limit of geometric invariants is initiated by E. Witten [39], who relates the adiabatic limit of the η-invariant to the holonomy of determinant line bundle, the so called "global anomaly". In this case the manifold is fibered over a circle and the metric is been blown up along the circle direction. Witten's result was given full mathematical treatment in [8], [9] and [13], see also [16]. In [4], J.-M. Bismut and J. Cheeger studied the adiabatic limit of the eta invariant for a general fibration of closed manifolds. Assuming the invertibility of the Dirac family along the fibers, they showed that the adiabatic limit of the η-invariant of a Dirac operator on the total space is expressible in terms of a canonically constructed differential form, $\tilde{\eta}$, on the base. The Bismut-Cheeger $\tilde{\eta}$ form is a higher dimensional analogue of the η-invariant and it is exactly the boundary correction term in the families index theorem for manifolds with boundary, [5], [6]. The families index theorem for manifolds with boundary has since been established in full generality by Melrose-Piazza in [31], [32].

Around the same time, Mazzeo and Melrose took on the analytic aspect of the adiabatic limit [27] and studied the uniform structure of the Green's operator of the Laplacian in the adiabatic limit. Their analysis enables the first author to prove the general adiabatic limit formula in [14]. The adiabatic limit formula is used in [7] to prove a generalization of the Hirzebruch conjecture on the signature defect (Cf. [1],[35]). Other applications of adiabatic limit technique can be found in [40], [18] and [36].

The main purpose of this paper is to study the uniform behavior of the heat kernel in the adiabatic limit. The adiabatic limit introduces degeneracy along the base directions and gives rise to new singularity for the heat kernel which interacts in a complicated way with the usual diagonal singularity. We resolve this difficulty by lifting the heat kernel to a larger space obtained by blowing up certain submanifolds of the usual carrier space of the heat kernel (times the adiabatic direction). The new space is a manifold with corner and the uniform structure of the adiabatic heat kernel can be expressed by stating that it gives rise to a polyhomogeneous conormal distribution on the new space.

More precisely, if $\phi : M \longrightarrow Y$ is a fibration with typical fibre F, the adiabatic metric is the one-parameter family of metrics $x^{-2}g_x$, with $g_x = \phi^* h + x^2 g$, on M, where h is a metric on Y and g a symmetric 2-tensor on M which restricts to Riemannian metrics on the fibers. Note that g_x collapses the fibration to the base space in the limit $x \to 0$. Our main object of study is the regularity of the heat

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kernel $\exp(-t\Delta_x)$ of the Laplacian for the metric g_x. We prefer to write the heat kernel as $\exp(-\frac{t}{x^2}\,{}^a\!\Delta)$ where ${}^a\!\Delta$ is the Laplacian of the adiabatic metric g_x/x^2. The techniques of [27] are extended to construct the 'adiabatic heat calculus,' of which this heat kernel is a fairly typical element. In particular ${}^a\!\Delta$ is considered as an operator on the rescaled bundle ${}^a\!\Lambda^*$ which is the bundle of exterior powers of the adiabatic cotangent bundle ${}^aT^*M_a$, $M_a = M \times [0,1]$. This rescaling and parabolic blow-up methods are used to define the 'adiabatic heat space,' M^2_A, from $[0,\infty)\times M^2 \times [0,1]$. The adiabatic heat calculus, $\Psi^{*,*,*}_A(M;{}^a\!\Lambda^*)$, is defined in terms of the Schwartz kernels of its elements which are smooth sections over M^2_A of the kernel bundle, a weighted version of the lift of the homomorphism bundle tensored with a density bundle. The operator $\exp(-t\Delta_x)$ is constructed in this calculus directly using the three symbol homomorphisms. Each of these maps is defined by evaluation of the Schwartz kernel at one of the boundary hypersurfaces of M^2_A.

The first of the symbol maps is just a parametrized version of the corresponding map for the ordinary heat calculus and is used in exactly the same way; in this case it takes values in the fibrewise operators on the bundle ${}^aTM_a$ over M_a. The second map is more global and takes values in fibrewise operators on the space $[0,\infty)\times\phi^*(TY)$ as a bundle over Y. In fact the image consists of elements of the heat calculus for the fibres $T_yY \times F_y = {}^YTM_{(y,z)}$ for each $y \in Y, z \in F$. The ordinary heat calculus can be used to invert these operators. The third map is the 'obvious' boundary map obtained by setting $x = 0$. In practice it is necessary to consider a 'reduced' normal operator at this face.

Our first result is

Theorem 0.1. *The heat kernel is an element*

$$\exp(-x^{-2}t\,{}^a\!\Delta) \in \Psi^{-2,-2,0}_{A,E}(M;{}^a\!\Lambda^k) \tag{0.1}$$

with normal operators

$$N_{h,-2} = (4\pi)^{-\frac{n}{2}}\exp(-\frac{1}{4}|v|^2_a) \tag{0.2}$$

$$N_{A,-2} = \exp(-T\Delta_A),\ T = x^{-2}t, \tag{0.3}$$

$$N_{a,0} = \exp(-t\Delta_Y) \tag{0.4}$$

where Δ_A is the fibrewise Laplacian on the bundle ${}^YTM$ and Δ_Y is the reduced Laplacian on Y, that is, the Laplacian on Y twisted by the flat bundle of the fiber cohomologies.

Here the subscript E indicates a refinement of the adiabatic heat calculus which will be discussed in §3. Note that the right hand side of (0.2) is the Euclidean heat kernel on the tangent space evaluated at time $t = 1$ and $(v, 0)$. In (0.3) the fibers of ${}^YTM$ are ${}^YTM_{(y,z)} = T_yY \times F_y$, hence the fiberwise Laplacian Δ_A consists of the Euclidean Laplacian on T_yY together with the Laplacian of F_y. The heat kernel for the Eucildean Laplacian here should be evaluated at $(v, 0)$. As we will see later, Theorem 0.1 contains all the usual uniform regularity property of the adiabatic heat kernel.

However, in applications to geometric problems such as the study of the eta invariant and the analytic torsion, one needs to incorporate the Getzler's rescaling [23], which has important implication for supertrace cancellations. We show how to build it into the calculus using bundle filtrations, resulting in the rescaled adiabatic heat calculus $\Psi^*_{A,G}(M;{}^a\Lambda^*)$. This enables us to refine Theorem 0.1 to obtain our main result.

Theorem 0.2. *The heat kernel is an element of the rescaled adiabatic heat calculus $\Psi^*_{A,G}(M;{}^a\Lambda^*)$,*

$$\exp(-\frac{t}{x^2}\,{}^a\Delta) \in \Psi^{-2,-2,0}_{A,G}(M;{}^a\Lambda^*). \tag{0.5}$$

Moreover, the normal operator at the temporal front face is given by

$$N_{h,G,-2} = e^{-[\mathcal{H}-\frac{1}{8}C(R)]} \tag{0.6}$$

where $\mathcal{H}$ is the generalized harmonic oscillator on the tangent space T_pM defined in (5.39) *and $C(R)$ is the quantization of the curvature tensor R defined in* (5.38)*. The normal operator at the adiabatic front face is*

$$N_{A,G,-2} = \exp(-\mathcal{H}_Y)\exp(\frac{1}{8}C(R_Y))\exp(-\mathcal{A}_T^2). \tag{0.7}$$

Here $\mathcal{H}_Y$ is the generalized harmonic oscillator on the fibres of ${}^xTY$, and $\mathcal{A}_T$ is the rescaled Bismut superconnection:

$$\mathcal{A}_T^2 = -T[\nabla_{e_i} + \frac{1}{2}T^{-\frac{1}{2}}\langle\nabla_{e_i}e_j, f_\alpha\rangle c_l(e_i)c_l(f_\alpha) + \frac{1}{4}\langle\nabla_{e_i}f_\alpha, f_\beta\rangle c_l(f_\alpha)c_l(f_\beta)]^2 + \frac{1}{4}TK_F, \tag{0.8}$$

where e_i is an orthonormal basis of the fibers and f_α that of the base, and K_F denotes the scalar curvature of the fibers.

We then apply this to the study of the adiabatic limit of the analytic torsion. The analytic torsion, $T_\rho(M,g)$, introduced by Ray and Singer [37], is a geometric invariant associated to each orthogonal representation, ρ, of the fundamental group of a compact manifold M with Riemann metric g (later extended to more general representations such as unimodular ones [34], [11]). It depends smoothly on g and is identically equal to 1 in even dimensions. As conjectured in [37] it has been identified with the Reidemeister torsion and this is the celebrated Cheeger-Müller theorem ([12], [33]. See also [34], [11] for generalizations.). Using the uniform behavior of the heat kernel, we show that the torsion (our normalization corresponds to the square of that in [37]) in the adiabatic limit satisfies

$$T_\rho(M,g_x) = x^{-2\alpha}b(x),\ \alpha\in\mathbb{N},\ b\in\mathcal{C}^\infty([0,1]). \tag{0.9}$$

Thus, whilst not necessarily smooth in the adiabatic limit, $x \downarrow 0$, the analytic torsion behaves quite simply.

The characteristic exponent in (0.9) can be expressed in terms of the Leray spectral sequence for the cohomology twisted by ρ as

$$\alpha = -\chi_2(M) + \chi_2(Y,\mathcal{H}^*(F)) + \sum_{r\geq 2}(r-1)[\chi_2(E_r) - \chi_2(E_{r+1})]. \tag{0.10}$$

Here if β_j is the dimension in degree j of the cohomology then $\chi_2(E_r) = \sum_j j(-1)^j\beta_j$. For $\chi_2(M)$ the β_j are the Betti numbers of M, for $\chi_2(Y,\mathcal{H}^*(F))$ they are the dimensions of the twisted cohomology spaces of Y and for E_r they are the dimensions, $\beta_{j,r}$, of the rth term, (E_r,d_r), of the spectral sequence.

The limiting value of the smooth factor in (0.9) depends on the parity of the dimension of the fibres. If the fibres are even-dimensional then

$$b(0) = \prod_{j=1}^{\dim F} \left[T_{\phi_*(\rho)\otimes\rho'_j}(Y, h) \right]^{(-1)^j j} \prod_{r\geq 2} \tau(E_r, d_r),\ \dim Y \text{ odd}, \tag{0.11}$$

where ρ'_j is the representation of $\pi_1(Y)$ associated to the flat bundle given by the fibre cohomology in dimension j, twisted by $\rho_{\restriction\pi_1(F)}$, and $\tau(E_r, d_r)$ is the torsion of the finite complex. The other case, when the fibres are odd-dimensional, is only a little more complicated. If R_Y is the curvature operator of the metric h on Y then the Gauss-Bonnet theorem states that the Euler characteristic of Y is given by the integral over Y of the Pfaffian density $(2\pi)^{-n}\operatorname{Pf}(R_Y)$, $n = \dim Y$. Consider the weighted integral

$$\chi_\rho(Y, \phi, g, h) = (2\pi)^{-n} \int_Y \operatorname{Pf}(R_Y) \log T_{\rho(y)}(F_y, g_y) \tag{0.12}$$

where g_y is the restriction of g to the fibre $F_y = \phi^{-1}(y)$, $y \in Y$, and $\rho(y)$ is the representation of $\pi_1(F_y)$ induced by ρ. Then (0.10) still holds and

$$b(0) = e^{\chi_\rho(Y,\phi,g,h)} \prod_{r\geq 2} \tau(E_r, d_r),\ \dim Y \text{ even}. \tag{0.13}$$

If $\dim Y$ is even the twisted torsion factor in (0.11) reduces to 1 and if $\dim Y$ is odd the weighted Euler characteristic in (0.12) is zero, so these two formulæ can be combined to give one in which the parity does not appear explicitly.

Theorem 0.3. *The analytic torsion of an adiabatic metric g_x for a fibration satisfies*

$$\begin{gathered} \log T_\rho(M, g_x) \\ = -2\left(-\chi_2(M) + \chi_2(Y, \mathcal{H}^*(F)) + \sum_{r\geq 2}(r-1)[\chi_2(E_r) - \chi_2(E_{r+1})] \right) \log x \\ +\chi_\rho(Y, \phi, g, h) + \sum_{j=1}^{\dim F} (-1)^j j \log T_{\phi_*(\rho)\otimes\rho'}(Y, h) + \sum_{r\geq 2} \log\tau(E_r, d_r) + xb'(x), \\ b' \in \mathcal{C}^\infty([0,1]). \end{gathered} \tag{0.14}$$

Remark 1 For the holomorphic analogue see [3]. It should be pointed out that, in our case, the analytic torsion form of [10] did appear in the formula. However the higher degree terms in the torsion form are cancelled by the Pffafian term. Our proof also extends to the holomorphic case.

Remark 2 Under appropriate acyclicity conditions formula (0.14) reduces to the purely topological formulas for the Reidemeister torsion obtained by D. Fried [21], D. Freed [22], and Lück-Schick-Thielman [26].

The analytic torsion is defined in terms of the torsion zeta function

$$\log T_\rho(M, g) = \zeta'_T(0) \tag{0.15}$$

where $\zeta_T(s)$ is a meromorphic function of $s \in \mathbb{C}$ which is regular at $s = 0$. For $\operatorname{Re} s >> 0$

$$\zeta_T(s) = \frac{1}{\Gamma(s)} \sum_{j=1}^{\dim M} (-1)^j j \int_0^\infty t^s \operatorname{Tr}\left(\exp(-t\underline{\Delta}_j)\right) \frac{dt}{t} \tag{0.16}$$

where $\underline{\Delta}_j$ is the Laplacian on j-forms with null space removed (i.e. acting on the orthocomplement of the harmonic forms). The proof of (0.14) is thus reduced to a sufficiently fine understanding of the heat kernel in the adiabatic limit.

The paper is organized as follows. In §1 we recall the construction of the heat kernel in the standard case of a compact manifold. This is done to introduce, in a simple context, the approach via parabolic blow-up, which is used here. The appropriate notion of parabolic blow up is described in §2. The important finite time properties are then summarized by the statement that the heat kernel is an element of order -2 of the even part of the heat calculus acting on the exterior bundle

$$\exp(-t\Delta) \in \Psi^{-2}_{h,E}(M; \Lambda^*). \tag{0.17}$$

This in turn is a regularity statement for the lift of the Schwartz kernel from the space $[0,\infty) \times M^2$, where it is usually defined, to the heat space, M^2_h, obtained by t-parabolic blow-up of the diagonal at $t = 0$. This is discussed in §3. The heat calculus has a 'symbol map', the normal homomorphism, into the homogeneous and translation-invariant part of the heat calculus on (the compactification of) the fibres of the tangent bundle to M. Under this map the heat kernel is carried to the family (over M) of heat kernels for the fibrewise Laplacian on TM :

$$\begin{gathered} N_{h,-2}(\exp(-t\Delta))(m,v) = \exp(-\Delta_m)(v), \\ \Delta_m = \sum_{j,k=1}^{\dim M} g^{jk}(m) D_{v^j} D_{v^k} \text{ on } T_m M. \end{gathered} \tag{0.18}$$

Conversely (0.18) allows an iterative construction of the heat kernel. The appropriate composition properties for the heat calculus, allowing this iterative approach, are also discussed in §3.

For a general Laplacian, P, without null space, the heat kernel is, for $t > 0$, a smoothing operator which decreases exponentially as $t \to \infty$. As a result the zeta function, obtained by Mellin transform of the heat kernel is, following Seeley [38], meromorphic with poles in $\operatorname{Re} s \geq 0$ only at $s = \frac{1}{2}\dim M - k$, $k = 0, 1, \ldots$. These poles come from the short-time asymptotics of the heat kernel. If Δ is the Laplacian on forms then (0.16) can be rewritten

$$\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{STr}\left(N \exp(-t\underline{\Delta})\right) \frac{dt}{t} \tag{0.19}$$

where N is the number operator, acting as k on Λ^k, and STr is the supertrace functional, i.e.

$$\operatorname{STr}(A) = \operatorname{Tr}(QA),\ Q = (-1)^k \text{ on } \Lambda^k. \tag{0.20}$$

The algebraic properties of the supertrace imply that there are actually no poles of $\zeta_T(s)$ in $s > \frac{1}{2}$.

To see this, in §5, the rescaling argument of Getzler [23] (see also [2]) is formulated in terms of the heat calculus. This is done by defining a 'rescaled' version of the homomorphism bundle over M^2_h, of which the heat kernel is a smooth section, and then showing that the kernel is again a smooth section of the rescaled bundle. The general process of rescaling a bundle is discussed in §4. Under this rescaling (of length $\dim M + 1$), defined by extension of the Clifford degree of a homomorphism of the exterior algebra, the (pointwise) supertrace functional lies in the maximal graded quotient. The number operator has degree two so it follows that

$$\begin{gathered}\operatorname{str}(N\exp(-t\varDelta)) \in t^{-\frac12}\mathcal{C}^\infty([0,\infty)\times M;\Omega M),\\ \operatorname{STr}(N\exp(-t\varDelta)) = \int_M \operatorname{str}(N\exp(-t\varDelta)) \in t^{-\frac12}\mathcal{C}^\infty([0,\infty)).\end{gathered}\tag{0.21}$$

Moreover the leading term can be deduced from the normal operator for the rescaled calculus:

$$\begin{gathered}t^{\frac12}\operatorname{str}(N\exp(-t\varDelta_j))_{\restriction t=0} = c(n)\sum_{k=1}^{\dim M}(-1)^k\operatorname{Pf}(R_k)\wedge\omega_k \in \mathcal{C}^\infty(M;\Omega M)\\ c(n) = 2i(-1)^{\frac12(n+1)}(16\pi)^{-\frac12 n}\end{gathered}\tag{0.22}$$

where with respect to any local orthonormal frame, ω_k of T^*M, R_k is the curvature operator with kth row and column deleted,

$$R_k = \sum_{i,j\neq k} R_{ijk\ell}\,\omega_i\omega_j,\tag{0.23}$$

and $\operatorname{Pf}(R_k)$ is its Pfaffian as an antisymmetric matrix. The cancellation formula (0.21) shows that $\zeta_T(s)$ has only one pole in $\operatorname{Re} s \geq 0$, at $s = \frac12$, and (0.22) gives its residue. It is thus straightforward to give a formula for $\zeta_T(s)$ which is explicitly regular near $s = 0$; see Corollary 5.2.

This representation is used to obtain (0.9). As already noted, the main step is a clear analysis of the regularity of the heat kernel $\exp(-t\varDelta_x)$ of the Laplacian for the metric g_x; this is carried out in §6-§9. In §10 the incorporation of Getzler's rescaling into the ordinary heat calculus is extended to the adiabatic heat calculus to give the cancellation effects for the supertrace. Two related but distinct rescaling are required, one just as in the standard case and the other at the adiabatic front face. For finite times this results in the following description of the regularity of the supertrace. Consider the space $Q = [0,\infty)_t \times [0,1]_x$ on the interior of which $\operatorname{STr}(\exp(-t\varDelta_x))$ is defined and $\mathcal{C}^\infty$. Let Q_2 be obtained by t-parabolic blow-up of the corner $\{t = x = 0\}$ with $\beta_1 : Q_2 \longrightarrow Q$ the blow-down map; the subscripts 1 and 2 here refer to the Leray spectral sequence. Then Q_2 has boundary lines tf, arising from $t = 0$, af from the blow-up, ab from $x = 0$ and ef from $x = 1$. For appropriate defining functions, ρ_F, for the boundary lines, F, the lift to Q_2 satisfies

$$\beta_1^*\operatorname{STr}(\exp(-t\varDelta_x)) \in \begin{cases}\rho_{\mathrm{tf}}^{-1}\rho_{\mathrm{af}}^{-1}\mathcal{C}^\infty(Q_2), & \dim Y \text{ odd}\\ \rho_{\mathrm{tf}}^{-1}\mathcal{C}^\infty(Q_2), & \dim Y \text{ even}.\end{cases}\tag{0.24}$$

We also need to discuss the behaviour of the heat kernel as $t \to \infty$. To do so we use results from [14]. The kernel decomposes as $t \to \infty$ into a part which is rapidly decreasing, and uniformly smoothing, plus finite rank parts corresponding to the

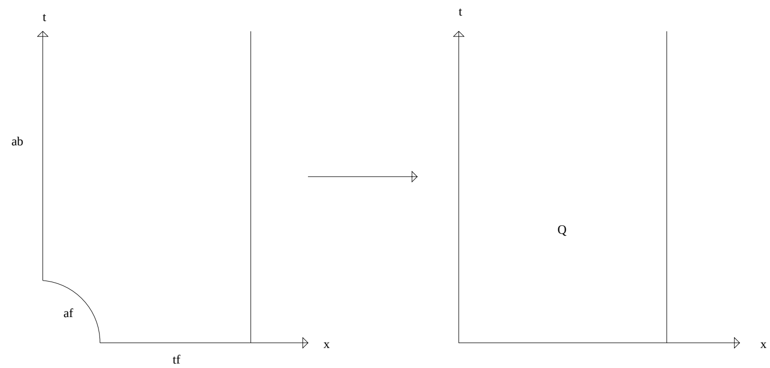

FIGURE 1. $\beta_1 : Q_2 \to Q$

small eigenvalues of the Laplacian. These are in turn associated to the individual terms (E_r, d_r), $r \geq 2$, of the Leray spectral sequence.

The leading terms at the boundary faces in (0.24) can be deduced from the construction of the heat kernel, and ultimately therefore from the solutions of the three model problems arising from the rescaled symbol maps. Together with the behaviour of the small eigenvalues this leads directly to (0.9), (0.10) and (0.14); the final derivation is given in §11.

In a continuation of this paper, [17], these results on the analytic torsion are extended to manifolds with boundary. In [30] there is a related discussion of the analytic torsion for a b-metric on a compact manifold with boundary.

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1. Hadamard's construction

To orient the reader towards our detailed description of the behaviour in the adiabatic limit of the heat kernel, and in particular its trace, we shall first recall the 'classical' case. Thus we shall show how to construct the heat kernel for an elliptic differential operator, P, of second order on a compact $\mathcal{C}^\infty$ manifold under the assumption that P has positive, diagonal, principal symbol. Such a construction is well-known and can certainly be carried out by the method developed by Hadamard [24] for the wave equation. The construction below proceeds rather formally, in terms of the heat calculus. The only novelty here is in the definition, and discussion, of the calculus itself in §3 in which a systematic use of the process of parabolic blow-up is made. As opposed to the standard construction of Hadamard this allows us to generalize, to the adiabatic limit and, later, to the case of boundary problems with limited changes.

The basic model for the heat kernel is the Euclidean case, $\mathit{\Delta}_E = D_1^2 + \cdots + D_n^2$ where $D_j = -i\partial/\partial x_j$ on $\mathbb{R}^n$. Then the function

$$\Phi'(t,x) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right),\ t > 0,\ x \in \mathbb{R}^n \tag{1.1}$$

has a unique extension to $\widetilde{\Phi}' \in \mathcal{S}'(\mathbb{R}^{1+n})$ which is locally integrable in t with values in $\mathcal{S}'(\mathbb{R}^n)$ and vanishes in $t<0$. Acting as a convolution operator this is the unique tempered forward inverse of $\partial_t + \Delta_E$.

The convolution operator defined by $\widetilde{\Phi}'$ can be embedded into a graded algebra of operators by generalizing (1.1). For $p<0$ the operators of order p in this Euclidean heat calculus, or perhaps more correctly homogeneous and translation-invariant heat calculus, have kernels of the form

$$K(t,x) = \begin{cases} t^{-\frac{n}{2}-1-\frac{p}{2}}\kappa(\frac{x}{t^{\frac{1}{2}}})|dx|, & t>0 \\ 0, & t \le 0 \end{cases} \tag{1.2}$$

where $\kappa \in \mathcal{S}(\mathbb{R}^n)$. Again K is locally integrable in t as a function with values in $\mathcal{S}'(\mathbb{R}^n)$ and so fixes a convolution operator on $\mathbb{R}^{1+n}$. We shall denote this space of operators as $\Psi^p_{\mathrm{th}}(\mathbb{R}^n)$. For $p=0$ there is a similar, but slightly more subtle, definition of the space of operators $\Psi^0_{\mathrm{th}}(\mathbb{R}^n)$. Namely this space is the span of the identity and the convolution operators given by distributions as in (1.2) with $p=0$ and satisfying in addition the condition

$$\int_{\mathbb{R}^n} \kappa(x)dx = 0. \tag{1.3}$$

In this case the kernel is not locally integrable as a function of t; nevertheless the mean value condition (1.3) means that

$$\langle K, \phi\rangle = \lim_{\delta\downarrow 0} \int\limits_{|x|^2+t>\delta^2,\ t\ge 0} \phi(t,x)\kappa(\frac{x}{t^{\frac{1}{2}}})t^{-\frac{n}{2}-1}dxdt,\ \phi\in\mathcal{S}(\mathbb{R}^{1+n}) \tag{1.4}$$

defines $K \in \mathcal{S}'(\mathbb{R}^{1+n})$. It is also straightforward to check that these spaces of operators are invariant under linear transformations of $\mathbb{R}^n$, so the spaces $\Psi^p_{\mathrm{th}}(V)$ are well defined for any vector space V and $p \le 0$. Essentially by definition we have 'normal operators' for the spaces which are isomorphisms

$$\begin{aligned} N_{h,p} :\ & \Psi^p_{\mathrm{th}}(V) \ni K \longrightarrow \kappa \in \mathcal{S}(V;\Omega),\ p<0 \\ N_{h,0} :\ & \Psi^0_{\mathrm{th}}(V) \ni (c\,\mathrm{Id} + K) \longrightarrow (c,\kappa) \in \mathbb{C}\oplus\overline{\mathcal{S}}(V;\Omega), \end{aligned} \tag{1.5}$$

where $\overline{\mathcal{S}}(V;\Omega)$ is the null space of the integral on $\mathcal{S}(V;\Omega)$.

For Schwartz densities on a vector space, V, we can define a two-parameter family of products in any linear coordinates by

$$\begin{aligned} & a\star_{p,q} b(x)|dx| = \\ & \int_0^1\int\limits_V a((1-s)^{-\frac{1}{2}}x-y)b(s^{-\frac{1}{2}}y)dy(1-s)^{-\frac{n}{2}-1+p}s^{-\frac{n}{2}-1+q}ds|dx|,\ p,q<0. \end{aligned} \tag{1.6}$$

If a (resp b) has mean value 0 the definition extends to $p=0$ (resp $q=0$); in case both p and q are zero the result has mean value zero. The product is extended to the sum in (1.5) by letting $\mathbb{C}$ act as multiples of the identity. These products give the composition law for the normal operators in the Euclidean heat calculus

$$\begin{aligned} & \Psi^p_{\mathrm{th}}(V)\circ\Psi^q_{\mathrm{th}}(V) \subset \Psi^{p+q}_{\mathrm{th}}(V),\ p,q\le 0 \\ & N_{h,p+q}(A\circ B) = N_{h,p}(A)\star_{p,q} N_{h,q}(B) \end{aligned} \tag{1.7}$$

as follows by a simple computation. Since the calculus and products are invariant under linear transformations both extend to the fibres of any vector bundle and can be further generalized to act on sections of the lift of another vector bundle over the base.

