The Intersection R-Torsion for Finite Cone

Xianzhe Dai* Xiaoling Huang†

1 Introduction

Torsion invariants were originally introduced in the 3-dimensional setting by K. Reidemeister [23] in 1935 who used them to give a homeomorphism classification of 3-dimensional lens spaces. The Reidemeister torsions (R-torsions for short) are defined using linear algebra and combinational topology. The salient feature of R-torsions is that it is not a homotopy invariant but rather a simple homotopy invariant; hence a homeomorphism invariant as well. From the index theoretic point of view, R-torsion is a secondary invariant with respect to the Euler characteristic. For geometric operators such as the Gauss-Bonnet and Dolbeault operator, the index is the Euler characteristic of certain cohomology groups. If these groups vanish, the Index Theorem has nothing to say, and secondary geometric and topological invariant, i.e., R-torsion, appears. The R-torsions were generalized to arbitrary dimensions by W. Franz [13] and later studied by many authors (Cf. [19]).

Analytic torsion (or Ray-Singer torsion), which is a certain combinations of determinants of Hodge Laplacians on k-forms, is an invariant of Riemannian manifolds defined by Ray and Singer [22] as an analytic analog of R-torsions. Based on the evidence presented by Ray and Singer, Cheeger [4] and Müller [20] proved the Ray-Singer conjecture, i.e., the equality of analytic and Reidemeister torsion, on closed manifolds using different techniques. Cheeger's proof uses surgery techniques to reduce the problem to the case of a sphere, while Müller's proof examines the convergence of the spectral theory of the combinatorial Laplacians to that of the smooth Laplacians as the mesh of the triangulation goes to zero. Vishik [25] gave a cutting and pasting proof based on ideas from topological quantum field theory, and Bismut and Zhang [1] had a proof based on Witten's proof of the Morse inequalities.

Further significant work includes that of Müller [21], which extended the theorem to unimodular representations, that of Bismut and Zhang [1], which treated general representations (in which interesting secondary invariants come in), and that of Burghelea-Friedlander-Kappeler-McDonald [3], which dealt with infinite dimensional representations.

It is a natural question wether the Ray-Singer conjecture/Cheeger-Müller theorem extends to singular manifolds. For manifolds with isolated conical singularity, both the R-torsion and analytic torsion have been defined by Dar [12], using respectively, the intersection homology of Goresky-MacPherson [14, 15] and Cheeger's theory of heat kernels for conical singularity [5]. There are several possible approaches to this question, among which the most natural one is to reduce the problem to three parts. One concerns manifolds with boundary, for which the question has been extensively studied [4, 17, 18, 9, 2]. The second part would be a finite cone. The last part deals with the Mayer-Vietories principle.

In this paper we concentrate on the intersection R-torsion side of the story. We will first study the intersection R-torsion of a finite cone. Our main result expresses it as a combination of determinants

*Math Dept, UCSB, Santa Barbara, CA 93106 Email: dai@math.ucsb.edu. Partially supported by NSF Grant # DMS-

†Math Dept, UCSB, Santa Barbara, CA 93106 Email: dai@math.ucsb.edu. Partially supported by NSF Grant # DMS-

of the combinatorial Laplacian on the cross section of the cone. We then study an analytic invariant which is obtained by replacing the combinatorial Laplacian by the Hodge Laplacian.

More specifically, consider the finite cone $X = C(Y)$ with the cross section Y a closed $(n-1)$-dimensional manifold. Let $I\tau^{\bar{p}}(X)$ denote the intersection R-torsion of X, where $\bar{p}$ is a given perversity. Then,

Theorem 1.1 *Let $\Delta^{(c)}$ denote the combinatorial Laplacian of the cross section Y. Then*

$$\ln I\tau^{\bar{p}}(X) = \sum_{p=0}^{n-p_n-1} (-1)^{p+1} p \ln\det \Delta_p^{(c)} + (n-p_n) \sum_{p=n-p_n}^{n-1} (-1)^{p+1} \ln\det \Delta_p^{(c)}.$$

This leads us to the following analytic invariant for an even dimensional manifolds. Thus, let Y be an even dimensional closed manifold with $m = \dim Y$. Let p be an integer such that $0 \leq p \leq m-1$ (p corresponds to p_n which is determined by a given perversity). Given an orthogonal representation $\rho : \pi_1(Y) \longrightarrow O(N)$, one has an associated flat vector bundle E_ρ with compatible metric on Y. Let Δ_k be the Laplacian acting on differential k forms on Y with coefficients in E_ρ. Then we define

$$\ln T_p(Y,\rho) = \frac{1}{2}\left[\sum_{k=0}^{m-p} (-1)^{k+1} k \ln\det(\Delta_k) + (m-p) \sum_{k=m-p+1}^{m} (-1)^{k+1} \ln\det(\Delta_k)\right].$$

For $p = 0$ this gives the usual analytic torsion which is trivial for dimensional reasons. Other values of p give nontrivial and more interesting analytic invariants. To investigate what kind of invariant $\ln T_p(Y,\rho)$ defines, we now look at its variation under metric change. Let $g(u)$ be a family of Riemannian metrics on Y and $\Delta_k(u)$ the corresponding Laplacian (when there is no ambiguity we will often write Δ_k instead of $\Delta_k(u)$. Let $\dot{\star} = d\star/du$ and $\alpha = \star^{-1}\dot{\star}$. Denote by $E_k(t) = e^{-