Replacing the Euclidean Laplacian by a differential operator P, as described above, acting on sections of an Hermitian vector bundle over the Riemann manifold M and having principal symbol $|\xi|_x^2 \operatorname{Id}$ at $(x,\xi) \in T^*M$ we wish to construct an analogue of $\widetilde{\Phi}'$; this will be the kernel of an operator which is still a convolution operator in t but not in the spatial variables. These operators, which will be discussed in §3, are defined directly in terms of their Schwartz' kernels. The kernels are sections of a density bundle over the manifold M_h^2, which is the product $[0,\infty)_t \times M^2$ with the product manifold $\{0\} \times \operatorname{Diag}$ blown up. Here $\operatorname{Diag} \subset M^2$ is the diagonal, so functions on M_h which are smooth up to the new boundary hypersurface (the 'front face') produced by the blow up are really singular, in the manner appropriate for the heat kernel, at $\{0\} \times \operatorname{Diag}$ when considered as functions on $(0,\infty) \times M^2$.

If U is a vector bundle over M we denote by $\Psi_h^{-k}(M;U)$ this space of heat operators, discussed in detail in §3, of order $-k$, for $k \in \mathbb{N}_0 = \{0,1,\dots\}$. These operators act on $\dot{\mathcal{C}}^\infty([0,\infty) \times M;U)$, which is the space of $\mathcal{C}^\infty$ sections of U, lifted to $[0,\infty) \times M$, vanishing with all derivatives at $\{0\} \times M$. There is a well-defined *normal operator*:

$$N_{h,-k} : \Psi_h^{-k}(M;U) \twoheadrightarrow \mathcal{S}(TM;\Omega_{\text{fibre}} \otimes \pi_M^* \hom(U)),\ k \in \mathbb{N}. \tag{1.8}$$

Here $\mathcal{S}$ denotes the (fibre) Schwartz space on TM. This normal operator is determined by the leading coefficient of the Scwartz kernel at the front face of M_h^2. For operators of order 0 the normal operator becomes

$$N_{h,0} : \Psi_h^0(M;U) \twoheadrightarrow \mathbb{C} \oplus \overline{\mathcal{S}}(TM;\Omega_{\text{fibre}} \otimes \pi_M^* \hom(U)), \tag{1.9}$$

where $\overline{\mathcal{S}}(TM;\Omega_{\text{fibre}})$ denotes the space of Schwartz fibre densities with mean value zero on each fibre. The following result is proved in §3.

Proposition 1.1. *The maps* (1.8) *and* (1.9) *extend to normal homomorphisms which filter* $\Psi_h^0(M;U)$ *as an asymptotically complete algebra of operators on* $\dot{\mathcal{C}}^\infty([0,\infty)\times M;U)$ *i.e. the null space of* $N_{h,p}$ *is exactly* $\Psi_h^{p-1}(M;U)$ *and*
(1.10)

$$A \in \Psi_h^{-k}(M;U),\ B \in \Psi_h^{-j}(M;U) \Longrightarrow A \circ B \in \Psi_h^{-k-j}(M;U),\ j,k \in \mathbb{N},\ \textit{with}$$
$$N_{h,-k-j}(A \circ B) = N_{h,-k}(A) \star_{k,j} N_{h,-j}(B).$$

Any element of $\operatorname{Id} + \Psi_h^{-1}(M;U)$ *is invertible with inverse in the same space.*

In view of (1.5), the maps in (1.8) and (1.9) can be interpreted as homomorphisms into the homogeneous and translation-invariant heat calculus on the fibres of TM.

This calculus of operators of non-positive order can be extended to non-positive real orders and positive orders as well, but all we need is the composition properties with differential operators. The following result, which follows easily from the definition, is also proved in §3.

Proposition 1.2. *If* $P \in \operatorname{Diff}^k(M;U)$ *and* $j \geq k$ *composition gives*

$$\begin{gathered} \Psi_h^{-j}(M;U) \ni A \longmapsto P \circ A \in \Psi_h^{k-j}(M;U) \\ N_{k-j}(P \circ A) = \sigma_k(P) N_{-j}(A) \end{gathered} \tag{1.11}$$

where the symbol of P is considered as a homogeneous differential operator with constant coefficients on the fibres of TM; similarly if V_r is the radial vector field on the fibres of TM then

$$\begin{gathered}\Psi_h^{-j}(M;U)\ni A\longmapsto D_t\circ A\in\Psi_h^{-j+2}(M;U) \text{ if } j\geq 2 \text{ with}\\ N_{h,-j+2}(D_tA)=\frac{i}{2}(V_r+n-j+2)N_{h,-j}(A),\ j>2\\ D_tA=a\operatorname{Id}+\overline{B},\ \overline{B}\in\overline{\Psi}_h^0(M;U),\ \text{if } A\in\Psi_h^{-2}(M;U) \text{ with}\\ a=-i\int\limits_{\text{fibre}}N_{h,-2}(A),\ N_0(\overline{B})=\frac{i}{2}(V_r+n)N_{h,-2}(A).\end{gathered}\tag{1.12}$$

This calculus allows us to give a direct construction of the heat kernel. Namely we look for $E\in\Psi_h^{-2}(M;U)$ satisfying

$$(\partial_t+P)E=\operatorname{Id}.\tag{1.13}$$

Here $\operatorname{Id}\in\Psi_h^0(M;U)$ has symbol 1. From (1.12) this imposes conditions on the 'normal operator' $N_{h,-2}(E_1)$, *viz*

$$\int\limits_{\text{fibre}}N_{h,-2}(E_1)=1,\ \left[\sigma_2(P)-\frac{1}{2}(V_r+n)\right]N_{h,-2}(E_1)=0.\tag{1.14}$$

This has a unique Schwartz solution, namely that derived from (1.1) in any linear coordinates on TM induced by a local orthonormal basis:

$$N_{h,-2}(E_1)=(4\pi)^{-\frac{n}{2}}\exp(-\frac{1}{4}|v|_x^2)\operatorname{Id}_U,\ \text{at } (x,v)\in TM.\tag{1.15}$$

The surjectivity of the map (1.8) means that we can indeed find E_1 with properties (1.14) which therefore satisfies

$$(\partial_t+P)E_1=\operatorname{Id}+R,\ R\in\Psi_h^{-1}(M;U).\tag{1.16}$$

From the last part of Proposition 1.1 we conclude that the inverse exists $(\operatorname{Id}+R)^{-1}=\operatorname{Id}+S$ with $S\in\Psi_h^{-1}(M;U)$ and so

$$E=E_1\circ(\operatorname{Id}+S)=E_1+E_1\circ S\in\Psi_h^{-2}(M;U)\tag{1.17}$$

satisfies (1.13).

There is a further refinement of the calculus. Namely let $\Psi_{h,E}^0(M;U)\subset\Psi_h^0(M;U)$ be the subspace which extends (by duality) to define an operator on $\mathcal{C}^\infty([0,\infty)\times M)$. This space is characterized in terms of kernels in §3. It is a filtered subalgebra with normal homomorphism taking values in the even subspace of the Schwartz space under the involution $v\longmapsto -v$ on the fibres of TM. Proposition 1.1 and Proposition 1.2 extend directly to this smaller algebra, so the construction of the heat kernel above actually shows that

$$\exp(-tP)\in\Psi_{h,E}^{-2}(M;U).\tag{1.18}$$

The point of this improvement is that the trace tensor on the kernel restricted to the diagonal gives in general

$$\operatorname{tr}:\ \Psi_h^{-k}(M;U)\longrightarrow t^{-\frac{1}{2}(\dim M-k+2)}\mathcal{C}^\infty([0,\infty)_{\frac{1}{2}}\times M)\tag{1.19}$$

where the subscript denotes that $t^{\frac{1}{2}}$ is the $\mathcal{C}^\infty$ variable. For the even subspace however

$$\operatorname{tr} : \Psi^{-k}_{h,E}(M;U) \longrightarrow t^{-\frac{1}{2}(\dim M-k+2)}\mathcal{C}^\infty([0,\infty)\times M) \tag{1.20}$$

and apart from the singular factor the trace is $\mathcal{C}^\infty$ in the usual sense up to $t=0$. This gives a rather complete description of the heat kernel for finite times.

We are also interested in the behaviour as $t\to\infty$. In the case of a compact manifold this is easily described. The variable $t/(1+t)\in[0,1]$ can be used to compactify the heat space near $t=\infty$. Thus $1/t$ is a $\mathcal{C}^\infty$ defining function for 'temporal infinity,' $\mathrm{ti}=\{1\}\times M^2$ in the new heat space; this leads to a new heat calculus $\Psi^0_H(M;U)$ with smoothness as $t\to\infty$ added. This refined heat calculus has another 'normal operator,' with values in the smoothing operators on M, namely

$$N_{h,\infty} : \Psi^0_H(M;U) \longrightarrow \Psi^{-\infty}(M;U). \tag{1.21}$$

Proposition 1.3. *If $P\in\operatorname{Diff}^2(M;U)$ is a differential operator on sections of U over M which is elliptic, non-negative self-adjoint and with diagonal principal symbol the heat kernel*

$$\exp(-tP)\in\Psi^{-2}_{H,E}(M;U) \tag{1.22}$$

has normal operator (1.15) *at* tf *and at* ti *has normal operator the orthogonal projection onto the zero eigenspace of P.*

It is this (well-known) result which we wish to generalize in various ways, in particular to the adiabatic limit.

2. Parabolic blow up

Since it is used extensively in the discussion of the heat kernel we give a brief description of the notion of the parabolic blow up of a submanifold of a manifold with corners. More specifically we define the notation

$$[X;Y,S],\quad \beta\colon [X;Y,S]\longrightarrow X,\quad \beta=\beta[X;Y,S], \tag{2.1}$$

where $Y\subset X$ is a submanifold and $S\subset N^*Y$ is a subbundle, satisfying certain extra conditions. The result, $[X;Y,S]$ is 'X blown up along Y with parabolic directions S.' A more extensive discussion of this operation will be given in [19], see also [20].

The basic model case we consider is $X=\mathbb{R}^{n,k}=[0,\infty)^k\times\mathbb{R}^{n-k}$ with coordinates x_i, $i=1,\dots,k$, y_j, $j=1,\dots,n-k$,

$$Y=Y_{l,p}=\{x_1=\cdots x_l=0,\ y_1=\cdots y_p=0\} \tag{2.2}$$

for some $1\le l\le k$, $1\le p\le n-k$ and

$$S=S_r=\operatorname{sp}\{dx_1,\dots dx_r\}\subset\mathbb{R}^n \tag{2.3}$$

for some $r\le l$. Then the blown-up manifold is, by definition, the product

$$[\mathbb{R}^{n,k};Y_{l,p},S_r]=\mathbb{S}^{l+p-1,l,r}_{\frac{1}{2}}\times[0,\infty)\times\mathbb{R}^{n-p-l,k-l} \tag{2.4}$$

where

$$\mathbb{S}^{l+p-1,l,r}_{\frac{1}{2}}=\left\{(x',x'',y')\in\mathbb{R}^{l+p,l};\ \sum_{1\le i\le r}x_i^2+\sum_{r<i\le l}x_i^4+\sum_{1\le j\le p}y_j^4=1\right\}. \tag{2.5}$$

The 'blow-down map' is by definition

(2.6)
$$\begin{gathered}\beta\colon [\mathbb{R}^{n,k}; Y_{l,p}, S_r] \longrightarrow \mathbb{R}^{n,k},\ \beta((x',x'',y'),r,(x''',y'')) = (r^2x', rx'', x''', ry', y''),\\ \text{where }\ x'=(x_1,\dots,x_r),\ x''=(x_{r+1},\dots,x_l),\ x'''=(x_{l+1}\dots,x_k),\\ y'=(y_1,\dots,y_p) \text{ and } y''=(y_{p+1},\dots,y_{n-k}).\end{gathered}$$

Let us note some properties of the triple of blown-up space, blow-down map and original space which are easily checked by direct computation. Both $\mathbb{R}^{n,k}$ and $[\mathbb{R}^{n,k}; Y_{l,p}, S_r]$ are manifolds with corners, $[\mathbb{R}^{n,k}; Y_{l,p}, S_r]$ has one more boundary hypersurface than $\mathbb{R}^{n,k}$. Clearly β is smooth and surjective, it is a diffeomorphism from $\mathbb{S}^{l+p-1,l,r} \times (0,\infty) \times \mathbb{R}^{n-p-l,k-l}$ onto $\mathbb{R}^{n,k} \setminus Y_{l,p}$ and a fibration from the 'front face'

$$\beta\colon \mathrm{ff}[\mathbb{R}^{n,k}; Y_{l,p}, S_r] = \mathbb{S}^{l+p-1,l,r}_{\frac12} \times \{0\} \times \mathbb{R}^{n-p-l,k-l} \longrightarrow \mathbb{R}^{n-p-l,k-l}. \tag{2.7}$$

Any smooth function on $\mathbb{R}^{n,k} \setminus Y_{l,p}$ which is homogeneous of non-negative integral degree under the $\mathbb{R}^+$ action

$$(x',x'',x''',y',y'') \longmapsto (s^2x', sx'', x''', sy', y''),\ s\in(0,\infty) \tag{2.8}$$

lifts to be $\mathcal{C}^\infty$ on $[\mathbb{R}^{n,k}; Y_{l,p}, S_r]$. Moreover these functions generate the $\mathcal{C}^\infty$ structure, i.e. give local coordinates near each point on the blown up manifold. In fact the front face, $\mathrm{ff}[\mathbb{R}^{n,k}; Y_{l,p}, S_r]$ can be identified with the quotient of $\mathbb{R}^{n,k} \setminus Y_{l,p}$ under this $\mathbb{R}^+$ action. As a set the blown up manifold can then be written

$$[\mathbb{R}^{n,k}; Y_{l,p}, S_r] = \mathrm{ff}[\mathbb{R}^{n,k}; Y_{l,p}, S_r] \sqcup \left(\mathbb{R}^{n,k} \setminus Y_{l,p}\right). \tag{2.9}$$

Any smooth vector field on $\mathbb{R}^{n,k}$, which is homogeneous of non-negative integral degree under (2.8) lifts to be smooth on $[\mathbb{R}^{n,k}; Y_{l,p}, S_r]$; if the vector field is tangent to all boundary hypersurfaces of $\mathbb{R}^{n,k}$ then the lift is tangent to all boundary hypersurfaces (including the new front face) of $[\mathbb{R}^{n,k}; Y_{l,p}, S_r]$. The lifts of these homogeneous vector fields tangent to all boundary hypersurfaces of $\mathbb{R}^{n,k} \setminus Y_{l,p}$ lift to span, over $\mathcal{C}^\infty([\mathbb{R}^{n,k}; Y_{l,p}, S_r])$, all smooth vector fields tangent to the boundary hypersurfaces of $[\mathbb{R}^{n,k}; Y_{l,p}, S_r]$. It follows from these results, or direct computation, that any local diffeomorphism on $\mathbb{R}^{n,k}$, $F\colon O \longrightarrow O' = F(O)$, which preserves both $Y_{l,p}$ and S_r in the sense that

$$F(O\cap Y_{l,p}) = O' \cap Y_{l,p},\ F^*(S_r) = S_r, \tag{2.10}$$

lifts to a diffeomorphism on the blown up space, i.e. there is a uniquely defined smooth diffeomorphism $\tilde F\colon \beta^{-1}(O) \longrightarrow \beta^{-1}(O')$ giving a commutative diagram

$$\begin{tikzcd}[\mathbb{R}^{n,k}; Y_{l,p}, S_r] \arrow[d, "\beta"'] & \beta^{-1}(O) \arrow[l, hook'] \arrow[r, "\tilde F"'] \arrow[d, "\beta"'] & \beta^{-1}(O') \arrow[r, hook] \arrow[d, "\beta"] & [\mathbb{R}^{n,k}; Y_{l,p}, S_r] \arrow[d, "\beta"'] \\ \mathbb{R}^{n,k} & O \arrow[l, hook'] \arrow[r, "F"'] & O' \arrow[r, hook] & \mathbb{R}^{n,k}\end{tikzcd} \tag{2.11}$$

This invariance allows the blow up to be defined more generally. Suppose that X is a manifold with corners (in particular each of the boundary hypersurfaces should be embedded.) Let $Y \subset X$ be a closed embedded submanifold which is of product type (a p-submanifold), in the sense that near each point of Y there is a local diffeomorphism of X to a neighborhood of $0 \in \mathbb{R}^{n,k}$ which reduces Y locally to some $Y_{l,p}$. In particular this means that the conormal bundle of Y is reduced to a product. Let $S \subset N^*Y$ be a subbundle which can be simultaneously reduced to S_r in (2.3) by such a diffeomorphism. The blown up manifold is then given as a set by the extension of (2.9)

$$[X;Y,S] = \mathrm{ff}[X;Y,S] \sqcup (X \setminus Y)\,. \tag{2.12}$$

Here the front face can be defined as the set of equivalence classes of curves with initial point on Y which are S-tangent to it. That is, consider the set of all curves

$$\begin{gathered}\chi\,[0,\epsilon),\ \epsilon>0,\ \mathcal{C}^\infty \text{ with}\\ \chi(0)\in Y,\ \frac{d(\chi^* f)}{ds}(0)=0 \text{ if } f\in\mathcal{C}^\infty(X),\ df(y)\in S_y\ \forall\ y\in Y.\end{gathered} \tag{2.13}$$

The first equivalence relation imposed on this set is that

$$\begin{gathered}\chi_1\sim\chi_2 \text{ if } \chi_1(0)=\chi_2(0),\ \frac{d(\chi_1^* f-\chi_2^* f)}{ds}(0)=0 \text{ and}\\ \frac{d^2(\chi_1^* f-\chi_2^* f)}{ds^2}(0)=0 \text{ if } f\in\mathcal{C}^\infty(X) \text{ has } df(y)\in S_y \text{ for } y\in Y.\end{gathered} \tag{2.14}$$

For each $y \in Y$ the curve with $\chi(s) \equiv y$ gives a base, or zero section. The second equivalence relation is on the curves which are non-zero in this sense, in which $\chi(ts) \sim \chi(s)$ for any $t > 0$. The resulting space $\mathrm{ff}[X;Y,S]$ is a fibre bundle over Y with fibre diffeomorphic to $\mathbb{S}_{\frac12}^{p-1,r}$. In particular it reduces to $\mathbb{S}_{\frac12}^{l+p-1,l,r}$ in the model case discussed above. The invariance properties just described show that local identification with $[\mathbb{R}^{n,k};Y_{l,p},S_r]$ leads to a $\mathcal{C}^\infty$ structure on $[X;Y,S]$. The blow down map is the obvious map from $[X;Y,S]$ to X, it has similar properties to those described above in the model case.

If $Y' \subset X$ is a closed submanifold the lift $\beta^*(Y') \subset [X;Y,S]$ is defined if $Y' \subset Y$ (respectively $Y' = \mathrm{cl}(Y \setminus Y'))$ to be $\beta^{-1}(Y')$ (resp. $\mathrm{cl}(\beta^{-1}(Y' \setminus Y))$. The lift of a subbundle $S' \subset N^*Y'$, denoted $\beta^*(S')$, is defined in these two cases as, respectively, $\beta^*(S')$ and the closure in $T^*[X;Y,S]$ of $\beta^*(S' \restriction (Y' \setminus Y)$. If this lifted manifold and the lift of S' satisfy the decomposition conditions introduced above then the iterated blow up is defined. In this case we use the notation

$$[X;Y,S;Y',S'] = [[X;Y,S];\beta^*(Y'),\beta^*(S')]. \tag{2.15}$$

3. Heat calculus

To define the heat calculus we shall extrapolate from the properties of the model convolution operator $\widetilde{\Phi}'$ considered in §1. The kernel of this operator, Φ' from (1.1), is 'simple' in a sense that is related to homogeneity under the transformation

$$\mu_s:\ (t,x)\longmapsto (s^2t,sx),\ \ s\in\mathbb{R}^+. \tag{3.1}$$

Thus, set $Z = [0,\infty) \times M^2$ and consider its t-parabolic blow-up along the submanifold

$$B = \{(0,x,x)\in Z; x\in M\}\,. \tag{3.2}$$

Following the notation for parabolic blow-up in (2) above, this can be written

$$M^2_h = [Z;B,S] \xrightarrow{\beta_h} Z \text{ where } S=\mathrm{sp}(dt)\subset N^*B. \tag{3.3}$$

In case $M = \mathbb{R}^n$ the space M_h^2 is easily identified. Let $\sim_\mu$ be the equivalence relation on $Z \setminus B$ generated by (3.1) in the sense that $p = (t, x, x') \sim_\mu p' = (r, y, y')$ if and only if $\mu_s(t, x) = (r, y)$ and $\mu_s(t, x') = (r, y')$ for some $s > 0$. Then

$$M_h^2 = [(Z \setminus B)/\sim_\mu] \sqcup [Z \setminus B] \,. \tag{3.4}$$

For $M = \mathbb{R}^n$ this space has a natural $\mathcal{C}^\infty$ structure, as a manifold with corners, which restricts to that on $Z \setminus B$ and which is generated by those $\mathcal{C}^\infty$ functions on $Z \setminus B$ which are homogeneous of non-negative integral degrees under μ_s (meaning this space of functions includes local coordinate systems). This $\mathcal{C}^\infty$ structure is independent of the coordinates in $\mathbb{R}^n$ and so is defined in the general case of a manifold M. In terms of the definition (2.12)

$$M_h^2 = {}^+SN\,\{Z; B, S\} \sqcup [Z \setminus B] \tag{3.5}$$

where ${}^+SN\,\{Z; B, S\}$ is the inward-pointing part of the S-parabolic normal bundle to B in Z. The first term in (3.5) forms the front face, denoted tf, the other boundary face will be denoted tb. Defining functions for these faces will be written ρ_{tf} and ρ_{tb}. Notice that

$$\beta_h^* t = \rho_{\mathrm{tf}}^2 \rho_{\mathrm{tb}} \tag{3.6}$$

for an appropriate choice of these defining functions.

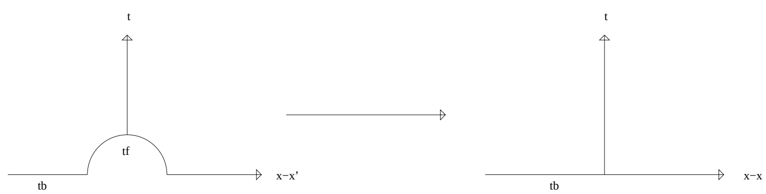

FIGURE 2. $\beta_h : Z_h = M_h^2 \to Z$

Since we wish to consider $\widetilde{\Phi}'$ as a convolution operator we need to consider density factors; for the usual reasons of simplicity we work with half-densities as the basic coefficient bundle. In case $M = \mathbb{R}^n$ the half-density

$$\Phi'' = \Phi'(t, x - x')|dtdxdx'|^{\frac{1}{2}}, \ x, x' \in \mathbb{R}^n, \tag{3.7}$$

lifts to M_h^2 to a smooth half-density away from tf which extends to be of the form

$$\Phi = \beta_h^* \Phi'' \in \rho_{\mathrm{tf}}^{-\frac{n}{2}+\frac{1}{2}} \rho_{\mathrm{tb}}^\infty \mathcal{C}^\infty(M_h^2; \Omega^{\frac{1}{2}}). \tag{3.8}$$

The notation here means that, for every $k \in \mathbb{N}$, $\rho_{\mathrm{tf}}^{\frac{n}{2}-\frac{1}{2}} \rho_{\mathrm{tb}}^{-k} \Phi \in \mathcal{C}^\infty(M_h^2; \Omega^{\frac{1}{2}})$. In particular Φ vanishes to infinite order at tb. We shall hide the singular factor of ρ_{tf} in (3.8), since it is of geometric origin, by defining a new bundle, the kernel density bundle KD, by the prescription

$$\mathcal{C}^\infty(M_h^2; \mathrm{KD}) = \rho_{\mathrm{tf}}^{-\frac{n}{2}-\frac{3}{2}} \mathcal{C}^\infty(M_h^2; \Omega^{\frac{1}{2}}). \tag{3.9}$$

The weighting here is chosen so that the identity is (for the moment formally) of order 0. Then we can write (3.8) in the form $\Phi \in \rho_{\mathrm{tf}}^2 \rho_{\mathrm{tb}}^\infty \mathcal{C}^\infty(M_h^2; \mathrm{KD})$. It is important to note that (3.9) does indeed define a new vector bundle. Since we shall use such constructions significantly below we give a general result of this type in §4. In particular Proposition 4.1 applied with the trivial filtration of $\Omega^{\frac12}$, shows that (3.9) defines the vector bundle KD. In this same sense we can write

$$\mathrm{KD} = \rho_{\mathrm{tb}}^{\frac{n}{2}+1}\beta_h^*\left[t^{-\frac{n}{2}-1}\Omega^{\frac12}(Z)\right]. \tag{3.10}$$

The singularity of KD at tb is not very important, since all the kernels vanish to infinite order there.

This discussion is all for the case $M=\mathbb{R}^n$. However the definition (3.9) extends to the general case and we simply set

$$\Psi_h^{-k}(M;\Omega^{\frac12}) = \rho_{\mathrm{tf}}^k\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty(M_h^2;\mathrm{KD}) \text{ for } k\in\mathbb{N}. \tag{3.11}$$

These are to be the elements of the 'heat calculus' of negative integral order. To define the elements of order zero observe that we can define a leading part of any element $A\in\rho_{\mathrm{tf}}^k\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty(M_h^2;\mathrm{KD})$ by setting

$$N_{\mathrm{tf}}(A) = t^{-k/2}A_{\restriction\mathrm{tf}} \in \dot{\mathcal{C}}^\infty(\mathrm{tf};\mathrm{KD}) \tag{3.12}$$

where the dot indicates that the resulting section of KD vanishes to infinite order at the boundary of the compact manifold with boundary $\mathrm{tf}(M_h^2)$. From the definition, (3.5), the front face fibres over $B\cong M$. The fibres are half-spheres (or balls) of dimension $n=\dim M$:

$$\begin{array}{ccc} \mathbb{S}^n_+ & \longrightarrow & \mathrm{tf}(M_h^2) \\ & & \big\downarrow{\scriptstyle \pi_{\mathrm{tf}}} \\ & & B\cong M. \end{array} \tag{3.13}$$

In fact the interiors of the fibres of (3.13) have natural linear structures, coming from the definition of ${}^+SN\{Z;B,S\}$. Namely tf is a compactification of the normal bundle to the diagonal, $\mathrm{Diag}\subset M^2$, which in turn is naturally isomorphic to TM so $TM\hookrightarrow \mathrm{tf}(M_h^2)$ is the interior. Using t, as in (3.10), to remove the singular powers from the kernel density bundle and noting that the lift of the density bundle on B is naturally isomorphic to the density bundle on TM this allows us to identify $\dot{\mathcal{C}}^\infty(\mathrm{tf}(M_h^2);\mathrm{KD})\longleftrightarrow\mathcal{S}(TM;\Omega_{\mathrm{fibre}})$. Thus for each k the normal map (3.12) can be regarded as a map

$$N_{h,-k}:\ \Psi_h^{-k}(M;\Omega^{\frac12})\twoheadrightarrow\mathcal{S}(TM;\Omega_{\mathrm{fibre}}),\ k\in\mathbb{N}. \tag{3.14}$$

The map also extends to the space defined by the right side of (3.11) for $k=0$. However, using the fact that the fibre integral is well-defined on the right image of (3.14), we actually set

$$\begin{gathered}\Psi_h^0(M;\Omega^{\frac12}) = \mathcal{C}^\infty(M)\,\mathrm{Id}\oplus\overline{\Psi}_h^0(M;\Omega^{\frac12}),\\ \overline{\Psi}_h^0(M;\Omega^{\frac12}) = \left\{A\in\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty(M_h^2;\mathrm{KD});\ \int\limits_{\mathrm{fibre}} N_{h,0}(A)=0\right\}.\end{gathered} \tag{3.15}$$

This of course is just the analogue of the usual 'mean value zero' condition for singular integrals.

To see how the operators $\Psi^k_h(M;\Omega^{\frac12})$ act, let $M_h=[0,\infty)\times M$ and consider the bilinear map

$$\dot{\mathcal{C}}^\infty_c(M_h;\Omega^{\frac12})\times\dot{\mathcal{C}}^\infty(M_h;\Omega^{\frac12})\ni(\phi,\psi)\longmapsto \phi\hat{*}_t\psi=\int_0^\infty\phi(t+t',x)\psi(t',x')dt'\in\mathcal{C}^\infty_c(Z;\Omega^{\frac12}). \tag{3.16}$$

Lifting to M^2_h we can define

$$\langle A\psi,\phi\rangle=\int_{M^2_h}A\cdot\beta_h^*(\phi\hat{*}_t\psi),\ A\in\Psi_h^{-k}(M;\Omega^{\frac12}),\ k\in\mathbb{N} \tag{3.17}$$

since the integrand is in the product

$$\rho^k_{\mathrm{tf}}\rho^\infty_{\mathrm{tb}}\mathcal{C}^\infty(M^2_h;\mathrm{KD})\cdot\beta_h^*\mathcal{C}^\infty_c(Z;\Omega^{\frac12})\subset\rho^{k-1}_{\mathrm{tf}}\mathcal{C}^\infty_c(M^2_h;\Omega)\subset L^1(M^2_h;\Omega)\text{ if }k\ge1. \tag{3.18}$$

For operators of order 0 the limiting form of the same definition applies to the second term in (3.15) since

$$A\in\overline{\Psi}^0_h(M;\Omega^{\frac12})\Longrightarrow\langle A\psi,\phi\rangle=\lim_{\epsilon\downarrow0}\int_{\{p\in M^2_h;\rho_{\mathrm{tf}}\ge\epsilon\}}A\cdot\beta_h^*(\phi\hat{*}_t\psi) \tag{3.19}$$

exists, independent of the choice of ρ_{tf} (which can be replaced by $t^{\frac12}$). Of course $\mathrm{Id}\in\Psi^0_h(M;\Omega^{\frac12})$ acts as the identity. Thus we find that

$$\begin{gathered}A\in\Psi^0_h(M;\Omega^{\frac12})\text{ defines an operator}\\ A:\dot{\mathcal{C}}^\infty([0,\infty)\times M;\Omega^{\frac12})\longrightarrow\mathcal{C}^{-\infty}([0,\infty)\times M;\Omega^{\frac12}).\end{gathered} \tag{3.20}$$

The image space here is just the dual space to $\dot{\mathcal{C}}^\infty([0,\infty)\times M;\Omega^{\frac12})$ and contains it as a dense subspace in the weak topology. In fact the range of A in (3.20) is always contained in $\dot{\mathcal{C}}^\infty([0,\infty)\times M;\Omega^{\frac12})$, as is shown in Lemma 3.1 below, so these operators can be composed. Before stating these results we note how to extend the discussion to general vector bundle coefficients.

Suppose that U and W are vector bundles over M. The (diagonal) homomorphism bundle from U to V over M is denoted $\hom(U,V)$, the (full) homomorphism bundle over M^2 is denoted $\mathrm{Hom}(U,V)$:

$$\hom(U,W)\cong\bigsqcup_{x\in M}W_x\otimes U'_x,\ \mathrm{Hom}(U,W)\cong\bigsqcup_{x,x'\in M}W_x\otimes U'_{x'}. \tag{3.21}$$

To ‘reduce’ general operators to operators on half-densities consider the bundle

$$\mathrm{Hom}_\Omega(U,W)=\bigsqcup_{x,x'\in M}(W_x\otimes\Omega_x^{-\frac12})\otimes(U'_{x'}\otimes\Omega^{\frac12}_{x'})\equiv\mathrm{Hom}(U\otimes\Omega^{-\frac12},W\otimes\Omega^{-\frac12}) \tag{3.22}$$

with the half-density bundles those on M. We define the general kernels by taking the tensor product, over $\mathcal{C}^\infty(M^2_h)$ of the space of $\mathcal{C}^\infty$ section of the lift of this bundle and the kernels already discussed:

$$\begin{aligned}\Psi_h^{-k}(M;U,W)=&\Psi_h^{-k}(M;\Omega^{\frac12})\otimes_{\mathcal{C}^\infty(M^2_h)}\mathcal{C}^\infty(M^2_h;\beta_h^*\mathrm{Hom}_\Omega(U,W))\\ =&\rho_{\mathrm{tf}}^{-k}\rho^\infty_{\mathrm{tb}}\mathcal{C}^\infty(M^2_h;\mathrm{KD}(U,W))\end{aligned} \tag{3.23}$$

with the modified kernel density bundle

$$\mathrm{KD}(U,W)=\mathrm{KD}\otimes\beta_h^*\operatorname{Hom}_\Omega(U,W). \tag{3.24}$$

If $U=W$, which is often the case, we denote the space as $\Psi_h^{-k}(M;U)$.

The boundary hypersurface tf lies above the diagonal so the additional density factors in (3.23) cancel there. The normal operator therefore extends to a surjective linear map

$$N_{h,-k}:\ \Psi_h^{-k}(M;U,W)\twoheadrightarrow\mathcal{S}(TM;\Omega_{\mathrm{fibre}}\otimes\pi^*\hom(U,W)),\ k\in\mathbb{N}. \tag{3.25}$$

We shall make a further small, but significant, refinement of this construction. The Taylor series at tf of $\mathcal{C}^\infty$ functions on M^2_h are generated by the homogeneous functions under (3.1) in any local coordinates. Now we can choose these local coordinates to be t, x_j-y_j and x_j+y_j where x and y are the same local coordinates in the two factors of M. Under the involution, J, on M^2, which interchanges the factors, x_j-y_j is odd and x_j+y_j is invariant. So consider the subspace $\mathcal{C}^\infty_E(M^2_h)\subset\mathcal{C}^\infty(M^2_h)$ fixed by the condition that its elements have Taylor series at tf with terms of even homogeneity invariant under J and terms of odd homogeneity odd under J. If $\rho=(t+|x-y|^2)^{\frac12}$ then the Taylor series at tf of a general $\mathcal{C}^\infty$ function on M^2_h is of the form

$$\sum_{k=0}^{\infty}\rho^kF_k(\frac{t}{\rho^2},\frac{x-y}{\rho},x+y) \tag{3.26}$$

where the F_k are $\mathcal{C}^\infty$ functions on $\mathbb{R}^{2n+1}$ away from 0. It is therefore clear that the space $\mathcal{C}^\infty_E(M^2_h)$ is well-defined independent of the choice of coordinates. Similarly the space $\mathcal{C}^\infty_O(M^2_h)\subset\mathcal{C}^\infty(M^2_h)$ is fixed by requiring the F_k to be odd or even in the second variables for k even or odd respectively. Then $\mathcal{C}^\infty_E(M^2_h)+\mathcal{C}^\infty_O(M^2_h)=\mathcal{C}^\infty(M^2_h)$ and the intersection $\mathcal{C}^\infty_E(M^2_h)\cap\mathcal{C}^\infty_O(M^2_h)=\rho_{\mathrm{tf}}^\infty\mathcal{C}^\infty(M^2_h)$ consists of the functions with trivial Taylor series at tf .

Notice that $\mathcal{C}^\infty(Z)$ lifts under β_h into $\mathcal{C}^\infty_E(M^2_h)$. This means that we can define the spaces $\mathcal{C}^\infty_E(M^2_h;\beta_h^*U)$ and $\mathcal{C}^\infty_O(M^2_h;\beta_h^*U)$ for any vector bundle over M^2. Since we can certainly choose defining functions $\rho_{\mathrm{tf}}\in\mathcal{C}^\infty_O(M^2_h)$, $\rho_{\mathrm{tb}}\in\mathcal{C}^\infty_E(M^2_h)$ and also $t^{\frac12}\in\mathcal{C}^\infty_O(M^2_h)$ this means we can define the odd and even parts of the heat calculus using (3.10) and (3.11). We define

$$\Psi^{-k}_{h,E}(M;U,W)=\begin{cases}\rho_{\mathrm{tf}}^k\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty_E(M^2_h;\mathrm{KD}(U,W)), & k\text{ even}\\ \rho_{\mathrm{tf}}^k\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty_O(M^2_h;\mathrm{KD}(U,W)), & k\text{ odd.}\end{cases} \tag{3.27}$$

For $k=0$ we define

$$\begin{gathered}\Psi^0_{h,E}(M;U,V)=\mathcal{C}^\infty(M)\operatorname{Id}\oplus\overline{\Psi}^0_{h,E}(M;U,V)\text{ with}\\ \overline{\Psi}^0_{h,E}(M;U,V)=\overline{\Psi}^0_h(M;U,V)\cap\mathcal{C}^\infty_E(M^2_h;\mathrm{KD}(U,W)).\end{gathered} \tag{3.28}$$

Let $[0,\infty)_{\frac12}$ be the half line with $t^{\frac12}$ as smooth variable.

Lemma 3.1. *Each element $A\in\Psi_h^{-k}(M;U,W)$, for a compact manifold M and any $k\geq0$, defines a continuous linear map*

$$A:\ \dot{\mathcal{C}}^\infty([0,\infty)\times M;U)\longrightarrow\dot{\mathcal{C}}^\infty([0,\infty)\times M;V) \tag{3.29}$$

and the same pairing, (3.17), *leads to a continuous linear map*

$$A:\ \mathcal{C}^\infty([0,\infty)\times M;U)\longrightarrow t^{\frac{k}{2}}\mathcal{C}^\infty([0,\infty)_{\frac12}\times M;V). \tag{3.30}$$

For an element $A \in \Psi^{-k}_{h,E}(M; U, V)$ *the operator* (3.30) *has range in* $t^{[\frac{k}{2}]}\mathcal{C}^\infty([0,\infty)\times M; V)$.

Here $S = [s]$ is the largest integer satisfying $S \le s$.

Proof. We give a rather 'geometric' proof of this regularity result, in the spirit of [28]. That is we introduce the singular coordinates needed to analyze the integral in the action of the operators by defining certain blown-up spaces. In the process of showing (3.30) we shall in essence work with the t-variable coefficient heat calculus.

Thus consider the product

$$\begin{aligned} Z_2 = [0,\infty)^2 \times M^2 &= [0,\infty)_{t-t'} \times M \times [0,\infty)_{t'} \times M \\ &= \{(t,t',q); q \in M^2, t, t' \in \mathbb{R}, t \ge t' \ge 0\}, \end{aligned} \tag{3.31}$$

with two 'time' variables. There are the three obvious projections,

$$\begin{gathered} \pi_L(t,t',m,m') = (t,m),\ \pi_R(t,t',m,m') = (t',m') \text{ and} \\ \pi_K(t,t',m,m') = (t-t',m,m'). \end{gathered} \tag{3.32}$$

These combine to give a diagram:

$$\begin{tikzcd} & & {[0,\infty) \times M} \\ Z & Z_2 \arrow[l, "\pi_K"'] \arrow[ur, "\pi_R"] \arrow[dr, "\pi_L"'] & \\ & & {[0,\infty) \times M.} \end{tikzcd} \tag{3.33}$$

The space $[0,\infty)^2 \times M^2$ is not symmetric in t, t' and this is reflected in the fact that the right projection is a fibration whereas the map π_L is not; in fact it is not even a b-map. To compensate for this asymmetry we need only blow up the submanifold (the corner) $t = t' = 0$; set

$$M^2_2 = [Z_2; \{t = t' = 0\}],\ \beta_2 : M^2_2 \longrightarrow Z_2. \tag{3.34}$$

After this blow up none of the three lifted projections is a fibration but all three are now b-fibrations.

Consider the lift under π_K of the submanifold B, in (3.2), blown up in (3.3). Under the blow up of $t = t' = 0$ this further lifts to two submanifolds:

$$\begin{gathered} B_1 = \operatorname{cl}\left\{\left(\pi_K^{-1}(B)\right) \setminus \{t = t' = 0\}\right\} \text{ in } M^2_2 \text{ and} \\ B_2 = \beta_2^{-1}(\{(0,0,q); q \in \operatorname{Diag} \subset M^2\}. \end{gathered} \tag{3.35}$$

These submanifolds are each contained in one boundary hypersurface; $B_1 \subset \mathrm{df}$ and $B_2 \subset \mathrm{ff}$ where df is the lift of $t = t'$. Then we further blow up the space, along these submanifolds and parabolically in the direction of the conormal to the respective boundary hypersurfaces:

$$M^2_{2h} = [M^2_2; B_2, N^*\,\mathrm{ff}; B_1, N^*\,\mathrm{df}]. \tag{3.36}$$

The order of blow up here is important.

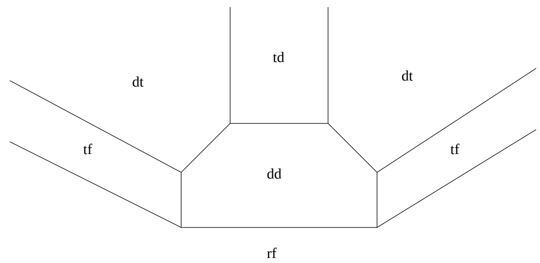

FIGURE 3. Faces of M^2_{2h}

Now the diagram of maps (3.33) lifts to a triple of b-fibrations:

$$
\begin{tikzcd}
 & & {[0,\infty)\times M} \\
M^2_h & M^2_{2h} \arrow[l, "\pi_{2,K}"'] \arrow[ur, "\pi_{2,R}"] \arrow[dr, "\pi_{2,L}"'] & \\
 & & {[0,\infty)\times M.}
\end{tikzcd}
\tag{3.37}
$$

Certainly the left and right lifted projections exist, since they are just composites of the blow-down map from M^2_{2h} with the maps in (3.33). That they are b-fibrations follows from the fact that the lifts to M^2_2 are b-fibrations and that each of the two subsequent blow ups in (3.37) is of a submanifold to which the corresponding map is b-transversal. Similarly the map to M^2_h arising from the lift of π_K is a b-fibration because of the lifing theorem for b-fibrations in [28], i.e. because the b-fibration from M^2_2 to $[0,\infty)\times M^2$ coming from the lift of π_K is transversal to B and the components of the lift of B under it, just B_1 and B_2, are blown up with the appropriate, lifted, parabolic directions.

Thus (3.37) is a diagram of b-fibrations. The space M^2_{2h} has five boundary hypersurfaces, two of them arising from the lifts of $t'=0$ and of $t=t'$ and the remaing three produced by the blow ups of $t=t'=0$, of B_2 and of B_1. We shall denote them rf, dt, tf, dd and td respectively. The lifts of defining functions are then easily computed and for appropriate choices of defining functions

$$
\begin{gathered}
\pi^*_{2,K}\rho_{\mathrm{tf}}=\rho_{\mathrm{dd}}\rho_{\mathrm{td}},\ \pi^*_{2,K}\rho_{\mathrm{tb}}=\rho_{\mathrm{dt}}\rho_{\mathrm{tf}}\\
\pi^*_{2,R}t'=\rho_{\mathrm{rf}}\rho_{\mathrm{tf}}\rho^2_{\mathrm{dd}} \text{ and } \pi^*_{2,L}t=\rho_{\mathrm{tf}}\rho^2_{\mathrm{dd}}.
\end{gathered}
\tag{3.38}
$$

Similarly the product of the lift of smooth positive half-densities from each of the images $[0,\infty)\times M$, in (3.37) with the lift of a smooth section of the kernel density bundle on M^2_2 and of $|dt|^{\frac12}$ is easily computed. It follows that

$$
\pi^*_{2,K}\,\mathrm{KD}\cdot\pi^*_{2,L}(\Omega^{\frac12}\otimes|dt|^{\frac12})\cdot\pi^*_{2,R}\Omega^{\frac12}=\rho_{\mathrm{tf}}\rho_{\mathrm{dd}}\rho^{-1}_{\mathrm{td}}\Omega
\tag{3.39}
$$

Now if $\phi\in\dot{\mathcal{C}}^\infty([0,\infty)\times M;\Omega^{\frac12})$ and $\psi\in\mathcal{C}^\infty([0,\infty)\times M;\Omega^{\frac12})$ the action of $A\in\Psi^k_h(M;\Omega^{\frac12})$, where for the moment we assume that $k>0$, on ϕ can be written

$$
A\phi\cdot\psi=(\pi_{2,L})_*\left((\pi_{2,K})^*A\cdot(\pi_{2,L})^*(\psi)\cdot(\pi_{2,R})^*\phi\right).
\tag{3.40}
$$

Applying (3.39) and (3.38) it can be seen that

$$A\phi\cdot\psi\subset(\pi_{2,L})_*\left(\rho_{\mathrm{tf}}^{\infty}\rho_{\mathrm{dt}}^{\infty}\rho_{\mathrm{dd}}^{\infty}\rho_{\mathrm{td}}^{k-1}\Omega\right)\subset\dot{\mathcal{C}}^{\infty}([0,\infty)\times M;\Omega)\tag{3.41}$$

which gives (3.29). The inclusion in (3.41) follows from the push-forward theorem for conormal distributions under b-fibrations, from [29]. If instead $\phi\in\mathcal{C}^{\infty}([0,\infty)\times M;\Omega^{\frac12})$, still with $k>0$, then a similar computation and the same theorem shows that

$$A\phi\cdot\psi\subset(\pi_{2,L})_*\left(\rho_{\mathrm{tf}}^{\infty}\rho_{\mathrm{dd}}^{k+2}\rho_{\mathrm{td}}^{k-1}\Omega\right)\subset t^{\frac{k}{2}}\mathcal{C}^{\infty}([0,\infty)_{\frac12}\times M;\Omega)\tag{3.42}$$

which proves (3.30).

The case $k=0$ is similar except that the integral implicit in (3.41) or (3.42) is not absolutely convergent at $t=t'$, i.e. $\mathrm{dd}(M^2_{2h})$. However the mean value condition in (3.15) makes the integral conditionally convergent and the same results, (3.29) and (3.30), follow.

The improved regularity in the case of an operator in the even part of the calculus follows from the fact that non-integral powers of t in the Taylor series expansion in $t^{\frac12}$ would arise from the odd part of the integrand and hence vanish. □

We now turn to

Proof of Proposition 1.1. To prove this composition result we proceed very much as above in the proof of Lemma 3.1. Thus we first construct a 'triple' space to which the two kernels can be simultaneously lifted. Set

$$Z_3=\{(t,t')\in\mathbb{R}^2;t'\geq 0,t\geq t'\}\times M^3\tag{3.43}$$

and consider the three maps:

$$\begin{gathered}\pi_o:\,Z_3\longrightarrow Z,\;\;o=f,c,s\\ \pi_f(t,t',x,x',x'')=(t',x',x'')\\ \pi_s(t,t',x,x',x'')=(t-t',x,x')\\ \pi_c(t,t',x,x',x'')=(t,x,x'').\end{gathered}\tag{3.44}$$

the first two of which are projections. The diagram:

$$\begin{tikzcd} & Z & \\ & Z_3 \arrow[u, "\pi_c"] \arrow[dl, "\pi_s"'] \arrow[dr, "\pi_f"] & \\ Z & & Z \end{tikzcd}\tag{3.45}$$

is a symbolic representation of the composition of operators $A,B\in\Psi_H^{-\infty}(M;\Omega^{\frac12})$ in the sense that if $C=A\circ B$ then

$$C=(\pi_c)_*\left[(\pi_s)^*A\cdot(\pi_f)^*B\right].\tag{3.46}$$

We define blown-up versions of Z_3 by defining the three partial diagonals:

$$B_o=\pi_o^{-1}(B),\;\;o=f,c,s\tag{3.47}$$

and the triple surface, which is the intersection of any pair in (3.47):

$$B_3=\{(0,0,x,x,x)\in Z_3\}.\tag{3.48}$$

Similarly set

$$S_o = \pi_o^*(S) \subset N^*(B_o) \Longrightarrow S_f = \mathrm{sp}(dt'),\ S_s = \mathrm{sp}(dt - dt'),\ S_c = \mathrm{sp}(dt) \tag{3.49}$$

and

$$S_3 = \mathrm{sp}(dt, dt') \text{ over } B_3. \tag{3.50}$$

Consider first the manifold with corners defined by iterated parabolic blow up (this is discussed in [19]Appendix B):

$$Z_{3,1} = [Z_3; B_3, S_3; B_f, S_f; B_s, S_s] \xrightarrow{\beta_{3,1}} Z_3. \tag{3.51}$$

The order of blow up amongst the last two submanifolds is immaterial since they lift to be disjoint in $[Z_3; B_3, S_3]$. In fact, since we can also interchange the blow up of B_3 and either B_s or B_f, we have natural $\mathcal{C}^\infty$ maps

(3.52)

$$\begin{gathered} Z_{3,1} \equiv [Z_3; B_f, S_f; B_3, S_3; B_s, S_s] \xrightarrow{\pi_{2,f}} Z_h (= M_h^2) \\ \pi_{2,f} : [Z_3; B_f, S_f; B_3, S_3; B_s, S_s] \longrightarrow [Z_3; B_f, S_f] \equiv Z_h \times [0,\infty) \times M \longrightarrow Z_h \\ Z_{3,1} \equiv [Z_3; B_s, S_s; B_3, S_3; B_f, S_f] \xrightarrow{\pi_{2,s}} Z_h \\ \pi_{2,s} : [Z_3; B_s, S_s; B_3, S_3; B_f, S_f] \longrightarrow [Z_3; B_s, S_s] \equiv Z_h \times [0,\infty) \times M \longrightarrow Z_h. \end{gathered}$$

These maps give a commutative diagram with the bottom part of (3.45):

$$\begin{tikzcd} Z_h \arrow[ddd, "\beta_h"'] & & Z_h \arrow[ddd, "\beta_h"] \\ & Z_{3,1} \arrow[ul, "\pi_{2,s}"'] \arrow[ur, "\pi_{2,f}"] \arrow[d, "\beta_{3,1}"] & \\ & Z_3 \arrow[dl, "\pi_s"'] \arrow[dr, "\pi_f"] & \\ Z & & Z. \end{tikzcd} \tag{3.53}$$

This allows us to lift the product of the kernels in (3.46) to $Z_{3,1}$ by lifting the individual kernels under $\pi_{2,f}$ and $\pi_{2,s}$:

$$\beta_{3,1}^* \left[(\pi_s)^* A \cdot (\pi_f)^* B\right] = (\pi_{2,s})^* A \cdot (\pi_{2,f})^* B. \tag{3.54}$$

Using (3.10) we can write the kernel as

$$B = bt^{-\frac{n}{2}-1+k/2}\nu,\ \nu \in \mathcal{C}^\infty(Z; \Omega^{\frac12}),\ b \in \mathcal{C}^\infty(Z_H),\ b \equiv 0 \text{ at } \mathrm{tb}\,. \tag{3.55}$$

The manifold $Z_{3,1}$ has five boundary hypersurfaces, the two 'trivial' faces tr and tl arising from the lifts of $t' = 0$ and $t = t'$ respectively and the three faces created by blow-up; namely tt arising from the blow-up of B_3, sf arising from the blow-up of $B_{2,f}$ and ss arising from the blow-up of $B_{2,s}$. Clearly

$$\begin{gathered} \mathcal{C}^\infty(Z_{3,1}) \ni (\pi_{2,f})^* b \equiv 0 \text{ at } \mathrm{tr} \\ \mathcal{C}^\infty(Z_{3,1}) \ni (\pi_{2,s})^* a \equiv 0 \text{ at } \mathrm{tl} \end{gathered} \tag{3.56}$$

Thus the product vanishes to infinite order at two of the boundary hypersurfaces, i.e. has non-trivial Taylor series only at sf, ss and tt. If we take into account the fact that $\pi_{2,f}t'$ and $\pi_{2,s}(t-t')$ vanish to second order at tt we conclude that the product in (3.54) is of the form

$$\begin{gathered}\beta_{3,1}^*\left[(\pi_s)^*A\cdot(\pi_f)^*B\right]=\rho_{\mathrm{tt}}^{-2n-4+j+k}\rho_{\mathrm{sf}}^{-n-1+k}\rho_{\mathrm{ss}}^{-n-1+j}c(\pi_{2,f})^*\nu(\pi_{2,s})^*\nu,\\ c\in\mathcal{C}^\infty_{\mathrm{tt}}(Z_{3,1}),\ \text{i.e. } \mathcal{C}^\infty(Z_{3,1})\ni c\equiv 0 \text{ at } \mathrm{tr}\cup\mathrm{tl}\,.\end{gathered}\tag{3.57}$$

In particular the product of the kernels vanishes to infinite order at the corner, B', produced by the intersection of tl and tr in $Z_{3,1}$. Consider the manifold, $Z_{3,2}$ defined by blowing this up, parabolically with respect to both normal directions:

$$Z_{3,2}=[Z_{3,1};B',N^*B'],\ \ F=\mathrm{tr}\cap\mathrm{tl}\,.\tag{3.58}$$

This adds another boundary hypersurface, td, but makes not essential difference to the kernel so that (3.57) becomes, with the same notation used for the other boundary hypersurfaces and their lifts,

$$\begin{gathered}\beta_{3,2}^*\left[(\pi_s)^*A\cdot(\pi_f)^*B\right]=\rho_{\mathrm{tt}}^{-2n-4+j+k}\rho_{\mathrm{sf}}^{-n-1+k}\rho_{\mathrm{ss}}^{-n-1+j}c(\pi_{2,f})^*\nu(\pi_{2,s})^*\nu,\\ c\in\mathcal{C}^\infty_{\mathrm{tt}}(Z_{3,1}),\ \text{i.e. } \mathcal{C}^\infty(Z_{3,2})\ni c\equiv 0 \text{ at } \mathrm{tr}\cup\mathrm{tl}\cup\mathrm{td}\,.\end{gathered}\tag{3.59}$$

Having arrived at $Z_{3,2}$ with a 'simple' kernel we need to map back to Z_h. The manifold $Z_{3,2}$ can be constructed in another way, using the commutability of appropriate blow ups. Thus, the final blow up in (3.58) does not meet ss or sf so can be performed after that of B_3 in (3.51). Furthermore, B_3 is then a submanifold of the corner, $Y=\{t=t'=0\}$ being blown up, with the same parabolic directions. The order can therefore be interchanged and so

$$Z_{3,2}=[Z_3;B',S_3;B_3,S_3;B_f,S_f;B_s,S_s]\xrightarrow{\beta_{3,2}}Z_3\tag{3.60}$$

This means that the third map in (3.44) lifts into a b-fibration from $Z_{3,2}$ to Z_H as we procced to show.

Indeed, consider the blown up space

$$\beta':\ Z_3'=[Z_3;B']\longrightarrow Z_3,\ \ B'=\{t=t'=0\}.\tag{3.61}$$

The composite map is then a fibration

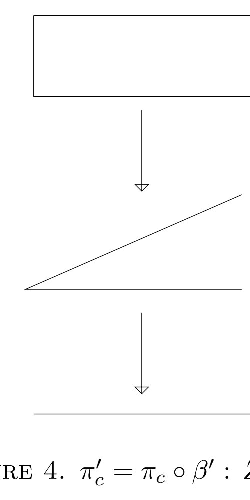

FIGURE 4. $\pi_c'=\pi_c\circ\beta':\ Z_3'\to Z$

$$\pi_c'=\pi_c\cdot\beta':\ Z_3'\longrightarrow Z.\tag{3.62}$$

Parabolically blowing up the lift, which we can denote $B_{2,c}$, to Z_3', of the submanifold B, in Z gives a further fibration

$$\pi_c'' \colon Z_{3,1}' = [Z_3; B'; B_{2,c}, S_{2,c}] \longrightarrow Z_h \tag{3.63}$$

where the lift of $S_{2,c}$ is just the intersection of the conormal bundle of $(\pi_c^1)^{-1}(B)$ with the conormal bundle to the front face of Z_3'. Consider next the blow-up in $Z_{3,1}'$ of the lift of B_3, which is a submanifold of $B_{2,c}$:

$$\beta_{3,2}' \colon Z_{3,2}' = [Z_3; B'; B_{2,c}, S_{2,c}; B_3, S_3] \longrightarrow Z_{3,1}'. \tag{3.64}$$

Now we can also blow up the other two partial diagonals, lifted to $Z_{3,2}'$, and again use commutation for non-itersecting submanifolds to write

(3.65)
$$Z_{3,2}'' = [Z_{3,2}'; B_{2,s}, S_{2,s}; B_{2,f}, S_{2,f}] \equiv [Z_3; B'; B_3, S_3; B_{2,s}, S_{2,s}; B_{2,f}, S_{2,f}; B_{2,c}, S_{2,c}]\,.$$

This means that there is a blow-down map (for the lift of $B_{2,c}$)

$$Z_{3,2}'' \longrightarrow Z_{3,2},\ \ Z_{3,2}'' \equiv [Z_{3,2}; B_{2,c}, S_{2,c}]. \tag{3.66}$$

Since the density in (3.59) vanishes to infinite order at the submanifold, $B_{2,c} \subset B'$, blown up in (3.66) we also conclude that

$$\begin{gathered} [(\pi_s)^* A \cdot (\pi_f)^* B] \ \text{ lifts to } \rho_{\text{tt}}^{-2n-4+j+k} \rho_{\text{sf}}^{-n-1+k} \rho_{\text{ss}}^{-n-1+j} c (\pi_{2,f})^* \nu (\pi_{2,s})^* \nu, \\ \mathcal{C}^\infty(Z_{3,2}'') \ni c \equiv 0 \text{ at } \operatorname{tr} \cup \operatorname{tl} \cup \operatorname{td} \cup \operatorname{sc}, \end{gathered} \tag{3.67}$$

where sc is the hypersurface produced by the blow up of $B_{2,c}$.

The last step is to consider the push-forward of this density under the map from $Z_{3,2}''$ to Z_h given by (3.63). We wish to consider the image, a half-density on Z_h, as a multiple of the lift of a smooth half-density on Z, as in (3.10), so simply multiply by the lift to Z_3 of ν under $\pi_{2,c}$. Lifting to $Z_{3,1}'$ this gives

$$\gamma(\pi_{2,c})^* \nu = (\pi_c')^* (t^{-n-2+\frac{j+k}{2}}) c'' \mu \tag{3.68}$$

where μ is the lift to $Z_{3,1}'$ of the product of ν from Z under the maps π_o. Thus it is a non-vanishing density on Z_3 and lifted to $Z_{3,1}'$ it is an element of

$$\rho_{\text{tt}}^{n+1} \mathcal{C}^\infty(Z_{3,1}'; \Omega). \tag{3.69}$$

Inserting this into (3.68), using the rapid vanishing at all faces except tt shows that

$$\gamma(\pi_{2,c})^* \nu \in (\pi_c')^* \left(t^{-(n+3)/2+(k+j)/2} \right) \dot{\mathcal{C}}^\infty_{\text{tt}}(Z_{3,1}'; \Omega). \tag{3.70}$$

The map π_c' in (3.63) is not a fibration but it is a b-fibration and from the push-forward results in [29] it follows that

$$(\pi_c')_* : \dot{\mathcal{C}}^\infty_{\text{tt}}(Z_{3,1}'; \Omega) \longrightarrow \dot{\mathcal{C}}^\infty_{\text{tf}}(Z_h; \Omega). \tag{3.71}$$

This proves the composition formula, since it shows that composite kernel is an element of $\Psi_h^{k+j}(M; \Omega^{\frac{1}{2}})$.

To see the last statement note that the composition formula shows that the Neumann series can be summed modulo a rapidly vanishing term. This reduces the consideration to $\operatorname{Id} + \Psi_h^{-\infty}(M; U)$, for which Duhamel' principle finishes the proof. $\square$

Proof of Proposition 1.2. It suffices to prove (1.11) for vector fields. Namely, we need to show that if V is a vector field acting through the connection on U then $\Psi_h^{-j}(M;U) \ni A \to V\circ A \in \Psi_h^{-j+1}(M;U)$, and $N_{-j+1}(V\circ A) = \sigma_1(V)N_{-j}(A)$.

We first assume that U is the trivial bundle $\mathbb{C}$. The projective coordinates

$$t,\ X = \frac{x-x'}{t^{\frac12}},\ x, \tag{3.72}$$

give a valid coordinate system near the front face, except at the corner, where $X=\infty$. Any $A\in\Psi_h^{-j}(M;U)$ can be written as

$$A = t^{\frac{j}{2}-\frac{n+2}{2}+\frac{n}{4}}\tilde{A}(t,X,x)|dtdXdx|^{\frac12}, \tag{3.73}$$

where $\tilde{A}$ is smooth and vanishes rapidly as $X\to\infty$.

Now if $\phi = \phi_0|dtdx|^{\frac12} \in \dot{\mathcal{C}}_c^\infty(M_h;\Omega^{\frac12})$ and $\psi = \psi_0|dtdx|^{\frac12} \in \dot{\mathcal{C}}^\infty(M_h;\Omega^{\frac12})$, we have

$$\langle A\psi,\phi\rangle = \int\limits_{M_h^2} t^{\frac{j}{2}-1}\tilde{A}\beta_h^*(\phi_0\hat{*}_t\psi_0)|dtdXdx|, \tag{3.74}$$

and $N_{-j}(A) = \tilde{A}(0,X,x)|dXdx|^{\frac12}$.

Also, if we let $V = a(x)\partial_x$ be a smooth vector field and V' denote its transpose:

$$\langle V\psi,\phi\rangle = -\langle\psi,V'\phi\rangle, \tag{3.75}$$

then $V' = \partial_x a(x)$, and $\beta_h^*(t^{\frac12}V') = t^{\frac12}\partial_x a(x) + a(x)\partial_X$.

Now

$$\begin{aligned}\langle (V\circ A)\psi,\phi\rangle &= -\,\langle A\psi, V'\phi\rangle\\ &= -\int\limits_{M_h^2} t^{\frac{j}{2}-1}\tilde{A}\beta_h^*((V'\phi_0)\hat{*}_t\psi_0)|dtdXdx|\\ &= -\int\limits_{M_h^2} t^{\frac{j}{2}-1}\tilde{A}\beta_h^*(V')[\beta_h^*(\phi_0\hat{*}_t\psi_0)]|dtdXdx|\\ &= -\lim_{\epsilon\downarrow 0}\int\limits_{t\ge\epsilon} t^{\frac{j-1}{2}-1}\tilde{A}\beta_h^*(t^{\frac12}V')[\beta_h^*(\phi_0\hat{*}_t\psi_0)]|dtdXdx|,\end{aligned} \tag{3.76}$$

since $j\ge 1$. Integration by part gives

$$\langle (V\circ A)\psi,\phi\rangle = \lim_{\epsilon\downarrow 0}\int\limits_{t\ge\epsilon}(\beta_h^*(t^{\frac12}V'))'[t^{\frac{j-1}{2}-1}\tilde{A}]\beta_h^*(\phi_0\hat{*}_t\psi_0)|dtdXdx|, \tag{3.77}$$

where there is no boundary contribution because there is no integration by parts in the t direction. Since $(\beta_h^*(t^{\frac12}V'))' = t^{\frac12}a(x)\partial_x + a(x)\partial_X$, it follows this integral reduces to

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\int\limits_{t\ge\epsilon}(t^{\frac{j}{2}-1}a(x)\partial_x\tilde{A} + t^{\frac{j-1}{2}-1}a(x)\partial_X\tilde{A})\beta_h^*(\phi_0\hat{*}_t\psi_0)|dtdXdx|\\ &= \int\limits_{M_h^2}(t^{\frac{j}{2}-1}a(x)\partial_x\tilde{A} + t^{\frac{j-1}{2}-1}a(x)\partial_X\tilde{A})\beta_h^*(\phi_0\hat{*}_t\psi_0)|dtdXdx|,\end{aligned} \tag{3.78}$$

where the integral converges since

$$\int a(x)\partial_X \tilde{A}\beta_h^*(\phi_0 \hat{*}_t \psi_0)|_{t=0}|dX| = 0. \tag{3.79}$$

Therefore

$$V \circ A = t^{\frac{j-1}{2}-\frac{n+2}{2}+\frac{n}{4}}(a(x)\partial_X \tilde{A} + t^{\frac{1}{2}}a(x)\partial_x \tilde{A})|dtdXdx|^{\frac{1}{2}} \tag{3.80}$$

is an elememt of $\Psi_h^{-j+1}(M;\Omega^{\frac{1}{2}})$. Moreover

$$N_{-j+1}(V \circ A) = a(x)\partial_X \tilde{A}(0,X,x)|dXdx|^{\frac{1}{2}} = \sigma_1(V)N_{-j}(A). \tag{3.81}$$

A similar computation works for D_t, except that

$$\beta_h^*(tD_t) = tD_t + \frac{i}{2}X\partial_X. \tag{3.82}$$

Therefore the integration by part will produce a boundary term, and

$$(\beta_h^*(tD_t))' = D_t t + \frac{i}{2}\partial_X X = D_t t + \frac{i}{2}(n + X\partial_X). \tag{3.83}$$

In fact, by carrying out the above computation for D_t, one find, for $k \geq 2$,

$$\begin{aligned} \langle (D_t \circ A)\psi, \phi\rangle =& \lim_{\epsilon\downarrow 0}\int\limits_{t\geq\epsilon} (\beta_h^*(tD_t))'[t^{\frac{j-2}{2}-1}\tilde{A}]\beta_h^*(\phi_0 \hat{*}_t \psi_0)|dtdXdx| \\ & - \lim_{\epsilon\downarrow 0}\int\limits_{t=\epsilon} it[t^{\frac{j-2}{2}-1}\tilde{A}]\beta_h^*(\phi_0 \hat{*}_t \psi_0)|dXdx|. \end{aligned} \tag{3.84}$$

The second term vanishes if $j > 2$, and becomes

$$-i\langle (\int N_{-2}(\tilde{A})|dX|)\psi, \phi\rangle, \tag{3.85}$$

if $j = 2$. This proves (1.12).

Now if U is not trivial, by linearity, we can assume that ϕ, ψ and A are supported in a small neighborhood where U is trivialized by an orthonormal basis $\{s_i\}$. Write

$$\phi = \phi_i s_i,\ \psi = \psi_i s_i,\ A = A_{ij}s_i^* \otimes s_j. \tag{3.86}$$

Then

$$\begin{aligned} \langle (V \circ A)\psi, \phi\rangle =& - \langle A\psi, \nabla_V \phi\rangle \\ =& - \langle A_{ij}\psi_i, V\phi_j + \Gamma_{kj}(V)\phi_k\rangle, \end{aligned} \tag{3.87}$$

where $\Gamma_{kj}(V) = \langle \nabla_V s_i, s_j\rangle$. This reduces to the scalar case and one sees that the connection produces only a lower order term. □

4. Bundle filtrations

Systematic use is made here of the 'geometrization' of a bundle filtration. Recall that for a vector bundle, E, over a compact manifold (possibly with corners) a *filtration* is a finite non-decreasing sequence of subbundles:

$$E^0 \subset E^1 \subset E^2 \subset \cdots \subset E^N = E \tag{4.1}$$

where the length of the filtration is N, so $N = 0$ corresponds to the trivial filtration. In particular we allow the same subbundle to be repeated. Another filtration $F^0 \subset F^1 \subset F^{N'} = E$ of E is said to be a *refinement* of (4.1) if there is a strictly increasing map $I : \{0,1,\ldots,N\} \longrightarrow \{0,1,\ldots,N'\}$ such that for each j $E^j = F^{I(j)}$. Near any point $p \in M$ we can always find a basis, $e_1,\ldots,e_n$, $n = \dim_{\text{fibre}} E$, of E which is *compatible* with the filtration in the sense that for each $k = 0, \dots N$ there is a subset $I(k) \subset \{1, \dots, n\}$ such that the e_i for $i \in I(k)$ span E^k. Of course we can even arrange that $I(k) = \{1, \dots, \dim_{\text{fibre}} E^k\}$ but it is more convenient not to demand this. Any collection of filtrations of a given bundle is said to be compatible if there is one filtration which is a refinement of each of them (it need not be one of the given filtrations).

We are typically interested in a vector bundle E, over a manifold with corners M, which is such that for a particular boundary hypersurface, $H \subset M$, the bundle $E_H = E_{\restriction H}$ has a filtration E^k. We then wish to define a 'rescaled' version of E, i.e. a new vector bundle $\widetilde{E}$ with the properties:

$$\widetilde{E} \cong E \text{ over } X \setminus H$$
$$\widetilde{E}_{\restriction H} \cong \bigoplus_{k=0}^{N} [E^k/E^{k-1}] \otimes [N^*H]^k, \ E^{-1} = \{0\}. \tag{4.2}$$

The second condition means that $\widetilde{E}$ is (naturally) isomorphic to the graded bundle associated to the filtration of E.

In fact the filtration alone does not fix the bundle $\widetilde{E}$ in a differential sense, except in the (important) case of filtrations of length one. To construct $\widetilde{E}$ the filtration should be extended to a *jet-filtration.* By a k-jet of subbundle of E at H we mean an equivalence class of subbundles in neighborhoods of H where the equivalence relation is $F \sim G$ if there is some neighborhood, P, of H in X such that

$$\mathcal{I}(E, F) \overset{\text{def}}{\to}= \mathcal{C}^\infty(P; F) + \rho_H^k \mathcal{C}^\infty(P; E) = \mathcal{C}^\infty(P; G) + \rho_H^k \mathcal{C}^\infty(P; E). \tag{4.3}$$

Here $\rho_H \in \mathcal{C}^\infty(X)$ is a defining function for H. If F is a k-jet of subbundle then the space $\mathcal{I}(E, F)$ determines F. If F and G are respectively a k-jet and a p-jet of subbundle of E at H then we write $F \subset G$ to mean that $k \geq p$ and F and G have representatives subbundles, F' and G', in some neighborhood of H with $F' \subset G'$. This relation can also be written $\mathcal{I}(E, F) \subset \mathcal{I}(E, G)$. By a jet-filtration of E at H we mean a sequence E^j of $N - j$-jets of subbundle satisfying (4.1) in this sense of inclusion.

Suppose that E^j is such a jet-filtration of the bundle E at H. Consider the space of sections of E :

$$\mathcal{D} = \sum_{p=0}^{N} \rho_H^p \mathcal{I}(E; E^p) \subset \mathcal{C}^\infty(X; E). \tag{4.4}$$

Away from H this consists, locally, of all sections of E. Thus if $\mathcal{I}_p \subset \mathcal{C}^\infty(X)$ is the ideal of functions vanishing at $p \in X$ then the vector spaces

$$\widetilde{E}_p = \mathcal{D}/\mathcal{I}_p \cdot \mathcal{D} \tag{4.5}$$

are canonically isomorphic to the fibres of E for $p \notin H$. Since $\mathcal{D} \subset \mathcal{C}^\infty(X; E)$ for any p there is a natural map

$$\widetilde{E}_p \longrightarrow E_p. \tag{4.6}$$

Proposition 4.1. *If E is a $\mathcal{C}^\infty$ vector bundle over a manifold with corners, X, with a jet-filtration at a boundary hypersurface H then*

$$\widetilde{E} = \bigsqcup_{p \in X} \widetilde{E}_p, \tag{4.7}$$

defined using (4.5)*, has a unique structure as a $\mathcal{C}^\infty$ vector bundle over X such that the map $\iota\, \widetilde{E} \longrightarrow E$ defined by* (4.6) *is a $\mathcal{C}^\infty$ bundle map,*

$$\iota^*\mathcal{D} = \mathcal{C}^\infty(X; \widetilde{E}) \tag{4.8}$$

and (4.2) *holds.*

Proof. Suppose $F \subset G$ are respectively a k-jet and a $k-1$-jet of subbundle of E at H. Then given any representative of F as a subbundle of E near H we can find a representative of G which contains it. Thus, starting at the bottom of the filtration we can find for each j a representative F^j of the $N-j$-jet of subbundle E^j such that $F^j \subset F^{j-1}$ as subbundles near H. The definition of $\mathcal{D}$ in (4.8) then becomes

$$\mathcal{D} = \left\{ u \in \mathcal{C}^\infty(X; E);\ \text{near } H,\ u = \sum_{j=0}^{N} \rho_H^j u_j,\ u_j \in \mathcal{C}^\infty(P; F^j) \right\} \tag{4.9}$$

where P is some neighborhood of H. Locally near any $p \in H$ we can choose a basis $e_1, \dots, e_N$ of E such that $e_1, \dots, e_{R(j)}$ is a basis of F^j, where $R(j)$ is the rank of F^j. Then, from (4.9), $\rho_H^j e_p$, where j is the smallest index such that $p \le R(j)$, is a basis for $\widetilde{E}$. This gives $\widetilde{E}$ its structure as a $\mathcal{C}^\infty$ vector bundle; it is clearly independent of choices and (4.8) holds by construction. □

In the main application above we need to carry out two such rescalings at two intersecting boundary hypersurfaces. Let H_1 and H_2 be the two boundary hypersurfaces of X equipped with the jet-filtrations E_1^j, and E_2^p. Naturally some compatibility conditions are required between the two. The rescaling at H_1 will be carried out first, so the compatibility conditions is just that the rescaling must induce a jet-filtration at H_2 of the rescaled bundle.

To see what this amounts to suppose first that E itself has a filtration, G^j over X. If this filtration is to induce a filtration on the rescaled bundle $\widetilde{E}$ with respect to some jet-filtration at a boundary hypersurface, H_1, F^p, it is necessary and sufficient that

$$G^j \cap F^p,\ p = 1 \dots, N \text{ be a jet-filtration of } G^j \text{ at } H_1. \tag{4.10}$$

In case the G^j only constitute a jet-filtration of E at a boundary hypersurface, H_2, we demand that (4.10) hold in the sense that the G^j have representative subbundles of E near H_2 which filter E and on which the F^p induce jet-filtrations at H_1 near H_2. If these conditions hold then we can define the doubly-rescaled bundle $\widetilde{\widetilde{E}}$ by first defining $\widetilde{E}$ with respect to the rescaling at H_1 and then rescaling $\widetilde{E}$ with respect to the jet-filtration on it at H_2 induced by the rescaling of the jet-filtration of E.

In practice the jet-filtrations are defined from local filtrations of the bundle obtained by normal translation of a filtration from the boundary hypersurface H. Thus suppose that E has a connection and that V is a real vector field which is transversal to H. Then any filtration E^j of E on H can be extended to a filtration near H by taking F^j to be the subbundle of E which is spanned (over $\mathcal{C}^\infty(X)$) by the sections satisfying

$$\nabla_V e = 0 \text{ near } H,\ e_{\restriction H} \in \mathcal{C}^\infty(H, E^j). \tag{4.11}$$

The connection will be a natural one, but the choice of normal vector field is less natural. It is also of interest to know the extent to which the rescaled bundle inherits a connection. The obvious condition is that the connection should preserve the filtration:

$$\nabla_W e_j \in \mathcal{C}^\infty(H; E^j), \quad \forall\ W \in \mathcal{C}^\infty(H; TH),\ e_j \in \mathcal{C}^\infty(H; E^j). \tag{4.12}$$

Proposition 4.2. *Suppose E is a vector bundle with connection over the $\mathcal{C}^\infty$ manifold with corners X and that on a boundary hypersurface H the connection preserves a filtration E^j in the sense of* (4.12)*, then if the covariant derivatives of the curvature of the connection satisfy*

(4.13)

$$(\nabla_{U_1}\cdots\nabla_{U_k} R)(W_1, W_2)\colon \mathcal{C}^\infty(H; E^j) \longrightarrow \mathcal{C}^\infty(H; E^{j+k-p+2})\ \forall\ k \le N - j - 2 + p,$$
$$\text{where } p = 0, 1 \text{ and if } p = 1,\ W_2 \text{ is tangent to } H$$

the jet-filtration defined by (4.12) *is independent of the choice of normal vector field and the rescaled bundle has a b-connection, i.e.*

$$\nabla_W\, \mathcal{C}^\infty(X; \widetilde{E}) \longrightarrow \mathcal{C}^\infty(X; \widetilde{E}) \text{ provided } W \text{ is tangent to } H. \tag{4.14}$$

Proof. If the jet-filtration is defined by (4.12) then $\mathcal{D} \subset \mathcal{C}^\infty(X; E)$ is characterized by the Taylor series of the action of the chosen normal vector field:

$$(\nabla_V)^j u_{\restriction H} \in \mathcal{C}^\infty(H; E^j) \text{ for } j = 0, \dots, N-1. \tag{4.15}$$

Suppose $W \in \mathcal{C}^\infty(X, TX)$ is tangent to H. Then the Taylor series of $\nabla_W u$, for $u \in \mathcal{D}$, in the sense of (4.15) can be written

$$\nabla_V^j(\nabla_W u) = \nabla_W(\nabla_V^j u) + \sum_{p<j} R_p(V, W_2^p)\nabla_V^p u \tag{4.16}$$

where R_p is a covariant derivative of order $s \le j - p - 1$ of the curvature operator and if $s = j - p - 1$ then $W_2^p = W$ is tangent to H. Thus from (4.15) and (4.13) it follows that $\nabla_W u$ also satisfies (4.15), i.e. (4.14) holds. Changing V by a nonvanishing multiple clearly does not change the jet filtration. If any vector field tangent to H is added to V it follows, using (4.13), that the content of (4.15) is unchanged. Thus the rescaling is independent of the normal vector field used to define it. □

5. Analytic torsion

Let M be a compact Riemann manifold, of odd dimension, with metric tensor g. If $\rho\, \pi_1(M) \longrightarrow U(k)$ is a unitary representation of the fundamental group let

$$L_\rho = \tilde{M} \otimes_\rho \mathbb{C}^k, \tag{5.1}$$

where $\tilde{M}$ is the universal cover of M, be the associated locally flat Hermitian bundle over M. Exterior differentiation extends to differential forms twisted by ρ

$$d : \mathcal{C}^\infty(M; \Lambda^* M \otimes L_\rho) \longrightarrow \mathcal{C}^\infty(M; \Lambda^* M \otimes L_\rho). \tag{5.2}$$

Using the Hermitian inner product on L, metric inner product on $\Lambda^* M$ and volume form on M the adjoint, δ, and hence the twisted Laplacian can be defined

$$\Delta = d\delta + \delta d,\ \Delta : \mathcal{C}^\infty(M; \Lambda^* \otimes L_\rho) \longrightarrow \mathcal{C}^\infty(M; \Lambda^* M \otimes L_\rho). \tag{5.3}$$

Let $Q \in \mathcal{C}^\infty(M; \hom(\Lambda^* M \otimes L_\rho))$ be the parity involution defined by $Q = (-1)^p$ on $\Lambda^p M \otimes L_\rho$ and let $\operatorname{str} A = \operatorname{tr}\ QA$, for $A \in \mathcal{C}^\infty(M; \hom(\Lambda^* M \otimes L_\rho))$ be the associated supertrace tensor. For a smoothing operator, defined by its Schwartz kernel

$$
\begin{aligned}
&B \in \Psi^{-\infty}(M; \Lambda^* M \otimes L_\rho) \\
&\Longleftrightarrow B \in \mathcal{C}^\infty(M^2; \operatorname{Hom}(\Lambda^* M \otimes L_\rho) \otimes \pi_R^* \Omega M)
\end{aligned}
\tag{5.4}
$$

the 'big' supertrace is defined by

$$
\operatorname{STr}(B) = \int_M \operatorname{str}(B_{\restriction \mathrm{Diag}}). \tag{5.5}
$$

By Lidsky's theorem the supertrace of a smoothing operator is given in terms of the operator trace by $\operatorname{Tr}(QB)$.

Consider the number operator $N = p$ on $\Lambda^p M \otimes L_\rho$. The supersymmetric zeta function is defined by

$$
\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{STr}(N e^{-t\underline{\Delta}}) \frac{dt}{t}, \ \operatorname{Re} s >> 0 \tag{5.6}
$$

where $\underline{\Delta}$ is the Laplacian restricted to the orthocomplement of its null space. That is, if Π_N is orthogonal projection off the null space of Δ then

$$
\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{STr}(N e^{-t\Delta} \Pi_N) \frac{dt}{t}. \tag{5.7}
$$

This zeta function extends to a meromorphic function on the entire complex plane with $s = 0$ a regular value as indeed follows from (1.20). We define the analytic torsion of Ray and Singer by

$$
\log T_\rho(M) = \zeta_T'(0). \tag{5.8}
$$

As defined here $T_\rho(M)$ is the square of the torsion defined in [37]. Formally it is a ratio of powers of determinants for the Laplacians $\underline{\Delta}_j$, restricted to $\Lambda^j M \otimes L_\rho$ and with null space removed:

$$
T_\rho(M) \sim \prod_{j=1}^n \left[\det \underline{\Delta}_j\right]^{(-1)^j j}, \ n = \dim M. \tag{5.9}
$$

To analyze $\zeta_T(s)$ near $s = 0$ the right side of (5.6) needs to be continued analytically. The integral decays exponentially as $t \to \infty$ so only the behaviour near 0 needs to be considered. In fact there is only one obstruction to convergence for s near 0 :

Theorem 5.1. *If M is an odd-dimensional Riemann manifold, as above, the pointwise supertrace of the weighted heat kernel has a uniform asymptotic expansion as $t \downarrow 0$*

$$
\operatorname{str}(N e^{t\Delta}) \sim a_{-\frac{1}{2}} t^{-\frac{1}{2}} + \sum_{j \geq 1, \ odd} t^{\frac{j}{2}} a_{\frac{j}{2}} \tag{5.10}
$$

with coefficients $a_k \in \mathcal{C}^\infty(M; \Omega M)$ and leading term

$$
a_{-\frac{1}{2}} = c(n) \sum_{k=1}^n (-1)^k \operatorname{Pf}(R_k) \wedge \omega_k, \ c(n) = 2i(-1)^{\frac{1}{2}(n+1)} (16\pi)^{-\frac{1}{2}n}. \tag{5.11}
$$

Here ω_k is an orthonormal frame for T^*M, R_k is obtained by deleting the kth row and column from the curvature matrix R in this frame and $\mathrm{Pf}(R_k)$ *is its Pfaffian.*

Corollary 5.2. *For any $\delta > 0$*

$$\begin{aligned}\log T_\rho(M) = &\int_0^\delta \left[\mathrm{STr}(Ne^{-t\varDelta}) - a_{-\frac12}(M,g)t^{-\frac12}\right]\frac{dt}{t}\\ &+\int_\delta^\infty \mathrm{STr}(Ne^{-t\varDelta})\frac{dt}{t} - 2\delta^{-\frac12}a_{-\frac12}(M,g) - (c+\log\delta)\chi_2(M,\rho)\end{aligned} \tag{5.12}$$

where $a_{-\frac12}(M,g) = \int_M a_{-\frac12}$ is given by (5.11), χ_2 is the twisted, weighted Euler characteristic

$$\chi_2(M,\rho) = \sum_{k=0}^{N}(-1)^k k b_k,\ \ b_k = \dim H^k(M;\rho) \tag{5.13}$$

and c is Euler's constant.

Proof of Corollary. Writing (5.6) in the form

$$\begin{aligned}\zeta_T(s) = &\frac{1}{\Gamma(s)}\int_\delta^\infty t^s\,\mathrm{STr}\left[Ne^{-t\varDelta}\right]\frac{dt}{t} + \frac{1}{\Gamma(s)}\int_0^\delta t^s\left[\mathrm{STr}(Ne^{-t\varDelta}) - a_{-\frac12}t^{-\frac12}\right]\frac{dt}{t}\\ &+\frac{1}{\Gamma(s)}\frac{\delta^{s-\frac12}}{(s-\frac12)}a_{-\frac12}(M,g) - \frac{1}{\Gamma(s+1)}\delta^s\chi_2(M;\rho)\end{aligned} \tag{5.14}$$

gives an explicitly regular formula near $s=0$ from which (5.12) follows by differentiation and evaluation at $s=0$. $\square$

To prove Theorem 5.1 we shall adapt Getzler's scaling argument to the odd-dimensional case, leading to the cancellation inherit in (5.10). We do so by making a global rescaling of the homomorphism bundle of $\varLambda^*M\otimes L_\rho$ near the front face of the heat space defined above. Since this is localized near the diagonal, L_ρ does not appear in the discussion. To get (5.10) we then show that the heat kernel lifts to the rescaled bundle.

Getzler's rescaling is defined by a decomposition of the homomorphism bundle in terms of Clifford multiplication. Let V be any Euclidean vector space. Let $\mathrm{C}\ell(V)$ be the associated Clifford algebra, the tensor algebra of V with the one relation

$$e\cdot f + f\cdot e = -2\langle e,f\rangle\,\mathrm{Id}\ \ \forall\ e,f\in V. \tag{5.15}$$

This algebra acts by Clifford multiplication on the exterior algebra, $\varLambda^*V$:

$$c_l(e)\ \varLambda^*V \longrightarrow \varLambda^*V,\ c_l(e) = \mathrm{ext}(e) - \mathrm{int}(e),\ e\in V, \tag{5.16}$$

where $\mathrm{ext}(e)$ is exterior (wedge) product with e and $\mathrm{int}(e)$ is contraction with the dual vector $v\in V^*$ to $e\in V$. This is *left* Clifford multiplication, we also consider right Clifford multiplication

$$c_r(e) = (\mathrm{ext}(e)+\mathrm{int}(e))\cdot Q \tag{5.17}$$

where Q is the parity operator for the natural grading of Λ^*V; the left and right actions commute.

In case W is an even-dimensional Euclidean vector space the complexified Clifford algebra $\mathbb{C}\ell(W) = \mathrm{C}\ell(W) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathrm{Gl}(2^k;\mathbb{C})$, $\dim W = 2k$. If V is odd-dimensional we shall exploit this by extending the left Clifford action on $\mathbb{C}\Lambda^*V = \Lambda^*V \otimes_{\mathbb{R}} \mathbb{C}$ to an action of $\mathbb{C}\ell(V \oplus \mathbb{R})$. Let $e_1, \dots, e_n$ be an orthonormal basis for V and, setting $e_0 = 1 \in \mathbb{R}$, consider the operator on $\mathbb{C}\Lambda^*(V \oplus \mathbb{R})$

$$\tilde{\tau} = i^{\frac{n+1}{2}} c_r(e_0) \cdot c_r(e_1) \dots c_r(e_n). \tag{5.18}$$

Lemma 5.3. *If V is an odd-dimensional Euclidean vector space and $\tilde{\tau}$ is defined by* (5.18) *the map*

$$E\, \mathbb{C}\Lambda^*V \ni \omega \longmapsto \frac{1}{2}(\omega + \tilde{\tau}\omega) \in \mathbb{C}\Lambda^*(V \oplus \mathbb{R}), \tag{5.19}$$

where $\Lambda^*V \hookrightarrow \Lambda^*(V \oplus \mathbb{R})$ is the natural embedding, embeds $\mathbb{C}\Lambda^*V$ as a subspace invariant under the left Clifford action of $\mathbb{C}\ell(V \oplus \mathbb{R})$ such that

$$E \cdot c_l(e) = c_l(e) \cdot E, \ \forall\, e \in V, \ \text{ and } E \cdot Q = Q \cdot E. \tag{5.20}$$

Proof. Clearly $\tilde{\tau}$ is an involution. Moreover E is injective and has range precisely the 1-eigenspace of $\tilde{\tau}$. The range of E is invariant under left Clifford multiplication by $\mathbb{C}\ell(V \oplus \mathbb{R})$ and E intertwines the action of $\mathbb{C}\ell(V)$ on $\mathbb{C}\Lambda^*V$ and as a subspace of $\mathbb{C}\ell(V \oplus \mathbb{R})$. Since $\tilde{\tau}\omega$ is a form of the same parity as ω, E also intertwines the super symmetries, Q, on Λ^*V and $\Lambda^*(V \oplus \mathbb{R})$. □

The Clifford action gives a decomposition of the endomorphism space:

Lemma 5.4. *For any odd-dimensional Euclidean vector space*

$$\hom(\mathbb{C}\Lambda^*V) = \mathbb{C}\ell(V \oplus \mathbb{R}) \otimes \hom'(\mathbb{C}\Lambda^*V) \tag{5.21}$$

where the second factor is the subspace commuting with the action of $\mathbb{C}\ell(V \oplus \mathbb{R})$, it is generated by the right Clifford action of $\mathbb{C}\ell(V)$.

Proof. For the even-dimensional case

$$\hom(\mathbb{C}\Lambda^*W) = \mathbb{C}\ell(W) \otimes \mathbb{C}\ell(W) \tag{5.22}$$

with the two factors acting by left and right Clifford multiplication. For $W = V \oplus \mathbb{R}$ we deduce (5.21) with the right factor being the subspace which preserves the 1-eigenspace of $\tilde{\tau}$. This is generated by the elements $c_l(e_0) \cdot c_l(e_j)$ $i, j = 1, \dots, n$, and this is the right Clifford action by $\mathbb{C}\ell(V)$.

Notice that the involution (5.18) depends only on the choice of orientation of V. Switching orientation replaces (5.19) by the embedding of $\mathbb{C}\Lambda^*V$ as the -1-eigenspace of $\tilde{\tau}$. However $c_r(e_0)$ interchanges the ±1-eigenspaces of $\tilde{\tau}$ and intertwines the left Clifford actions on them, so the decomposition (5.21) is completely natural. □

Using (5.21) the filtration of the Clifford algebra, by minimal degree in the generators, induces a filtration of the endomorphism space

$$\hom^{[k]}(\mathbb{C}\Lambda^*V) = \mathbb{C}\ell^{[k]}(V \oplus \mathbb{R}) \otimes \hom'(\mathbb{C}\Lambda^*V), \ k = 0, \dots, n+1. \tag{5.23}$$

To find the decomposition of operators on $\mathbb{C}\Lambda^*V$ in this sense we only need find their action, on the image of E in (5.19), in terms of left and right Clifford multiplication on $\Lambda^*(V \oplus \mathbb{R})$.

For any orthonormal basis e_i, $i = 1, \dots, n$,

$$E \cdot \operatorname{ext}(e_i) = [\operatorname{ext}(e_i) \operatorname{int}(e_0) \operatorname{ext}(e_0) - \operatorname{int}(e_i) \operatorname{ext}(e_0) \operatorname{int}(e_0)] \cdot E. \tag{5.24}$$

On $\Lambda^*(V \oplus \mathbb{R})$ we have

$$\begin{aligned} \operatorname{ext}(e_i) &= \frac{1}{2}\left[c_l(e_i) + c_r(e_i)Q\right] \\ \operatorname{int}(e_i) &= \frac{1}{2}\left[-c_l(e_i) + c_r(e_i)Q\right] \end{aligned} \tag{5.25}$$

Inserting these in (5.24) gives the decompositions

$$\begin{aligned} E \cdot \operatorname{ext}(e_i) &= \left[\frac{1}{2}c_l(e_i) \otimes \operatorname{Id} - \frac{1}{2}c_l(e_0) \otimes (c_r(e_i)c_r(e_0))\right] \cdot E \\ E \cdot \operatorname{int}(e_i) &= \left[-\frac{1}{2}c_l(e_i) \otimes \operatorname{Id} - \frac{1}{2}c_l(e_0) \otimes (c_r(e_i)c_r(e_0))\right] \cdot E. \end{aligned} \tag{5.26}$$

Thus both exterior and interior multiplication are operators of order 1,

$$\operatorname{ext}(v),\ \operatorname{int}(v) \in \hom^{[1]}(\mathbb{C}\Lambda^* V),\ \forall\ v \in V. \tag{5.27}$$

Similarly we decompose the number operator by writing it on the image of E,

$$N = \sum_{k=1}^{n} [\operatorname{ext}(e_i) \operatorname{int}(e_i) \operatorname{int}(e_0) \operatorname{ext}(e_0) + \operatorname{int}(e_i) \operatorname{ext}(e_i) \operatorname{ext}(e_0) \operatorname{int}(e_0)]. \tag{5.28}$$

Again using (5.25) this becomes

$$N = \frac{1}{2}\sum_{k=1}^{n} (\operatorname{Id} - c_l(e_k)c_l(e_0)c_r(e_k)c_r(e_0)). \tag{5.29}$$

Thus $N \in \hom^{[2]}(\mathbb{C}\Lambda^* V)$.

The parity involution, Q, can be written

$$Q = c_l(e_0)c_l(e_1) \dots c_l(e_n)c_r(e_0)c_r(e_1) \dots c_r(e_n). \tag{5.30}$$

That this involution has maximal order is, together with the following fundamental observation of Patodi, the main reason for introducing the filtration.

Lemma 5.5. *The supertrace functional annihilates* $\hom^{[n]}(\mathbb{C}\Lambda^* V)$ *in* (5.23) *and*

$$\operatorname{str}(Q) = 2^n. \tag{5.31}$$

Proof. Of course (5.31) is immediate. Taking an orthonormal basis for V and considering the basis elements of $\mathbb{C}\ell(V \oplus \mathbb{R})$, the odd elements anticommute with Q and hence have zero trace after composition with Q. For an element $\mu = \prod_{1 \le r \le k} c_l(e_{j_r}) \otimes A$, $0 \le j_1 < j_2 < \dots < j_k$, with $k \le n$ there exists e_q, $q \ne j_\ell$ for $1 \le \ell \le k$. Then μ commutes with Q and e_q, which interchanges the $+1$ and -1-eigenspaces of Q, so $\operatorname{tr}(Q\mu) = 0$. □

If M is an odd-dimensional Riemann manifold the naturality of (5.23) means that it extends to give a filtration of the endomorphism bundle

$$\hom^{[k]}(\Lambda^* M \otimes L_\rho) = \mathbb{C}\ell^{[k]}(T^* M \oplus \mathbb{R}) \otimes \hom'(\Lambda^* M \otimes L_\rho) \tag{5.32}$$

where $\hom'$ is the subbundle of elements commuting with the Clifford action. The 'full' homomorphism bundle over M^2

$$\operatorname{Hom}(\Lambda^* M \otimes L_\rho) = \bigsqcup_{(x,x') \in M^2} \hom\left((\Lambda^* M \otimes L_\rho)_{x'}, (\Lambda^* M \otimes L_\rho)_x\right) \tag{5.33}$$

has the property that its restriction to the diagonal is canonically isomorphic to $\hom(\Lambda^* M \otimes L_\rho)$ over M.

Thus over the diagonal $\mathrm{Hom}(\Lambda^* M \otimes L_\rho)$ has the filtration (5.32). The extension of the filtration off the diagonal is discussed in (4). In order to apply Proposition 4.2 we need to show that the curvature functional has the appropriate order with respect to the filtration. That is, if R is the curvature operator on $\mathrm{Hom}(\Lambda^* M \otimes L_\rho)$ and $V, W, U_1, \dots, U_p$ are $\mathcal{C}^\infty$ vector fields on M^2 near the diagonal we need to show that

$$\nabla_{U_1} \dots \nabla_{U_p} R(V, W)|_{\mathrm{Diag}} \text{ has order } p + 2 - \ell \tag{5.34}$$

where $\ell \leq p + 2$ is the number of vector fields which are tangent to the diagonal.

Since the action of the curvature operator, and its covariant derivatives, is always given by a sum of products of interior and exterior multiplication it follows from (5.26) that its order can never be greater than 2. Thus (5.34) certainly holds when $p+2-\ell \geq 2$, i.e. $p \geq \ell$. It therefore suffices to consider the case when either all the vector fields are tangent to the diagonal, or all but one are so tangent. In the first case the curvature operator and its covariant derivatives are of order zero, since the Levi-Civita connection on M preserves the filtration (5.32). In the second case the fact that the diagonal is geodesically flat means that the operator (5.34) vanishes.

Thus Proposition 4.2 applies and the rescaled bundle, GHom $(\Lambda^* M \otimes L_\rho)$, and rescaled heat calculus, $\Psi^*_{h,G}(M; \Lambda^* M \otimes L_\rho)$, are therefore defined. We wish to show that the heat kernel

$$\exp(-t\Delta) \in \Psi^{-2}_{h,G}(M; \Lambda^* M \otimes L_\rho), \tag{5.35}$$

for the twisted Laplacian.

Following the discussion in §4 it suffices to show that Δ acts on the rescaled bundle and to compute the normal operator in the rescaled calculus. This follows from the Lichnerowicz/Weitzenböck formula.

Proposition 5.6. *For the twisted Laplacian*

$$\Psi^{-k}_{h,G}(M; \Lambda^* M \otimes L_\rho) \ni A \longmapsto \Delta \cdot A \in \Psi^{-k+2}_{h,G}(M; \Lambda^* M \otimes L_\rho) \tag{5.36}$$

and

$$N_{h,G,-k+2}(\Delta A) = \left[\mathcal{H} - \frac{1}{8} C(R)\right] \cdot N_{h,G,-k}(A) \tag{5.37}$$

where

$$C(R) = \sum_{i,j,s,t} R_{ijst} c_l(e_i) c_l(e_j) c_r(e_s) c_r(e_t) \tag{5.38}$$

with respect to any orthonormal frame of $T^* M$, and $\mathcal{H}$ is the generalized harmonic oscillator:

$$\mathcal{H} = -\sum_i (\sigma_1(e_i) + \frac{1}{8} R(e_i, V_r))^2. \tag{5.39}$$

Here V_r denotes the radial vector field on TM.

Proof. The Weitzenböck formula for the action of the Laplacian on $\Lambda^* M$ is

$$\Delta = \Delta^c - \sum_{i,j,k,\ell} R_{ijk\ell} \operatorname{ext}(e_i) \operatorname{int}(e_j) \operatorname{ext}(e_k) \operatorname{int}(e_\ell). \tag{5.40}$$

Here $\varDelta^c$ is the connection Laplacian; with respect to any local orthonormal frame of TM it is

$$\varDelta^c = -\sum_i \nabla^2_{v_i}. \tag{5.41}$$

Inserting (5.26) into the tensorial term in (5.40) and using the symmetries of the Riemann curvature tensor we find

$$\begin{aligned}&\sum_{i,j,k,\ell} R_{ijk\ell}\operatorname{ext}(e_i)\operatorname{int}(e_j)\operatorname{ext}(e_k)\operatorname{int}(e_\ell)\\ &=\frac{1}{16}\sum_{i,j,k\ell} R_{ijk\ell}[c_l(e_i)c_l(e_j)+c_r(e_i)c_r(e_j)][c_l(e_k)c_l(e_\ell)+c_r(e_k)c_r(e_\ell)]\end{aligned} \tag{5.42}$$

is equal to $\frac{1}{8}C(R)-\frac{1}{4}S$ where the first term is given by (5.38) and S is the scalar curvature. As a scalar S is of order 0 with respect to the filtration so does not contribute to the normal operator.

As for the connection Laplacian term, we first show that

$$\Psi^{-k}_{h,G}(M;\varLambda^*M\otimes L_\rho)\ni A\longmapsto \nabla_V\cdot A\in\Psi^{-k+1}_{h,G}(M;\varLambda^*M\otimes L_\rho) \tag{5.43}$$

and

$$N_{h,G,-k+1}(\nabla_V\cdot A)=(\sigma_1(V)+\frac{1}{8}R(V,V_r))N_{-k}(A). \tag{5.44}$$

The proof is similar to the proof of Proposition 1.2 in (3). In fact, from the computation there we obtain the following formula for the action of A. If we write

$$A=t^{\frac{k}{2}-\frac{n+2}{2}+\frac{n}{4}}\tilde{A}|dtdXdx|^{\frac{1}{2}},\ \Psi=\Psi_0|dtdx|^{\frac{1}{2}}\text{ and }A\Psi=(A\Psi)_0|dtdx|^{\frac{1}{2}} \tag{5.45}$$

then

$$(A\Psi)_0(t,x)=\int(t')^{\frac{k}{2}-1}\tilde{A}(t',X,x)\Psi_0(t-t',x-(t')^{\frac{1}{2}}X)|dt'dx|. \tag{5.46}$$

Since the rescaling does not involve L_ρ, the same argument as in the proof of Proposition 1.2 shows that it produces only a lower order term. So we need only deal with $\varLambda^*M$.

For this we trivialize $\varLambda^*M$ near the diagonal by parallel translating from each x $(=(x,x)\in\varDelta(M)\subset M\otimes M)$ along the radial direction. This gives an identification

$$\operatorname{Hom}(\varLambda^*M)\equiv\operatorname{hom}(\varLambda^*M) \tag{5.47}$$

near the front face. Now if $\{e_i\}$ is an orthonormal frame at x, parallel translated to a neighborhood around x, and

$$\alpha=(\alpha_1,\cdots,\alpha_n) \tag{5.48}$$

is a multi-index, then we can write

$$\tilde{A}=\tilde{A}_{\alpha\beta}t^{\frac{|\alpha|}{2}}c_l(e_\alpha)c_r(e_\beta), \tag{5.49}$$

where $\tilde{A}_{\alpha\beta}(t,X,x)$ is a smooth function vanishing rapidly as $X\to\infty$.

Similarly if we use $\{s_i\}$ to denote the corresponding orthonormal basis for $\varLambda^*M$, we can write $\Psi_0=\Psi_is_i$. With this notation, we have

$$(A\Psi)_0(t,x)=\int(t')^{\frac{k}{2}-1+\frac{|\alpha|}{2}}\tilde{A}_{\alpha\beta}(t',X,x)\Psi_i(t-t',x-(t')^{\frac{1}{2}}X)c_l(e_\alpha)c_r(e_\beta)s_idt'dX. \tag{5.50}$$

Therefore

$$((\nabla_V \cdot A)\Psi)_0(t,x) = \int (t')^{\frac{k}{2}-1+\frac{|\alpha|}{2}} (V\tilde{A}_{\alpha\beta}\Psi_i + \tilde{A}_{\alpha\beta}V\Psi_i)c_l(e_\alpha)c_r(e_\beta)s_i + \int (t')^{\frac{k}{2}-1+\frac{|\alpha|}{2}} \tilde{A}_{\alpha\beta}\Psi_i\nabla_V(c_l(e_\alpha)c_r(e_\beta)s_i)dt'dX. \tag{5.51}$$

The first term can be handled exactly as in the proof of Proposition 1.2, which produces, in the normal operator, the term $\sigma_1(V)N_{-k}(A)$.

We now look at the second term. We have

$$\nabla_V(c_l(e_\alpha)c_r(e_\beta)s_i) = c_l(\nabla_V e_\alpha)c_r(e_\beta)s_i + c_l(e_\alpha)c_r(\nabla_V e_\beta)s_i + c_l(e_\alpha)c_r(e_\beta)\nabla_V s_i. \tag{5.52}$$

Clearly the first two terms will only produce lower order terms so we can happily ignore them. Now

$$\begin{aligned} \nabla_V s_i =& - \langle \nabla_V e_k, e_l\rangle \operatorname{ext}(e_k)\operatorname{int}(e_l)s_i \\ =& \frac{1}{4}\Gamma_{kl}(V)[c_l(e_k)c_l(e_l) + c_r(e_k)c_r(e_l)]s_i. \end{aligned} \tag{5.53}$$

Once again, only the first term matters so we need to consider

$$\int (t')^{\frac{k}{2}-1+\frac{|\alpha|}{2}} \tilde{A}_{\alpha\beta}(t',X,x)\Psi_i(t-t',x-(t')^{\frac{1}{2}}X)\frac{1}{4}\Gamma_{kl}(V)(x-(t')^{\frac{1}{2}}X) \times c_l(e_\alpha)c_r(e_\beta)c_l(e_k)c_l(e_l)s_i dt'dX. \tag{5.54}$$

This appears to be an operator of order $-k+2$ but is really of order $-k+1$ since

$$\Gamma_{kl}(V)(x-(t')^{\frac{1}{2}}X) = \frac{1}{2}R(V,V_r)(t')^{\frac{1}{2}}X + O(|(t')^{\frac{1}{2}}X|^2). \tag{5.55}$$

Moreover its contribution to the normal operator is $\frac{1}{8}R(V,V_r)N_{-k}(A)$. □

Recall that the normal operator in the rescaled heat calculus is a section of the rescaled homomorphism bundle over the front face of the heat space. This is just the associated graded bundle to the filtration (5.31), i.e.

$$\operatorname{GHom}(\mathrm{tf}; \Lambda^*M\otimes L_\rho) = \mathbb{C}\Lambda^*(M\oplus\mathbb{R})\otimes \hom'(\Lambda^*M\otimes L_\rho) \tag{5.56}$$

lifted from M to $\mathrm{tf}(M_h^2) \cong TM$, over its interior. Thus the left Clifford multiplication in (5.38) acts as exterior multiplication and $C(R)$ is therefore nilpotent.

Moreover ∂_t acts in the same way as before. From the discussion in §4 we conclude not only that (5.35) holds but that its normal operator is given by

$$N_{h,G,-2} = \exp(-\mathcal{H})\cdot\exp(\frac{1}{8}C(R)) \tag{5.57}$$

since the two terms commute.

Now we can finally turn to the

Proof of Theorem 5.1. Recalling the formula, (5.29), for the number operator we see that $tNe^{-t\Delta} \in \Psi_{h,G}^{-2}(M;\Lambda^*M\otimes L_\rho)$ has normal operator

$$-\frac{1}{2}\sum_{k=1}^{n} c_l(e_k)c_l(e_0)c_r(e_k)c_r(e_0)\exp(\frac{1}{8}C(R))\times\exp(-\mathcal{H}). \tag{5.58}$$

As noted in Lemma 5.5 only the maximal order term contributes to the supertrace. Thus we conclude directly that

$$\operatorname{str}(tNe^{-t\Delta}) \sim t^{\frac{n+1}{2}-\frac{n}{2}}(a_{-\frac{1}{2}} + \sum_{j\geq 1 \text{ odd}} t^{\frac{j}{2}+\frac{1}{2}} a_{\frac{j}{2}}) \tag{5.59}$$

where the factor of $t^{\frac{n+1}{2}}$ comes from the rescaling of the bundle and $t^{-\frac{n}{2}}$ from the normalization in the heat calculus. We also use Proposition 1.3 to deduce that there are only odd powers of $t^{\frac{1}{2}}$ in the expansion. Dividing by t gives (5.10).

Furthermore the leading part in (5.58), in terms of the filtration, is a multiple of Q. Using (5.31) we find

$$a_{-\frac{1}{2}} = c(n)\sum_{k=1}^{n}(-1)^k \operatorname{Pf}(R_k) e_1 \wedge \cdots \wedge e_n. \tag{5.60}$$

This is a well-defined density, the Pfaffian being (by definition) the term of degree $n-1$ in $\exp(R_k)$, $R_k = \sum\limits_{p,q\neq k} R_{ijpq} e^p \wedge e^q$ as an operator on span $\{e_j, j \neq k\}$. This completes the proof of Theorem 5.1. □

6. Adiabatic scaling

As in the introduction, consider a fibration of compact manifolds

$$\begin{array}{ccc} F & \text{——} & M \\ & & \downarrow{\scriptstyle\phi} \\ & & Y. \end{array} \tag{6.1}$$

On M consider the 1-parameter family of Riemannian metrics $g_x = \phi^* h + x^2 g$ and the conformal metric

$${}^a g = g + \frac{1}{x^2}\phi^* h \tag{6.2}$$

where g is a metric on M (or at least a non-negative smooth 2-cotensor inducing a metric on each fibre of ϕ) and h is a metric on Y.

As in [27] we first rescale the vector bundle to make ${}^a g$ a fibre metric. Thus on the manifold $M_a = M \times [0,1]_x$ consider first the lift of the tangent bundle from M, the sections of which are simply vector fields on M depending on x as a parameter; we shall denote this bundle ${}^M TM_a$. At the boundary hypersurface $\mathrm{ab} = \{x = 0\} \subset M_a$, which we identify with M, consider the filtration given by the subspace of fibre vector fields

$${}^\phi TM = \bigsqcup_{p\in M} T_p \phi^{-1}(\phi(p)) \subset TM \equiv {}^M T_{\mathrm{ab}} M_a \tag{6.3}$$

and let $\pi_a : {}^a TM_a \longrightarrow M_a$ be the vector bundle over M_a defined by Proposition 4.1 from ${}^M TM_a$ and this filtration. Thus in local coordinates in M_a, (x, y, z) where (y, z), are coordinates in M with the y_i coordinates in Y, $\partial_{z_k}, x\partial_{y_i}$ is a local basis for ${}^a TM_a$. We shall denote the space of $\mathcal{C}^\infty$ sections of ${}^a TM_a$ by

$$\mathcal{V}_a(M_a) = \mathcal{C}^\infty(M_a; {}^a T) = \left\{ u \in \mathcal{C}^\infty(M_a; {}^M TM_a); u_{\restriction \mathrm{ab}} \in \mathcal{C}^\infty(M; {}^\phi TM) \right\}. \tag{6.4}$$

An a-differential operator on a vector bundle F over M_a is one which can be written in a (any) local basis of F as a matrix of operators each entry of which is a sum of up to k-fold products of elements of $\mathcal{V}_a(M_a)$. The order is then k; the space of these is denoted $\mathrm{Diff}^k_{\mathrm{a}}(M_a;F)$ and more generally $\mathrm{Diff}^k_{\mathrm{a}}(M_a;E,F)$ is the space of such operators from sections of E to sections of F. The principal symbol of such an operator is a homogeneous polynomial of degree k on ${}^aT^*M_a$, the dual of ${}^aTM_a$, with values in the lift of the homomorphism bundle of F

$$\sigma^a_k : \mathrm{Diff}^k_{\mathrm{a}}(M_a;F) \twoheadrightarrow P^k({}^aT^*M_a;\pi^*_a \hom(F)). \tag{6.5}$$

An a-differential operator is elliptic if $\sigma^a_k(P)$ is invertible on ${}^aT^*M_a\setminus 0$. The basic example of an a-differential operator is the Laplacian, discussed in [27]. Let ${}^a\Lambda^k M_a$, $k = 1, \dots, \dim M$ be the exterior powers of ${}^aT^*M_a$. These bundles can also be identified with bundles constructed using Proposition 4.1. In $x > 0$ the Laplacian of the metric (6.2) acts on these bundles and in fact

Lemma 6.1. *([27]) The Laplacian ${}^a\Delta$ of the metric (6.2) is an elliptic element of the ring of a-differential operators,* $\mathrm{Diff}^2_a(M_a;{}^a\Lambda^k M_a)$.

Proof. One can write exterior differentiation on the x-fibres of M_a as

$$df = \sum_{i=1}^{\dim Y} x\partial_{y_i} f \frac{dy_i}{x} + \sum_{j=1}^{\dim F} \partial_{z_j} f dz_j. \tag{6.6}$$

This shows that $d \in \mathrm{Diff}^1_{\mathrm{a}}(M_a;{}^a\Lambda^0 M_a, {}^a\Lambda^1 M_a)$ and by Leibniz' formula d extends to an element of $\mathrm{Diff}^1_{\mathrm{a}}(M_a;{}^a\Lambda^k M_a, {}^a\Lambda^{k+1} M_a)$ for each k. Directly from the definition of $\mathcal{V}_a(M_a)$ it follows that on taking adjoints with respect to ${}^ag$, for any Hermitian bundles E and F, $A \longmapsto A^*$ is an isomorphism of $\mathrm{Diff}^m_{\mathrm{a}}(M_a;E,F)$ onto $\mathrm{Diff}^m_{\mathrm{a}}(M_a;F,E)$. Thus, for any k, $\delta \in \mathrm{Diff}^1_{\mathrm{a}}(M_a;{}^a\Lambda^{k+1} M_a, {}^a\Lambda^k M_a)$. Since composition gives

$$\mathrm{Diff}^m_{\mathrm{a}}(M_a;E,F) \cdot \mathrm{Diff}^{m'}_{\mathrm{a}}(M_a;G,E) \subset \mathrm{Diff}^{m+m'}_{\mathrm{a}}(M_a;G,F) \tag{6.7}$$

we conclude that $\Delta = d\delta + \delta d \in \mathrm{Diff}^2_{\mathrm{a}}(M_a;{}^a\Lambda^k M_a)$. Ellipticity is a consequence of the usual computation of symbols, that of d being $i\xi\wedge$, $\xi \in {}^aT^*M_a$, so the symbol of δ is $-i\,\mathrm{int}(\xi)$ and hence ${}^a\sigma_2(\Delta) = |\xi|^2$ on ${}^aT^*M_a$. □

Another way to prove Lemma 6.1 is to observe that the Levi-Civita connection on the x-fibres of M_a, for $x > 0$, extends by continuity to a connection on ${}^aT^*_{\mathrm{ab}}M_a$. That it extends to an a-connection, i.e. defines covariant differentiation by elements of $\mathcal{V}_a$ over ab is immediate; the fact that covariant differentiation is a differential operator

$$\nabla : \mathcal{C}^\infty(\mathrm{ab};{}^aT^*M_a) \longrightarrow \mathcal{C}^\infty(\mathrm{ab};{}^aT^*M_a \otimes T^*\,\mathrm{ab}) \tag{6.8}$$

in the usual sense follows from the product nature of the metric (6.2). Since the Hodge $*$ operator is well-defined on the a-form bundles the fact that both d and δ are a-differential operators follows, even from the weaker result that the Levi-Civita connection is an a-connection.

Next we recall, and slightly refine, the results of [27] which follow from this description of ${}^a\Delta$ and the use of a-pseudodifferential calculus introduced there. We can suppress the factor L_ρ since it makes no difference, except notational, to the discussion. The fibre cotangent bundle over M, with fibre over $p \in M$ equal to T^*F_y, $y = \phi(p)$, is a natural subbundle of ${}^aT^*_{\mathrm{ab}}M_a$, the restriction of ${}^aT^*M_a$ to $\mathrm{ab}(M_a) = \{x = 0\} \equiv M$. Since ${}^ag$ defines a non-degenerate metric on ${}^aT^*M_a$

the orthocomplement of the fibre cotangent bundle is a bundle which is naturally identified with $\frac{1}{x}\phi^*T^*Y$. This gives the decomposition of the a-form bundle as

$$ {}^{a}\Lambda^*_{\mathrm{ab}}M_a = {}^{F}\Lambda^* \otimes \phi^*(\frac{1}{x}\Lambda^*Y). \tag{6.9}$$

This decomposition is preserved by the Levi-Civita connection (6.8). For any smooth section $u \in \mathcal{C}^\infty(M_a; {}^{a}\Lambda^*)$

$$ ({}^{a}\Delta u)_{\restriction\mathrm{ab}} = {}^{F}\Delta(u_{\restriction\mathrm{ab}}) \tag{6.10}$$

is given by the the fibre Laplacian, acting as ${}^{F}\Delta \otimes 1$ in terms of (6.9). Thus

$$ E_1 = \{v \in \mathcal{C}^\infty(\mathrm{ab}(M_a); {}^{a}\Lambda^*); \exists\ u \in \mathcal{C}^\infty(M_a; {}^{a}\Lambda^*), u_{\restriction\mathrm{ab}} = v,\ {}^{a}\Delta u \in x\mathcal{C}^\infty(M_a; {}^{a}\Lambda^*)\} \tag{6.11}$$

is the space of fibre-harmonic forms. It is important that E_1 can be realized as a $\mathcal{C}^\infty$ vector bundle over Y :

$$ E_1 = H^*_{\mathrm{Ho}}(F) \otimes \mathcal{C}^\infty(Y; \Lambda^*) \tag{6.12}$$

where the fibre of $H^*_{\mathrm{Ho}}(F)$ at $y \in Y$ is $H^*_{\mathrm{Ho}}(F_y)$, the Hodge cohomology of F_y with respect to the metric g_y.

Using formal Hodge theory it can be seen that the space (6.11) can also be obtained as the case $k = 1$ of

$$ \begin{aligned} E_k = \big\{v \in \mathcal{C}^\infty(\mathrm{ab}(M_a); {}^{a}\Lambda^*); \exists\ u \in \mathcal{C}^\infty(M_a; {}^{a}\Lambda^*),& \\ u_{\restriction\mathrm{ab}} = v,\ {}^{a}\Delta u \in x^{2k}\mathcal{C}^\infty(M_a; {}^{a}\Lambda^*)\big\},& \end{aligned} \tag{6.13}$$

i.e. the error term in (6.11) can always be improved to $O(x^2)$. These spaces give a Hodge-theoretic form of the Leray spectral sequence for the cohomology of M :

Proposition 6.2. ([27]) *For k sufficiently large E_k is isomorphic to $H^*(M)$, the deRham cohomology of the total space M of the fibration.*

In fact (see [27]) for each $k \geq 0$ one obtains the same space in (6.13) by weakening the condition to ${}^{a}\Delta u \in x^{2k-1}\mathcal{C}^\infty(M_a; {}^{a}\Lambda^*)$. For each k, let Π_k be the orthogonal projection with respect to ${}^{a}g$ from $E_0 = L^2(\mathrm{ab}; {}^{a}\Lambda^*)$ to the closure of the subspace E_k in L^2. The Hodge-theoretic arguments in [27] show that $\Pi_k\,\mathcal{C}^\infty(\mathrm{ab}; {}^{a}\Lambda^*) \longrightarrow E_k$ for each k. Moreover if $v \in E_k$ then choosing u as in (6.13) it follows that $du, \delta u \in x^k\mathcal{C}^\infty(M_a; {}^{a}\Lambda^*)$ and the operators

$$ d_kv = \Pi_k(x^{-k}du_{\restriction\mathrm{ab}}),\ \delta_kv = \Pi_k(x^{-k}\delta u_{\restriction\mathrm{ab}}) \tag{6.14}$$

are well-defined, independent of the choice of u, are adjoints of each other with respect to the L^2 inner product on E_k and are such that

$$ \begin{gathered} d_k^2 = 0,\ \delta_k^2 = 0,\ \text{and} \\ E_{k+1} = \{v \in E_k; d_kv = \delta_kv = 0\} = \{v \in E_k; \Delta_kv = 0, \Delta_k = d_k\delta_k + \delta_kd_k\}\,. \end{gathered} \tag{6.15}$$

For $k = 1$ the operator d_1 is just the differential on Y, in the sense of (5.2), for the representation of $\pi_1(Y)$ on the fibre cohomology. The differential complexes (E_k, d_k) are precisely the Leray spectral sequence for the cohomology of the fibration. For $k \geq 2$ the spaces E_k are finite dimensional. If $E^j_k = E_k \cap \mathcal{C}^\infty(\mathrm{ab}; {}^{a}\Lambda^j)$ is the part of E_k in degree j then the torsion for the complex

$$ E^0_k \xrightarrow{d_k} E^1_k \dots \xrightarrow{d_k} E^{\dim M}_k \tag{6.16}$$

is by definition

$$\tau(E_k, d_k) = \prod_{j=0}^{\dim M} (\det \Delta'_k)^{-1^j j} \tag{6.17}$$

where Δ'_k is the restriction of Δ_k to $E_k \ominus E_{k+1}$.

From [14] it follows that the E_k have another representation in terms of the Laplacian Δ_x. Namely, for $\epsilon > 0$ small enough and each $k \geq 2$,

$$(\widetilde{E}_k)_x = \mathrm{sp}\{u \in \mathcal{C}^\infty(M; \Lambda^*); \Delta_x u = \lambda_x u,\ \lambda_x \in \mathbb{R}, 0 \leq \lambda_x < \epsilon^{-2} x^k\},\ 0 < x < \epsilon \tag{6.18}$$

is a vector space of dimension independent of x. Thought of as subspaces of $\mathcal{C}^\infty(M_a; {}^a\Lambda^*)$ over $(0,\epsilon) \times M$ these form subbundles which are smooth down to $x = 0$, with the limiting space exactly E_k. That is each element of E_k can be extended to a smooth a-form over $[0,\epsilon) \times M$ which is a sum of eigenvectors of Δ_x with eigenvalues $O(x^{2k})$ as $x \downarrow 0$. All other eigenvalues of Δ_x are bounded away from 0. Moreover

$$\lim_{x \downarrow 0} x^{-2k}\, {}^a\Delta_{\restriction \widetilde{E}_k} = \Delta_k,\ k \geq 2. \tag{6.19}$$

This alternative representation of E_k as the span of the boundary values at $x = 0$ of the eigenforms of Δ_x corresponding to x^{2k}-small eigenvalues arises in the long-time asymptotics of the heat kernel in §11.

7. Heat kernel for the adiabatic metric

We wish to consider the heat kernel of $x^{-2}P$ where P is a self-adjoint elliptic a-differential operator, acting on some bundle F, with diagonal principal part with symbol dual to (6.2), i.e. given by the fibre metric on ${}^aT^*M_a$. Thus we seek a distribution

$$\begin{gathered} E \in \mathcal{C}^{-\infty}(\mathbb{R}_t \times M^2 \times [0,1]_x; \mathrm{Hom}(F) \otimes \pi_R^*\Omega) \text{ satisfying} \\ (\partial_t + \frac{1}{x^2}P)E = \delta(t) \otimes \mathrm{Id}_F,\ E = 0 \text{ in } t < 0. \end{gathered} \tag{7.1}$$

As is usual in such analysis we treat the case of the half-density bundle first, to get the bundles right, and then comment on the changes needed for the general case. In §10 the further modifications corresponding to Getzler's rescaling are considered.

To construct, and analyze, E we first guess the space on which it should be reasonably simple. Set $Z = [0,\infty) \times M^2 \times [0,1]$ and consider the submanifolds

$$\begin{gathered} B_h = \{(0, m, m, x) \in Z; m \in M\} \\ B_a = \{(0, m, m', 0); m, m' \in M,\ \phi(m) = \phi(m')\}\,. \end{gathered} \tag{7.2}$$

In both cases consider $S = \mathrm{sp}(dt)$ as a subbundle of the conormal bundle. Then, in terms of parabolic blow-up as described in §2 and [19], we put

$$M_A^2 = [Z; B_a, S; B_h, S]\,,\ \beta_A : M_A^2 \longrightarrow Z. \tag{7.3}$$

Thus M^2_A is a manifold with corners, having five boundary hypersurfaces:

$$\begin{gathered}\mathrm{eb}(M^2_A) = \beta_A^{-1}\{x=1\} \text{ the `extension' or trivial boundary}\\ \mathrm{tb}(M^2_A) = \mathrm{cl}\,\beta_A^{-1}\left(\{t=0\}\setminus B_h\right) \text{ the temporal boundary}\\ \mathrm{ab}(M^2_A) = \mathrm{cl}\,\beta_A^{-1}\left(\{x=0\}\setminus (B_a\cap B_h)\right) \text{ the adiabatic boundary}\\ \mathrm{tf}(M^2_A) = \beta_A^*(B_h) \text{ the temporal front face}\\ \mathrm{af}(M^2_A) = \beta_A^*(B_a) \text{ the adiabatic front face}\end{gathered} \tag{7.4}$$

at each of which there will be a model operator. Of course eb can be freely ignored. Moreover all the kernels we shall consider vanish to infinite order at $\mathrm{tb}(M^2_A)$ so we shall build this into the calculus. As usual we let ρ_F denote a defining function for the boundary hypersurface F for $F = \mathrm{tb}$, ab, tf or af .

On M^2_A consider the kernel density bundle

$$\mathrm{KD}_A = \rho_{\mathrm{af}}^{-\frac{n}{2}-1}\rho_{\mathrm{tf}}^{-\frac{N}{2}-\frac{3}{2}}\Omega^{\frac12},\ n=\dim Y,\ N=\dim M \tag{7.5}$$

and the spaces of kernels

$$\Psi_A^{-j,-k,-p}(M;\Omega^{\frac12}) = \rho_{\mathrm{tf}}^j\rho_{\mathrm{af}}^k\rho_{\mathrm{ab}}^p\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty(M^2_A;\mathrm{KD}_A),\ j,k,p\in\mathbb{N}. \tag{7.6}$$

As in the ordinary heat calculus there are invariantly defined subspaces of the space of $\mathcal{C}^\infty$ functions on M^2_A corresponding to involutions around the submanifolds which are blown up to define it. If y, z are coordinates in M, near p with the y_j coordinates in Y and y', z' are coordinates near p', with $\phi(p) = \phi(p')$ and $y = y'$ as coordinates in Y, consider the coordinates t, x, y, z, y', z' in Z near $(0, 0, p, p') \in B_a$. Then, with $\rho_{\mathrm{af}} = (t + x^2 + |y-y'|^2)^{\frac12}$, we can consider the space of $\mathcal{C}^\infty$ functions on $[Z; B_a, S]$ with Taylor series at af, the front face defined in the blow-up of the form

$$\sum_k \rho_{\mathrm{af}}^k F_k\Big(\frac{t}{\rho_{\mathrm{af}}^2}, \frac{x}{\rho_{\mathrm{af}}}, \frac{y-y'}{\rho_{\mathrm{af}}}, y+y', z, z'\Big) \tag{7.7}$$

with F_k even or odd in the second two sets of variables as k is even or odd. The lift of any $\mathcal{C}^\infty$ function on Z satisfies this condition. The further blow-up of B_h is just a parametrized form of the definition of the ordinary heat space, and so even functions at tf can be defined as before. Again the $\mathcal{C}^\infty$ functions on $[Z; B_a, S]$ all lift to be even. Moreover the evenness conditions at the two front faces are independent so four subspaces $\mathcal{C}^\infty_{E,E}(M^2_A)$, $\mathcal{C}^\infty_{E,O}(M^2_A)$, $\mathcal{C}^\infty_{O,E}(M^2_A)$ and $\mathcal{C}^\infty_{O,O}(M^2_A) \subset \mathcal{C}^\infty(M^2_A)$ are all well-defined, where the first subscript refers to tf and the second to af . Choosing, as we can

$$\rho_{\mathrm{af}} \in \mathcal{C}^\infty_{E,O}(M^2_A) \text{ and } \rho_{\mathrm{tf}} \in \mathcal{C}^\infty_{O,E}(M^2_A) \tag{7.8}$$

gives

$$\begin{gathered}\mathcal{C}^\infty_{O,E}(M^2_A) = \rho_{\mathrm{tf}}\mathcal{C}^\infty_{E,E}(M^2_A),\ \mathcal{C}^\infty_{E,O}(M^2_A) = \rho_{\mathrm{af}}\mathcal{C}^\infty_{E,E}(M^2_A),\\ \mathcal{C}^\infty_{O,O}(M^2_A) = \rho_{\mathrm{tf}}\rho_{\mathrm{af}}\mathcal{C}^\infty_{E,E}(M^2_A).\end{gathered} \tag{7.9}$$

As already noted, $\mathcal{C}^\infty(Z)$ lifts into $\mathcal{C}^\infty_{E,E}(M^2_A)$ so we can define the corresponding space of sections $\mathcal{C}^\infty_{E,E}(M^2_A; \beta_A^* U)$ for any $\mathcal{C}^\infty$ vector bundle over Z. Then we refine (7.6) to

$$\Psi_{A,E}^{-j,-k,-p}(M;\Omega^{\frac12}) = \rho_{\mathrm{tf}}^j\rho_{\mathrm{af}}^k\rho_{\mathrm{ab}}^p\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty_{E,E}(M^2_A;\mathrm{KD}_A),\ j,k,p\in\mathbb{N} \tag{7.10}$$

subject to (7.8), and similarly for action on general vector bundles over M. In §9 it is shown that these kernels define operators (by convolution in t) on $\dot{\mathcal{C}}^\infty(X;\Omega^{\frac12})$, with $X=[0,\infty)\times M\times[0,1]$. Composition results for these, and related, operators are presented (although to get a general composition formula we allow logarithmic terms at ab). This allows the solution to (7.1) to be constructed in the same spirit as in §1.

To describe the results of this construction, which is actually carried out in §9, consider the normal operators associated to an element of $\Psi_A^{-j,-k,-p}(M;\Omega^{\frac12})$ at the boundary hypersurfaces tf, af and ab; by *fiat* the normal operator at tb is trivial. These give maps into simpler calculi.

The normal operator at tf is just a parametrized version of the normal operator in the heat calculus discussed in §1 and §3. To see this, first consider the structure of tf. This is the boundary face produced, in (7.3), by the blow up of the lift of B_h, which we denote for the moment by B_h'. The submanifold B_h' lies in the lift, to $[Z;B_a,S]$, of $t=0$ and the parabolic direction for the blow up is just the conormal bundle to this boundary hypersurface. Thus tf can be canonically identified as a fibre-by-fibre compactification of the normal bundle to B_h' in the boundary hypersurface. Within $t=0$, B_h is the diagonal and the blow up of B_a is the blow up of the fibre diagonal over $x=0$, just as in the definition of M_A^2. From this it follows that the normal bundle to B_h' is canonically identified with ${}^aTM_a$ so tf is the fibre-by-fibre compactification of the vector bundle ${}^aTM_a$, using as 'trivial' time variable $T=x^{-2}t$. Note that $T^{\frac12}$ is a defining function for $\mathrm{tf}(M_A^2)$ in a neighborhood of tf except at tb; apart from a square-root singularity at tb, it blows up as ρ_{ab}^{-1} at $\mathrm{ab}(M_A^2)$, but this hypersurface is disjoint from $\mathrm{tf}(M_A^2)$. The boundary hypersurface $\mathrm{tf}(M_A^2)$ has boundary hypersurfaces which we can denote eb, af and tb from their intersections with the boundary faces of M_A^2. If $A\in\Psi_A^{-j,-k,-p}(M;\Omega^{\frac12})$, multiplication of the kernel by $T^{\frac12(N-j)+1}$, followed by evaluation at tf gives

$$N_{h,-j}:\ \Psi_A^{-j,-k,-p}(M;\Omega^{\frac12})\twoheadrightarrow \rho_{\mathrm{af}}^k\rho_{\mathrm{tb}}^\infty\mathcal{C}^\infty({}^aTM_a;\Omega_{\mathrm{fibre}});\tag{7.11}$$

obviously the null space of this map is $\Psi_A^{-j-1,-k,-p}(M;\Omega^{\frac12})$.

At af the normal operator maps into the fibre heat calculus, which is described in §8. The boundary hypersurface af is just the lift from $[Z;B_a,S]$ of the boundary hypersurface, af$'$, produced by this first blow up. Consider the ϕ-fibred product of M with itself, this is the manifold M_ϕ^2 which is fibred over Y with fibres $F_y\times F_y$. Clearly $B_a\equiv M_\phi^2$. The interior of af$'$ is canonically isomorphic to ${}^YT(M_\phi^2)\times(0,\infty)_T$ where the first factor is the lift of TY under the projection and in the second factor the global variable is $T=t/x^2$. The boundary hypersurface af$'$ of $[Z;B_a,S]$ is a fibre-by-fibre compactification of ${}^YT(M_\phi^2)\times[0,\infty)$ over M_ϕ^2. The fibre $T_yY\times(0,\infty)$ is compactified to a non-round quarter sphere which can be identified smoothly with

$$\begin{gathered}\mathrm{HM}(T_yY)=([0,\infty)\times[0,\infty)\times T_yY\setminus\{0\})\,/\sim,\\ (T,x,v)\sim(T',x',v')\Longrightarrow(T',x',v')=(s^2T,sx,sv)\text{ for some }s>0\end{gathered}\tag{7.12}$$

As discussed in §8 this quarter-sphere is closely assoicated to the Euclidean heat space. Thus af$'$ is quarter-sphere bundle over M^2_ϕ :

$$\begin{tikzcd} F \times F \arrow[r] & \mathrm{af}' \arrow[d, "\phi\times\phi"] \\ \mathrm{HM}_q \arrow[r] & \mathrm{HM}(TY) \arrow[d, "\pi_Y"] \\ & Y. \end{tikzcd} \tag{7.13}$$

Now, the effect on af$'$ of the additional blow up of B_h, to define M^2_A, reduces to the parabolic blow up of the surface B'_h which is the intersection of af$'$ and the lift of B_h. Explicitly, in terms of the projective coordinates $T, Y = (y-y')/x, y, Z = z-z', z, x$, this is the part of $\{Y=0, Z=0\}$ lying above the diagonal part of the fibration of M^2_ϕ over Y. This then describes the boundary hypersurface af of M^2_A :

$$\mathrm{af} = [\mathrm{af}'; B'_h, S]. \tag{7.14}$$

It has three boundary hypersurfaces, coming from intersections with the other boundary hypersurface of M^2_A and denoted accordingly tf, tb and ab.

Thus if we consider the heat calculus, $\Psi^*_{h,\mathrm{fibre}}({}^YTM;\Omega^{\frac12})$ on the fibres of $[0,\infty)\times {}^YTM$ as a bundle over Y, with the action being invariant under translations we get a map

$$N_{A,-k}\ \Psi_A^{-j,-k,-p}(M;\Omega^{\frac12}) \twoheadrightarrow \Psi^{-j,-p+k}_{H,\mathrm{fibre}}({}^YTM;\Omega^{\frac12}). \tag{7.15}$$

Here, for simplicity, we have denoted ${}^YTM = {}^YT(M^2_\phi)$.

At the end of §8 the heat calculus on the base of a fibration, with values in the smoothing operators on the fibres, is discussed. The normal operator at ab takes values in this calculus.

Proposition 7.1. *If ${}^a\!\Delta$ is the Laplacian of an adiabatic metric,* (6.2)*, as in Lemma 6.1 then the heat kernel, the unique solution to* (7.1) *in $x>0$, is an element*

$$\exp(-x^{-2}t\,{}^a\!\Delta) \in \Psi^{-2,-2,0}_{A,E}(M;{}^a\!\Lambda^k) \tag{7.16}$$

with normal operators

$$N_{h,-2} = (4\pi)^{-\frac{n}{2}}\exp(-\frac14|v|^2_a) \tag{7.17}$$

$$N_{A,-2} = \exp(-T\Delta_A),\ T = x^{-2}t, \tag{7.18}$$

$$N_{a,0} = \exp(-t\Delta_Y) \tag{7.19}$$

where Δ_A is the fibrewise Laplacian on the bundle ${}^YTM$ and Δ_Y is the reduced Laplacian on Y.

The proof of this main regularity result for the adiabatic heat kernel is given §9, after some preparation in the next section.

Consider what this result shows about the restriction of the heat kernel to the spatial diagonal. The spatial diagonal is embedded by

(7.20)

$$[0,\infty)\times M\times[0,1] = \widetilde{\mathrm{Diag}} \hookrightarrow Z = [0,\infty)\times M^2\times[0,1],\ (t,m,x)\longmapsto(t,m,m,x).$$

Let tb, ab and eb be the three boundary hypersurfaces of $\widetilde{\mathrm{Diag}}$, equal to the intersections of $\widetilde{\mathrm{Diag}}$ in the image of (7.20) with the corresponding boundary hypersurfaces of Z. The first blow-up in (7.3), of B_a, results in the blow-up of $\widetilde{\mathrm{Diag}}$ by

$$\widetilde{\mathrm{Diag}} \cap B_a = \{0\} \times M \times \{0\} \tag{7.21}$$

parabolically in the t-direction. The second blow-up is the parabolic blow-up of the boundary surface, $t = 0$, so if we set

$$\widetilde{\mathrm{Diag}}_A = [\widetilde{\mathrm{Diag}}; \{0\} \times M \times \{0\}, S; \mathrm{tb}, S]; \ \widetilde{\beta}_A : \widetilde{\mathrm{Diag}}_A \longrightarrow \widetilde{\mathrm{Diag}} \tag{7.22}$$

and $Z_A = [Z, B_a; S]$ the first blow up in producing M^2_A, then embedding (7.20) lifts to give a commutative diagram:

$$\begin{tikzcd} \widetilde{\mathrm{Diag}}_A \arrow[r, "\iota_A"] \arrow[d, "\widetilde{\beta}_A"] & Z_A \arrow[d, "\beta_A"'] \\ \widetilde{\mathrm{Diag}} \arrow[r, "\iota"'] & Z. \end{tikzcd} \tag{7.23}$$

This results in $\widetilde{\mathrm{Diag}}_A$ having four bounding hypersurfaces, tf, af, ab and eb where af results from the first blow up and tf is only different from tb in that the manifold has the square root $\mathcal{C}^\infty$ structure there. Again the bounding hypersurfaces are equal to the intersections of the image of $\widetilde{\mathrm{Diag}}_A$ under ι_A with the corresponding boundary hypersurfaces of Z_A.

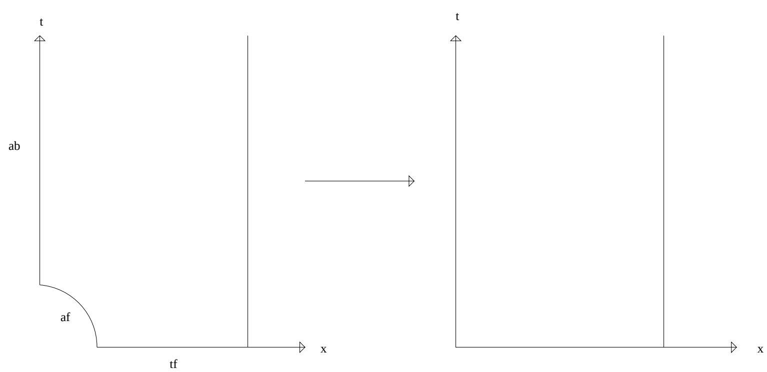

FIGURE 5. $\tilde{\beta}_A : \widetilde{\mathrm{Diag}}_A \to \widetilde{\mathrm{Diag}}$

Directly from Proposition 7.1 we conclude that the restriction to the spatial diagonal is such that, for each k,

$$\widetilde{\beta}^*_A \left[\exp(-x^{-2}tP)_{\restriction \widetilde{\mathrm{Diag}}} \otimes |dt|^{\frac{1}{2}} \right] \in \rho_{\mathrm{tf}}^{-N} \rho_{\mathrm{af}}^{-n} \mathcal{C}^\infty(\widetilde{\mathrm{Diag}}_A; {}^a\Lambda^k \otimes \Omega). \tag{7.24}$$

To see this note that the half-density bundle on Z_A at $\widetilde{\mathrm{Diag}}_A$ decomposes into the normal half-density bundle to $\widetilde{\mathrm{Diag}}_A$ tensored with the half-density bundle on $\widetilde{\mathrm{Diag}}_A$ itself. This gives the natural identification

$$\Omega^{\frac{1}{2}}(Z_A)_{\restriction \widetilde{\mathrm{Diag}}_A} \otimes |dt|^{\frac{1}{2}} \equiv \rho_{\mathrm{tf}}^{-\frac{N}{2} - \frac{1}{2}} \rho_{\mathrm{af}}^{-\frac{n}{2} - 1} \Omega(\widetilde{\mathrm{Diag}}_A) \tag{7.25}$$

which leads to (7.24). This is certainly a uniform expansion for the restriction to the diagonal and is optimal for the Laplacian in general.

Notice that $\widetilde{\mathrm{Diag}}_A = Q_2 \times M$ where Q_2 is defined at the end of the introduction. Integration of (7.24) will therefore give

$$\mathrm{Tr}(\exp(-\frac{t}{x^2}P)) \in t^{-\frac{N}{2}}\rho_{\mathrm{af}}^{-n}\mathcal{C}^\infty(Q_2). \tag{7.26}$$

This has to be considerably improved to get (0.24).

8. Euclidean and fibre heat calculus

In §1 the Euclidean heat calculus is briefly described. A slightly different description of the global regularity of these kernels is useful below.

Consider again the function Φ' in (1.1). The regularity of this kernel can be described in terms of a blown up version of the space introduced in (7.12). Let the two boundary hypersurfaces of $\mathrm{HM}(\mathbb{R}^n)$ be denoted tb and ti, where the first arises from $t = 0$ and the second from '$t = \infty$.' Set $Y = (0, 0, \mathbb{S}^{n-1}) \subset \mathrm{HM}(\mathbb{R}^n)$ be the corner of this manifold with corners and let S_Y be the conormal to the temporal boundary hypersurface, tb, which contains it. The compact manifold with corners

$$\mathrm{HHM}\,\mathbb{R}^n = [\mathrm{HM}(\mathbb{R}^n); Y, S_Y] \tag{8.1}$$

is the natural carrier of the Euclidean heat kernel, as a convolution operator. Thus, denoting by tf the new boundary hypersurface produced by the blow up in (8.1), the heat kernel lifts under blow up to an element

$$\Phi' \in \rho_{\mathrm{tf}}^{-\frac{n}{2}+1}\rho_{\mathrm{tb}}^{\infty}\rho_{\mathrm{ti}}^{-\frac{n}{2}+1}\mathcal{C}^\infty(\mathrm{HHM}(\mathbb{R}^n)). \tag{8.2}$$

More generally the convolution kernels in $\Psi^p_{\mathrm{th}}(\mathbb{R}^n)$ can be identified as the subspaces

$$\Psi^p_{\mathrm{th}}(\mathbb{R}^n) \subset \rho_{\mathrm{tf}}^{-\frac{n}{2}+1+p}\rho_{\mathrm{tb}}^{\infty}\rho_{\mathrm{ti}}^{-\frac{n}{2}+1}\mathcal{C}^\infty(\mathrm{HHM}(\mathbb{R}^n)),\ p < 0, \tag{8.3}$$

consisting of the elements which are homogeneous of degree $-\frac{n}{2}+1$ under the global $\mathbb{R}^+$ action.

Notice that this construction is independent of the basis of $\mathbb{R}^n$ so is defined for any vector space. Indeed if V is a $\mathcal{C}^\infty$ Euclidean vector bundle of rank n over some compact manifold Y then the construction can be carried out fibre-by-fibre to give a compact manifold $\mathrm{HHM}(V)$ which fibres over Y with fibre diffeomorphic to $\mathrm{HHM}(\mathbb{R}^n)$. This manifold is the natural carrier for the collective heat kernels of the flat Laplacians on the fibres, in the sense that, with the same notation for boundary faces

$$\exp(-t\Delta_{\mathrm{fibre}}) \in \rho_{\mathrm{tf}}^{-n}\rho_{\mathrm{tb}}^{\infty}\rho_{\mathrm{ab}}^{n}\mathcal{C}^\infty(\mathrm{HHM}(V)). \tag{8.4}$$

This function can also be considered as the kernel for the heat semigroup acting on half-densities, with the fibre-metric half-density used to trivialize the bundle of half densities.

For the product $\mathbb{R}^n \times F$ of Euclidean space and a compact manifold without boundary the natural heat space is obtained by combining this construction with that of (3). Thus consider the product $\mathrm{HM}(\mathbb{R}^n) \times F^2$. The appropriate heat space is

$$\mathrm{HHM}(\mathbb{R}^n \times F) = [\mathrm{HM}(\mathbb{R}^n) \times F^2; B_h, S_h] \tag{8.5}$$

where $B_h = \mathrm{tb}(\mathrm{HM}(\mathbb{R}^n)) \times \mathrm{Diag}$ and S_p is the conormal to the boundary hypersurface of $\mathrm{HM}(\mathbb{R}^n)$. For the product metric, coming from the Euclidean metric on

$\mathbb{R}^n$ and the metric on F, the heat kernel is the product $\exp(-t\Delta) = \exp(-t\Delta_F) \cdot \exp(-t\Delta_E)$.

Lemma 8.1. *The heat kernel on $\mathbb{R}^n \times F$, as a convolution kernel in the first variables, lifts to an element*

$$\exp(-t\Delta) \in \rho_{\mathrm{tf}}^{-\frac{N}{2}+1} \rho_{\mathrm{tb}}^{\infty} \rho_{\mathrm{ti}}^{-1} \mathcal{C}^\infty(\mathrm{HHM}(\mathbb{R}^n \times F); \Omega^{\frac{1}{2}}), \;\; N = n + \dim F. \tag{8.6}$$

Proof. Since the heat kernel is the product of the heat kernels, the regularity of the lifted kernel away from $t = 0$ is immediate from the separate discussions in the Euclidean and compact cases. □

To describe the normal operator at the ab face of the adiabatic heat calculus, we discuss here the heat calculus on the base with values in the smoothing operators on the fibers. Thus let M be the total space of a fibration, with the base Y and the fibers F, and set $Z = [0,\infty) \times M^2$. Now consider its t-parabolic blow-up along the submanifold B_a (defined in (7.2)) instead of the usual diagonal:

$$M^2_{h,\phi} = [Z; B_a, S]\,. \tag{8.7}$$

Note that B_a is the fibered diagonal of the fibration, M^2_ϕ.

As usual we denote by tf and tb its temporal front face and temporal boundary face respectively. Recall that n is the dimension of the base manifold. The kernel density bundle KD is now defined by the prescription

$$\mathcal{C}^\infty(M^2_{h,\phi}; \mathrm{KD}) = \rho_{\mathrm{tf}}^{-\frac{n}{2}-\frac{3}{2}} \mathcal{C}^\infty(M^2_{h,\phi}; \Omega^{\frac{1}{2}}). \tag{8.8}$$

Finally the heat calculus on the base with values in the smoothing operators on the fibers is now defined by

$$\Psi_h^{-k}(M,\phi;\Omega^{\frac{1}{2}}) = \rho_{\mathrm{tf}}^{k} \rho_{\mathrm{tb}}^{\infty} \mathcal{C}^\infty(M^2_{h,\phi}; \mathrm{KD}) \text{ for } k \in \mathbb{N}. \tag{8.9}$$

By definition the normal operator at ab of the adiabatic heat calculus, which is the multiplication of the kernel by X^{-p} ($X = x/t^{\frac{1}{2}}$) followed by the evaluation at ab, takes values in this calculus:

$$N_{a,p} : \; \Psi_A^{-j,-k,-p}(M;\Omega^{\frac{1}{2}}) \twoheadrightarrow \Psi_h^{-k}(M,\phi;\Omega^{\frac{1}{2}}). \tag{8.10}$$

Lemma 8.2. *The heat kernel of the reduced Laplacian lifts to an element*

$$\exp(-t\Delta_Y) \in \Psi_h^{-2}(M,\phi;\Omega^{\frac{1}{2}}). \tag{8.11}$$

Proof. The heat kernel of the reduced Laplacian is an element of the heat calculus of the base manifold Y. By using a partition of unity we can decompose it into the sum of two parts; the first is supported away from the front face and the second near the front face. Both can be lifted, fiberwise constantly, to an element of the base heat calculus, as described above. The first part is an effectively a smoothing operator on Y, that is, its Schwartz kernel is a smooth function on $Y \times Y$. Clearly this lifts to a smooth function on $M \times M$. Similarly, since the second piece is supported near the front face we can effectively think of it as a smooth function on af$\times[0,1]$ (say) multiplied by a singular density factor. Thus, once again it lifts to a function of the same type near the front face of the base heat space (Cf. the analysis of the adiabatic front face of the adiabatic heat calculus). □

9. Adiabatic heat calculus

In this section we generalize the results of §3 for heat calculus to the adiabatic heat calculus. As discussed in §4, the adiabatic heat calculus is defined so that the statement, in Proposition 7.1, that the heat kernel for the adiabatic Laplacian lies in the calculus gives a rather precise description of the degeneracy at $x=0$.

Formulæ (3.17) and (3.19) still define the action of $\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12})$ on $\dot{\mathcal{C}}^\infty(X,\Omega^{\frac12})$, where $X\equiv[0,\infty)\times M\times[0,1]$. We first note the result of composing these operators with differential operators. For simplicity of notation here we write the action of a vector field V through Lie derivation of half-densities simply as V.

Proposition 9.1. *Let $A\in\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12})$. If V is any smooth vector field, then*

$$(t^{\frac12}V)\circ A\in\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12}) \tag{9.1}$$

and with $\sigma_1(xV)$ the symbol of xV as an adiabatic vector field,

$$\begin{aligned} N_{h,-j}(t^{\frac12}V\circ A)&=\sigma_1(xV)N_{h,-j}(A),\\ N_{A,-k}(t^{\frac12}V\circ A)&=\sigma_1(xV)N_{A,-k}(A),\\ N_{a,-p}(t^{\frac12}V\circ A)&=(t^{\frac12}V)N_{a,-p}(A).\end{aligned} \tag{9.2}$$

If W is a vertical vector field and $T=t/x^2$, then

$$(T^{\frac12}W)\circ A\in\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12}) \tag{9.3}$$

$$\begin{aligned} N_{h,-j}[(T^{\frac12}W)\circ A]&=\sigma_1(W)N_{h,-j}(A),\\ N_{A,-k}[(T^{\frac12}W)\circ A]&=(W)N_{A,-k}(A),\\ N_{a,-p}[(T^{\frac12}W)\circ A]&=(T^{\frac12}W)N_{a,-p}(A).\end{aligned} \tag{9.4}$$

Finally, $(t\partial_t\circ A)\in\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12})$ and

$$\begin{aligned} N_{h,-j}(t\partial_t\circ A)&=[\frac{j}{2}-1-\frac12(N+R_M)]N_{h,-j}(A),\\ N_{A,-k}(t\partial_t\circ A)&=T\partial_T N_{A,-k}(A),\\ N_{a,-p}(t\partial_t\circ A)&=t\partial_t N_{a,-p}(A).\end{aligned} \tag{9.5}$$

Proof. As a vector field on the left factor of M, $t^{\frac12}V$ lifts from Z to M^2_H to a vector field of the form $\rho_{\mathrm{tb}}\tilde V$ where $\tilde V\in\mathcal{V}_{\mathrm b}(M^2_H)$. From this (9.1) follows. Similarly if W is a vertical vector field, i.e. is tangent to the fibres, then the same lift is of the form $\rho_{\mathrm{tb}}\rho_{\mathrm{af}}\tilde W$ with $\tilde W\in\mathcal{V}_{\mathrm b}(M^2_H)$, from which (9.3) follows. Similarly $t\partial_t$ lifts into $\mathcal{V}_{\mathrm b}(M^2_H)$.

To compute the normal operator at tf, we can use the projective coordinates

$$s=t^{\frac12}/x=T^{\frac12},\quad y,\quad \bar Y=\frac{y-y'}{t^{\frac12}},\quad z,\quad \bar Z=\frac{z-z'}{t^{\frac12}/x},\quad x \tag{9.6}$$

Then $A=T^{\frac{j}{2}-\frac{N+4}{4}}x^{k-\frac{n}{2}-1}a(T,y,\bar Y,z,\bar Z,x)|dTdyd\bar Ydzd\bar Zdx|^{\frac12}$, where a is a smooth function of $(T^{\frac12},y,\bar Y,z,\bar Z,x)$ and vanishes rapidly as $\bar Y\to\infty$ or $\bar Z\to\infty$.

Since $\beta_A^*(t^{\frac12}V)=t^{\frac12}V+\sigma_1(xV)$, a computation completely similar to that of the proof of Proposition 1.2 shows that

$$(t^{\frac12}V)\circ A\in\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12})\text{ and}\\ N_{h,-j}(t^{\frac12}V\circ A)=\sigma_1(xV)N_{h,-j}(A). \tag{9.7}$$

Similarly we have for a vertical vector field W

$$(T^{\frac12}W)\circ A\in\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12})\text{ and}\\ N_{h,-j}(T^{\frac12}W\circ A)=\sigma_1(W)N_{h,-j}(A). \tag{9.8}$$

and

$$t\partial_t\circ A\in\Psi_A^{-j,-k,-p}(M,\Omega^{\frac12})\\ N_{h,-j}(t\partial_t\circ A)=[\frac{j}{2}-1-\frac12(N+R_M)]N_{h,-j}(A), \tag{9.9}$$

where R_M denote the radial vector field on TM.

To compute the normal operator at af, we use the coordinates

$$T=t/x^2,\quad y,\quad Y=\frac{y-y'}{x},\quad z,\quad z',\quad x. \tag{9.10}$$

The same computation shows that

$$N_{A,-k}(t^{\frac12}V\circ A)=\sigma_1(xV)N_{A,-k}(A),\\ N_{A,-k}[(T^{\frac12}W)\circ A]=(W)N_{A,-k}(A),\\ N_{A,-k}(t\partial_t\circ A)=T\partial_TN_{A,-k}(A). \tag{9.11}$$

□

To prove composition results for adiabatic heat operators we shall use an 'adiabatic triple space'. In

$$W=\mathbb{R}^2\times M^3\times[0,1] \tag{9.12}$$

consider the three adiabatic fibre diagonals with the associated parabolic directions:

$$B_F=\{(t,0,m,m',m'',0);\phi(m')=\phi(m'')\}\,,\ S_F=\mathrm{sp}(dt')\\ B_S=\{(t,t,m,m',m'',0);\phi(m)=\phi(m')\}\,,\ S_S=\mathrm{sp}(dt-dt')\\ B_C=\{(0,t',m,m',m'',0);\phi(m)=\phi(m'')\}\,,\ S_C=\mathrm{sp}(dt) \tag{9.13}$$

and the triple fibre diagonal

$$B_T=\{(0,0,m,m',m'',0);\phi(m)=\phi(m')=\phi(m'')\}\,,\ S_T=\mathrm{sp}(dt,dt'). \tag{9.14}$$

Then set

$$W_A=[W;B_T,S_T;B_F,S_F;B_S,S_S;B_C,S_C]\,,\ \beta_A^3:W_A\longrightarrow W. \tag{9.15}$$

For this triple product we have the 'usual' results (recall that $Z=[0,\infty)\times M^2\times[0,1]$ and $Z_A=[Z,B_a;S]$):

Proposition 9.2. *The three projections*

$$\pi_F^2(t,t',m,m',m'',x)=(t',m',m'',x)\\ \pi_S^2(t,t',m,m',m'',x)=(t-t',m,m',x)\\ \pi_C^2(t,t',m,m',m'',x)=(t,m,m'',x) \tag{9.16}$$

lift to b-fibrations

$$\pi^2_{O,A} : W_A \longrightarrow Z_A,\ O = F, S, C \tag{9.17}$$

giving a commutative diagram

$$\begin{tikzcd}
 & & Z & \\
 & Z_A \arrow[ur, "\beta^2_A"] & & \\
 & & W \arrow[uu, "\pi^2_C"'] \arrow[ddl, "\pi^2_S"] \arrow[ddr, "\pi^2_F"] & \\
 & W_A \arrow[uu, "\pi^2_{C,A}"] \arrow[ur, "\beta^3_A"] \arrow[ddl, "\pi^2_{S,A}"'] \arrow[ddr, "\pi^2_{F,A}"] & & \\
 & Z & & Z \\
Z_A \arrow[ur, "\beta^2_A"'] & & Z_A. \arrow[ur, "\beta^2_A"] &
\end{tikzcd} \tag{9.18}$$

Proof. We need first to show that the 'stretched projections' $\pi^2_{O,A}$, for $O = F, S, C$ exist and are $\mathcal{C}^\infty$. Then we need to check that they are b-fibrations according to the definition given in [28]. There is sufficient symmetry (using $t \longleftrightarrow t'$ and sign reversal) that it is enough to consider one case, say $O = C$. The existence of the stretched projection follows from results on the commutation of blow-up. In this case [19, Appendix C] can be used to rewrite the definition, (9.15), in the form:

$$W_A = [W; B_C, S_C; B_T, S_T; B_F, S_F; B_S, S_S]\,. \tag{9.19}$$

The intermediate space

$$[W; B_C, S_C] \equiv Z_A \times (\mathbb{R} \times M) \tag{9.20}$$

so the iterated blow-up (9.19) gives $\pi^2_{C,A}$ as the product of a blow-down map and the projection from (9.20)

$$[W; B_C, S_C; B_T, S_T; B_F, S_F; B_S, S_S] \longrightarrow [W; B_C, S_C] \longrightarrow Z_A. \tag{9.21}$$

Since both the blow-down map and the projection are b-maps so is $\pi^2_{C,A}$; clearly it is surjective. Thus it remains only to show that $\pi^2_{C,A}$ is a b-fibration.

Let $f : X \longrightarrow Y$ be a b-map between manifolds with corners, i.e. a $\mathcal{C}^\infty$ map such that if $\rho'_i \in \mathcal{C}^\infty(Y)$, $i = 1, \dots, N'$ are defining functions for the boundary hypersurfaces of Y and $\rho_j \in \mathcal{C}^\infty(X)$, $j = 1, \dots, N$ are defining functions for the boundary hypersurfaces of X then

$$f^*\rho'_i = a_i \prod_{j=1} \rho_j^{k(i,j)},\ 0 < a_i \in \mathcal{C}^\infty(X). \tag{9.22}$$

The non-negative integers $k(i,j)$ are the boundary exponents of f. The condition that f be a b-fibration can be expressed as two conditions on the map, that it be a 'tangential submersion' and 'b-normal'. For any point $p \in X$ in a manifold with boundary let $\mathrm{BH}_X(p)$ be the smallest boundary face containing p. The first condition is the requirement that

$$f_*\left(T_p\,\mathrm{BH}_X(p)\right) = T_{f(p)}\,\mathrm{BH}_Y(f(p)) \quad \forall\, p \in X. \tag{9.23}$$

The condition of b-normality is the requirement on the boundary indices:

$$\text{For each } j,\ k(i,j) \neq 0 \text{ for at most one } i. \tag{9.24}$$

To check (9.23) for $\pi^2_{C,A}$ we note that under the iterated blow-down map in (9.21) the image of $T_p\,\mathrm{BH}(p)$ is always the tangent space at the image point to the smallest submanifold formed by the intersection of the boundary faces of the image and the submanifolds blown up. It follows easily that $\pi^2_{C,A}$ is a tangential submersion. To check that it is b-normal we simply compute the boundary indices. In table 1 the boundary exponents of all three of the stretched projections are recorded. This completes the proof of the proposition.

		at	asF	asC	asS	ab
$\pi^2_{F,A}$	af	1	1	0	0	0
	ab	0	0	1	1	1
$\pi^2_{S,A}$	af	1	0	0	1	0
	ab	0	1	1	0	1
$\pi^2_{C,A}$	af	1	0	1	0	0
	ab	0	1	0	1	1
	ν	0	0	n+2	0	0

Table 1 : Boundary exponents

□

Also in Table 1 there is a 'density row', labelled 'ν' which is important in the description of the composition results. These exponents are fixed by the natural identification of density bundles:

$$(\pi^2_{F,A})^*\,\mathrm{KD} \otimes (\pi^2_{S,A})^*\,\mathrm{KD} \otimes (\pi^2_{C,A})^*(\mathrm{KD}') \otimes |dt|^{\frac12} \cong \prod_{F=\mathrm{at,asF,asC,asS,ab}} \rho_F^{\nu_F} \cdot \Omega \text{ on } W_A. \tag{9.25}$$

Here KD' is the half-density bundle with the opposite weighting to KD so that $\mathrm{KD}' \otimes \mathrm{KD} \cong \Omega$. A straightforward computation gives the results as stated, i.e.

$$\nu_{\mathrm{at}} = \nu_{\mathrm{asF}} = \nu_{\mathrm{asS}} = \nu_{\mathrm{ab}} = 0,\ \nu_{\mathrm{asC}} = n+2. \tag{9.26}$$

The table can be used to give an 'upper bound' for the singularities of the composite of two operators using a general push-forward theorem from [28] (see also [19]) which applies because of Proposition 9.2. Thus if $\mathcal{E} = (E_{\mathrm{af}}, E_{\mathrm{ab}})$ is an index family for M^2_A, assumed trivial at tf and tb then let

$$\Psi_A^{-\infty,\mathcal{E}}(M, \Omega^{\frac12}) = \mathcal{A}_{\mathrm{phg}}^{(\infty,\infty,E_{\mathrm{af}},E_{\mathrm{ab}})}(M^2_A; \mathrm{KD}) \tag{9.27}$$

be the space of polyhomogeneous conormal distributions on M^2_A which vanish rapidly at tf and tb and have expansions at af and ab with exponents from E_{af} and E_{ab} respectively.

Proposition 9.3. *Composition, being convolution in t, gives, for any index families $\mathcal{E}$ and $\mathcal{F}$*

$$\Psi_A^{-\infty,\mathcal{E}}(M,\Omega^{\frac{1}{2}}) \circ \Psi_A^{-\infty,\mathcal{F}}(M,\Omega^{\frac{1}{2}}) \subset \Psi_A^{-\infty,\mathcal{G}}(M,\Omega^{\frac{1}{2}}) \tag{9.28}$$

where

$$\begin{aligned} G_{\mathrm{af}} &= [E_{\mathrm{af}} + F_{\mathrm{af}}] \overline{\cup} [E_{\mathrm{ab}} + F_{\mathrm{ab}} + n + 2] \\ G_{\mathrm{ab}} &= [E_{\mathrm{af}} + F_{\mathrm{ab}}] \overline{\cup} [E_{\mathrm{ab}} + F_{\mathrm{af}}] \overline{\cup} [E_{\mathrm{ab}} + F_{\mathrm{ab}}] \end{aligned} \tag{9.29}$$

Proof. Let $A \in \Psi_A^{-\infty,\mathcal{E}}(M;\Omega^{\frac{1}{2}}),\ B \in \Psi_A^{-\infty,\mathcal{F}}(M,\Omega^{\frac{1}{2}})$. The composition $C = A \circ B$ can be written in terms of their Schwartz kernels via the b-fibrations $\pi^2_{O,A}$:
(9.30)
$$\kappa_C KD' = (\pi^2_{C,A})_*[(\pi^2_{S,A})^*\kappa_A \cdot (\pi^2_{F,A})^*\kappa_B \cdot (\pi^2_{C,A})^*(KD') \cdot (\tilde{\pi}^2_{C,A})^*(|dt'|^{\frac{1}{2}}|dx|^{-1/2})],$$

where $\tilde{\pi}^2_{C,A} = \pi^2_C \circ \beta^3_A : W_A \to Z$.

By the push-forward theorem [28] (see also [19])

$$C \in \Psi_A^{-\infty,\mathcal{G}}(M,\Omega^{\frac{1}{2}}) \tag{9.31}$$

for some index set $\mathcal{G}$. The index set $\mathcal{G} = (G_{af}, G_{ab})$ can be computed by the Mellin transform

$$\begin{aligned} \langle \kappa_C \cdot KD', \rho_{af}^{z_1}\rho_{ab}^{z_2}\rangle &= \langle (\pi^2_{F,A})^*(\tilde{\kappa}_A \cdot (\pi^2_{S,A})^*(\tilde{\kappa}_B) \\ &\quad \times \prod_{F=at,asF,asC,asS,ab} \rho_F^{\nu_F} \cdot \Omega(W_A), (\pi^2_{C,A})^*(\rho_{af}^{z_1}\rho_{ab}^{z_2})\rangle, \end{aligned} \tag{9.32}$$

where $\kappa_A = \tilde{\kappa}_A \cdot KD,\ \kappa_B = \tilde{\kappa}_B \cdot KD$. From Table 1 and (9.26) we obtain (9.29). □

Proof of Theorem 1. Clearly (7.17), (7.18), (7.19) are compatible therefore there is $G_1 \in \Psi_A^{-2,-2,0}(M, {}^a\Lambda^k)$ whose normal operators are given by (7.17)-(7.19). Now by the composition formulas,

$$(t\partial_t + \frac{t}{x^2}\Delta_x)G_1 = t\,\mathrm{Id} - tR_1, \tag{9.33}$$

where $tR_1 \in \Psi_A^{-3,-3,-1}(M, {}^a\Lambda^k)$ or $R_1 \in \Psi_A^{-1,-1,-1}(M, {}^a\Lambda^k)$. Thus G_1 is already a parametrix. We now modify G_1. Using the heat calculus we can find a $G_0 \in \Psi_A^{-2,-2,-\infty}(M, {}^a\Lambda^k)$ such that

$$(t\partial_t + \frac{t}{x^2}\Delta_x)G_0 = t\,\mathrm{Id} - tR_0, \tag{9.34}$$

where $R_0 \in \Psi_A^{-\infty,0,-\infty}(M, {}^a\Lambda^k)$.

It follows that there is a correction term $G_0' \in \Psi_A^{-3,-1,-1}(M, {}^a\Lambda^k)$ such that the modification of the parametrix $G_2 = G_1 - G_0'$ still has the normal operator (7.17)-(7.19) and is a parametrix in the strong sense that

$$(t\partial_t + \frac{t}{x^2}\Delta_x)G_2 = t\,\mathrm{Id} - tR_2, \tag{9.35}$$

where $R_2 \in \Psi_A^{-\infty,-1,-1}(M, {}^a\Lambda^k)$.

By Proposition 9.3,

$$(R_2)^k \in \Psi_A^{-\infty,-k,-\alpha(k)}(M, {}^a\Lambda^k), \tag{9.36}$$

where $\alpha(k) = \{(-k, k-1)\}$. Thus the Neumann series $\sum_{k=0}^{\infty}(R_2)^k$ can be summed modulo a term vanishing rapidly at both af and ab, i.e. there exists $S' \in \Psi_A^{-\infty,-1,A}$ with $A = \cup\alpha(k)$ such that

$$(\mathrm{Id} - S')(\mathrm{Id} - R_2) = \mathrm{Id} - R_3, \quad R_3 \in \Psi_A^{-\infty,-\infty,-\infty}. \tag{9.37}$$

In other words, R_3 is a Volterra operator vanishing rapidly at all the boundaries. Thus $\mathrm{Id} - R_3$ can be inverted with an operator of the same type. It follows that $\mathrm{Id} - R_2$ has a two-side inverse $\mathrm{Id} - S$, $S \in \Psi_A^{-\infty,-1,A}(M, {}^a\Lambda^k)$. This in turn means we have

$$\exp(-\frac{t}{x^2}\Delta_x) = G_1(\mathrm{Id} - S) = G_1 - G_1 \circ S \tag{9.38}$$

and by Proposition B.2, $G_1 \circ S \in \Psi_A^{-\infty,-1,A}(M, {}^a\Lambda^k)$. That is, $\mathcal{C}^\infty$ except for logarithmic terms at ab.

To show that $\exp(-\frac{t}{x^2}\Delta_x) \in \Psi_A^{-2,-2,0}(M, {}^a\Lambda^k)$ we show that the logarithmic terms are actually zero. To this end we consider the behavior of the leading log term at the boundary of ab. Near ab, we can use the coordinates

$$t, y, Y = \frac{y - y'}{t^{\frac{1}{2}}}, z, z', X = \frac{x}{t^{\frac{1}{2}}}. \tag{9.39}$$

The boundary of ab is $\{t = 0, X = 0\}$. Let $t^{\frac{1}{2}(k-\frac{n}{2}-1)}X^p(\log X)^l u(y, Y, z, z')$ be the leading log term of $\exp(-\frac{t}{x^2}\Delta_x)$. By its explicit construction we have $1 \le p$ and $k \ge p + 2$. Since $\exp(-\frac{t}{x^2}\Delta_x)$ satisfies the heat equation, we find that the Taylor coefficients of u at $t = 0$ (we still denote by u) satisfies

$$(\Delta_Y - \frac{1}{2}Y\partial_Y + \frac{1}{2}(k-1) - \frac{n}{4} - \frac{p}{2})u = 0, \tag{9.40}$$

where $u = u(y, Y, z, z')$ is smooth in all variables and vanishes rapidly as $Y \to \infty$. Multiplying the equation by u and integrate by part

$$\int_R |\nabla u|^2 - \frac{1}{2}\int (Y\partial_Y u)u + d\int u^2 = 0. \tag{9.41}$$

But $\int(Y\partial_Y u)u = \int u\partial_Y(Yu) = -\int Yu \cdot \partial_Y u - n\int u^2$. Therefore

$$\int_R |\nabla u|^2 + c\int u^2 = 0 \tag{9.42}$$

with $c = \frac{1}{2}(k-1) - \frac{p}{2} \ge \frac{1}{2}$. Hence $u \equiv 0$.

This shows that the leading log term vanishes rapidly at the boundary of ab and therefore can be blown down to a smooth solution of the heat equation for the base manifold with zero initial data. It must be zero identically. $\square$

10. The rescaled adiabatic calculus and supertrace

With the hard work done for constructing the adiabatic heat calculus, we are now ready to show how to modify it to incorporate the Getzler's rescaling. We will first indicate the modification necessary in constructing the rescaled adiabatic heat calculus. Then a proof is given for Theorem 0.2. Finally we turn to the two lemmas in preparation of the application to the analytic torsion.

To construct the rescaled adiabatic heat calculus, the only thing different from the discussions in the previous sections is that we rescale the homomorphism bundle at both af and tf . Following the discussion in §4 we do so by giving filtrations for the homomorphism bundle over both the submanifolds, B_a and B_h, which are blown up in the construction of M^2_A. These filtrations need to satisfy the appropriate compatibility condition at $B_a \cap B_h$.

As noted in §6 over $\mathrm{ab}(M_a) = \{x = 0\}$ the exterior algebra decomposes into

$$ {}^a\Lambda^*_p M = {}^x\Lambda^*_y Y \otimes \Lambda^*_p F_y,\ p \in \mathrm{ab}(M_a),\ y = \phi(p). \tag{10.1}$$

Here ${}^x\Lambda^*_y Y = \Lambda^*({}^xT^*_y Y)$ so ${}^x\Lambda^k_y Y = x^{-k}\Lambda^k_y Y$. Since B_a lies above the fibre diagonal (10.1) leads to a decomposition of the homomorphism bundle

$$ \mathrm{Hom}_q({}^a\Lambda^* M) = \mathrm{hom}({}^x\Lambda^*_y Y) \otimes \mathrm{Hom}_q(\Lambda^* F_y) \text{ at } q \in F_y \times F_y \subset B_a. \tag{10.2}$$

We now separate into tow cases as the parity of the base dimension makes a difference here. If Y is odd-dimensional, the discussion in §2, leading to (5.32), applies to Clifford multiplication by ${}^xT^*Y$ and gives the filtration

$$ \mathrm{hom}^{[k]}({}^x\Lambda^*_y Y) = \mathbb{C}\ell^{[k]}({}^xT^*_y Y \oplus \mathbb{R}) \otimes \mathrm{hom}'({}^x\Lambda^*_y Y),\ y \in Y. \tag{10.3}$$

This lifts to give the desired filtration over B_a :

$$ \mathrm{Hom}^{[k]}_{B_a}({}^a\Lambda^* M) = \mathbb{C}\ell^{[k]}({}^xT^*Y \oplus \mathbb{R}) \otimes \mathrm{Hom}'({}^a\Lambda^* M). \tag{10.4}$$

which has length $\dim Y + 1$.

Since B_h, defined in (7.2), lies over the diagonal we can use the natural extension of the filtration (5.32). Namely left Clifford multiplication by the rescaled bundle ${}^aT^*M$ extends to give

$$ \mathrm{hom}({}^a\Lambda^* M) = \mathbb{C}\ell^{[k]}({}^aT^*M \oplus \mathbb{R}) \otimes \mathrm{hom}'({}^a\Lambda^* M) \text{ over } B_h \tag{10.5}$$

where this filtration has length $\dim M + 1$. This second filtration is consistent with the first over the intersection $B_a \cap B_h$ in the sense that (10.5) induces a filtration on each of the subspaces (10.4).

For the discussion in §4 to apply we need to show that the extension of the connection in (6.8) preserves the filtrations and that the curvature operator has the order property (4.13). These conditions follow as in §5, from the fact that exterior and interior multiplication have order one. Thus, the rescaled adiabatic heat calculus $\Psi^*_{A,G}(M; {}^a\Lambda^*)$ is defined in this case.

If Y is even dimensional, we use a different filtration of the homomorphism bundle over B_a. Suppose first that W is an even-dimensional Euclidean vector space. The complexified homomorphism bundle has the decomposition (5.22) in terms of left and right Clifford multiplication, defined by (5.16) and (5.17). Consider a different 'right' Clifford multiplication defined by

$$ \tilde{c}_r(e) = (\mathrm{ext}(e) + \mathrm{int}(e)) \cdot \tau,\ \tau = \tau_l = i^{n(2n-1)} c_l(e_1) \cdots c_l(e_{2n}) \tag{10.6}$$

in terms of an orthonormal basis $e_1, \dots, e_{2n}$ of W. The involution τ_l is, up to a power of i, the Hodge $*$ operator and so is independent of the choice of basis; we shall use it as the parity operator defining a (new) superbundle structure on the exterior algebra. Since $\tilde{c}_r(e)$ again commutes with the left Clifford multiplication this action actually gives the same decomposition (5.22) but we write it

$$ \mathrm{hom}(\mathbb{C}\Lambda^* W) = \mathbb{C}\ell(W) \otimes \widetilde{\mathbb{C}\ell}(W) \tag{10.7}$$

to emphasize that the action is through (10.6). We consider the filtration corresponding to this action:

$$ \mathrm{hom}^{[k]}(\mathbb{C}\Lambda^* W) = \mathbb{C}\ell^{[k]}(W) \otimes \widetilde{\mathbb{C}\ell}(W). \tag{10.8}$$

The true parity operator on forms can be written

$$Q = c_l(e_1)\cdots c_l(e_{2n})c_r(e_1)\cdots c_r(e_{2n}) = \tau_l\tilde{\tau}_r \qquad (10.9)$$
$$\tilde{\tau}_r = i^{n(2n-1)}\tilde{c}_r(e_1)\cdots\tilde{c}_r(e_{2n})$$

which shows it to be a homomorphism of maximal order.

The filtration (10.8) is independent of the choice of orientation, so extends to the homomorphism bundle of any even-dimensional Riemann manifold. For the fibration with even-dimensional base we consider in place of (10.3)

$$\hom^{[k]}({}^x\Lambda^*_yY) = \mathbb{C}\ell^{[k]}({}^xT^*Y)\otimes\hom'({}^x\Lambda^*_yY),\ y\in Y. \qquad (10.10)$$

and then, in place of (10.4)

$$\operatorname{Hom}^{[k]}_{B_a}({}^a\Lambda^*M) = \mathbb{C}\ell^{[k]}({}^xT^*Y)\otimes\operatorname{Hom}'({}^a\Lambda^*M). \qquad (10.11)$$

Hence, the rescaled adiabatic heat calculus $\Psi^*_{A,G}(M;{}^a\Lambda^*)$ is also defined in this case.

To prove Theorem 0.2, it remains to analyze the behaviour of the Laplacian and and to compute its normal operators. From the Weitzenböck formula (5.40) it again follows that the Laplacian acts on the rescaled bundle and hence

$$\exp(-\frac{t}{x^2}\,{}^a\Delta)\in\Psi^{-2,-2,0}_{A,G}(M;{}^a\Lambda^*). \qquad (10.12)$$

The rescaled normal operator at tf is still given by (5.57). Moreover the rescaled normal operator of the Laplacian at af is just given by (5.57), with $n=\dim Y$ and an additional term coming from the fibre as

(10.13)
$$N_{A,G,-k}((t\partial_t-\frac{t}{x^2}\,{}^a\Delta)A) = \left[\mathcal{A}_T^2+\mathcal{H}_Y+\frac{1}{2}(V_r+n+k-2)-\frac{1}{8}C(R_Y)\right]\cdot N_{A,G,-k}(A).$$

Here $\mathcal{H}_Y$ is the generalized harmonic oscillator on the fibres of ${}^xTY$, and $\mathcal{A}_T$ is the rescaled Bismut superconnection:

(10.14)
$$\mathcal{A}_T^2 = -T[\nabla_{e_i}+\frac{1}{2}T^{-\frac{1}{2}}\langle\nabla_{e_i}e_j,f_\alpha\rangle c_l(e_i)c_l(f_\alpha)+\frac{1}{4}\langle\nabla_{e_i}f_\alpha,f_\beta\rangle c_l(f_\alpha)c_l(f_\beta)]^2+\frac{1}{4}TK_F,$$

where e_i is an orthonormal basis of the fibers and f_α that of the base, and K_F denotes the scalar curvature of the fibers. We remark in passing that in the above formula the Clifford action of the base variables is acting really by exterior multiplication since at the front face they act on the graded space of the filtration. From this it follows that at the adiabatic front face the rescaled normal operator is just

$$\exp(-\mathcal{H}_Y)\exp(\frac{1}{8}C(R_Y))\exp(-\mathcal{A}_T^2). \qquad (10.15)$$

We have now finished the proof of Theorem 0.2. In what follows we shall show, by use of the rescaling adiabatic heat calculus, that not only does the analogue of (5.10) hold uniformly in x but there is additional cancellation at the adiabatic front face when (7.24) is used to compute the supertrace. The parity of the dimension of fibre and base makes a considerable difference to the argument so we treat the two cases separately.

Lemma 10.1. *If the fibres of* (6.1) *are even-dimensional then*

$$h=\operatorname{str}\left[N\exp(-\frac{t}{x^2}\,{}^a\Delta)\right]\in\rho_{\mathrm{tf}}^{-1}\rho_{\mathrm{af}}^{-1}\mathcal{C}^\infty_{E,E}(\widetilde{\mathrm{Diag}}_A;\Omega M) \qquad (10.16)$$

and if $a_{-\frac12}\in\mathcal{C}^\infty(M\times[0,1];\Omega M)$ *is given by* (5.11) *then*

$$h-a_{-\frac12}t^{-\frac12}\in\rho_{\mathrm{tf}}\rho_{\mathrm{af}}\mathcal{C}^\infty_{E,E}(\widetilde{\mathrm{Diag}}_A;\Omega M). \tag{10.17}$$

Proof. Observe that the number operator and involution decompose over af as

$$N=N_Y\otimes 1+1\otimes N_F,\ \ Q=Q_Y\otimes Q_F. \tag{10.18}$$

It follows as before that N has order 2 in terms of the rescaling at af and hence that (10.16) follows. Moreover the leading term at af is

$$t^{-\frac12}\operatorname{tr}(Q_Y\operatorname{Pf}(R_k))\operatorname{tr}(Q_F\exp(-T^F\Delta)). \tag{10.19}$$

Since the fibres are even-dimensional

$$\int\limits_{F_y}\operatorname{tr}(Q_F\exp(-T^F\Delta))=\chi(F) \tag{10.20}$$

is independent of both T and y. Thus (10.19) is independent of T and it must therefore be just $t^{-\frac12}a_{-\frac12}$. This proves (10.17) and the lemma. □

Turning to the case where the base is even-dimensional we have a similar result except that the supertrace is less singular at af :

Lemma 10.2. *If the fibres of* (6.1) *are odd-dimensional then*

$$h=\operatorname{str}\left[N\exp(-\frac{t}{x^2}\,{}^a\!\Delta)\right]\in\rho_{\mathrm{tf}}^{-1}\mathcal{C}^\infty_{E,E}(\widetilde{\mathrm{Diag}}_A;\Omega M) \tag{10.21}$$

and

$$h_{\restriction\mathrm{af}}=\operatorname{Pf}(R_Y)\operatorname{tr}_s(N_F\exp(-T\Delta_F)). \tag{10.22}$$

Proof. We proceed as for Lemma 10.1. As in the odd dimensional case, we still have $\exp(-x^{-2}t\,{}^a\!\Delta)\in\Psi^{-2,-2,0}_{A,G}(M;{}^a\!\Lambda^*)$ with the rescaled normal operator

$$\exp(-\mathcal{H}_Y)\exp(\frac18C(R_Y))\exp(-\mathcal{A}_T^2), \tag{10.23}$$

since the rescaled normal operator of the Laplacian is

$$N_{h,G,-k}(x^{-2}t\,{}^a\!\Delta)=\mathcal{A}_T^2+\mathcal{H}_Y-\frac18C(R_Y). \tag{10.24}$$

On the other hand, and this is the difference between even and odd dimensional cases, the number operator in this case has order 1 since

$$\begin{aligned}N_Y=&\sum_i\operatorname{ext}(e_i)\operatorname{int}(e_i)\\=&\frac14\sum_i(c_l(e_i)+c_r(e_i)Q)(-c_l(e_i)+c_r(e_i)Q)\\=&\frac12\sum_i(1+c_l(e_i)c_r(e_i)Q)\\=&\frac12\sum_i(1+c_l(e_i)\tilde c_r(e_i)\tilde\tau_r).\end{aligned} \tag{10.25}$$

It follows that

$$\operatorname{str}\left[N_Y\exp(-\frac{t}{x^2}\,{}^a\!\Delta)\right]\in\rho_{\mathrm{tf}}^{-1}\rho_{\mathrm{af}}^{-1}\mathcal{C}^\infty_{E,E}(\widetilde{\mathrm{Diag}}_A;\Omega M) \tag{10.26}$$

with the leading term at af equal to

$$\rho_{\mathrm{af}}^{-1}\operatorname{tr}(\sum_i c_l(e_i)\tilde{c}_r(e_i)\tilde{\tau}_r\exp(\frac{1}{8}C(R_Y))\exp(-T\Delta_F)\equiv 0 \tag{10.27}$$

since the whole expression involves an odd number of factor of $c_l(e_i)$. This proves (10.21). Moreover, $\operatorname{tr}_s(\sum_i c_l(e_i)\tilde{c}_r(e_i)\tilde{\tau}_r\exp(-x^{-2}t\,{}^a\!\Delta))$ contributes no constant term at af as it follows from (10.26), leaving us with the simpler

$$\operatorname{tr}_s[(\frac{n}{2}\operatorname{Id}+1\otimes N_F)\exp(-x^{-2}t\,{}^a\!\Delta)] \tag{10.28}$$

to evaluate. The leading term for this is
(10.29)

$$\begin{aligned}&\operatorname{tr}_s[(\frac{n}{2}\operatorname{Id}+1\otimes N_F)\exp(\frac{1}{8}C(R_Y))\exp(-T\Delta_F)]\\ =\ &\frac{n}{2}\operatorname{Pf}(R_Y)\chi(F)+\operatorname{Pf}(R_Y)\operatorname{tr}_s(N_F\exp(-T\Delta_F))=\operatorname{Pf}(R_Y)\operatorname{tr}_s(N_F\exp(-T\Delta_F)),\end{aligned}$$

giving (10.22). □

11. Adiabatic limit of analytic torsion

We show the existence of the expansions (0.14) separately in the two cases, starting with the assumption that $\dim Y$ is odd.

Consider the application of (5.12) to Δ_x as $x\downarrow 0$. Of the four terms let us start with the second. Since $t\geq\delta$ on the integrand the supertrace is smooth down to $x=0$, locally uniformly in t. We are therefore mainly concerned with the long-time behaviour. Let Π_1 be orthogonal projection onto the null space of ${}^F\!\Delta$, Π_2 the orthogonal projection onto the null space of Δ_Y and in general for $k\geq 3$ let Π_k be the orthogonal projection onto E_k, i.e. the null space of Δ_{k-1}. Thus for small x

$${}^a\underline{\Delta}\Pi_1=x^2\underline{\Delta}_Y+\prod_{k\geq 2}x^{2k}\underline{\Delta}_k \tag{11.1}$$

and hence

$$\begin{aligned}&\int_\delta^\infty \operatorname{STr}(Ne^{-t\,{}^a\underline{\Delta}})\frac{dt}{t}\\ &=\int_\delta^\infty\operatorname{STr}(Ne^{-t\underline{\Delta}_Y})\frac{dt}{t}+\sum_{k\geq 2}\int_\delta^\infty\operatorname{STr}(Ne^{-tx^{2(k-1)}\underline{\Delta}_k})\frac{dt}{t}+O(x).\end{aligned} \tag{11.2}$$

The terms in the sum each have an expansion

$$\begin{aligned}&\int_{x^{2(k-1)}\delta}^\infty \operatorname{STr}(Ne^{-T\underline{\Delta}_k})\frac{dT}{T}=\\ &-\log(\delta x^{2(k-1)})[\chi_2(E_k,d_k)-\chi_2(E_{k+1},d_{k+1})]+\log\tau(E_k,d_k)+O(x).\end{aligned} \tag{11.3}$$

Next consider the first term in (5.12). Dividing it at some arbitrary point

(11.4)
$$\int_0^\delta \left[\mathrm{STr}(Ne^{-t\Delta_x}) - a_{-\frac{1}{2}}(M,g_x)t^{-\frac{1}{2}}\right]\frac{dt}{t} =$$
$$\int_0^{\lambda x^2} \left[\mathrm{STr}(Ne^{-t\Delta_x}) - a_{-\frac{1}{2}}(M,g_x)t^{-\frac{1}{2}}\right]\frac{dt}{t} + \int_{\lambda x^2}^{\delta} \left[\mathrm{STr}(Ne^{-t\Delta_x}) - a_{-\frac{1}{2}}(M,g_x)t^{-\frac{1}{2}}\right]\frac{dt}{t}$$

allows either the coordinates $t/x^2, x$ or $t^{\frac{1}{2}}, xt^{-\frac{1}{2}}$ to be used in the two pieces. It then follows directly from Lemma 10.1 that the first term on the right is of the form

$$\int_0^{\lambda x^2} \left[g(\frac{t^{\frac{1}{2}}}{x}, x)\right]\frac{dt}{t} = \int_0^{\lambda}\left[g(T^{\frac{1}{2}}, x)\right]\frac{dT}{T} \tag{11.5}$$

where g is $\mathcal{C}^\infty$ and vanishes if either the first or the second argument vanishes. The integral is therefore $\mathcal{C}^\infty$ in x and vanishes at $x=0$. Similarly the second is term in (11.4) can be written

$$\int_{\lambda x^2}^{\delta} t^{\frac{1}{2}} g'(t, \frac{x}{t^{\frac{1}{2}}})\frac{dt}{t} \tag{11.6}$$

where g' is $\mathcal{C}^\infty$ and vanishes where the first argument vanishes. This is again $\mathcal{C}^\infty$ in x and converges to

$$\int_0^\delta \sum_{j=1}^{\dim F}\left[\mathrm{STr}_Y(N_Y e^{-t\Delta_{Y,j}}) - a_{-\frac{1}{2}}(Y)t^{-\frac{1}{2}}\right]\frac{dt}{t} \tag{11.7}$$

where $a_{-\frac{1}{2}}(Y)$ is necessarily the coefficient which makes the integral converge. Here $\Delta_{Y,j}$ is the Laplacian acting on the ρ-twisted fibre cohomology in dimension j. There is an extra term involving N_F which however vanishes by Poincaré duality.

Since the terms involving $\delta^{-\frac{1}{2}}$ and $\log\delta$ in (5.12) are just those needed to ensure the independence of δ, the first term on the right in (11.2) combines with (11.7) and the remaining two terms to give, in the limit as $x \downarrow 0$ the logarithm of the first factor in (0.11). In brief we have proved (0.10) and (0.11) in case Y is odd-dimensional.

The case of an even-dimensional base is quite similar. The analysis of the second term in (5.12) is exactly the same, so consider the first term. The kernel certainly behaves differently. Taking the decomposition (11.4) we get in place of (11.5) an integral

$$\int_0^{\lambda x^2} g(\frac{t^{\frac{1}{2}}}{x}, x)\frac{dt}{t} = \int_0^{\lambda} g(T^{\frac{1}{2}}, x)\frac{dT}{T} \tag{11.8}$$

where now g is $\mathcal{C}^\infty$ but vanishes only when the first argument vanishes.

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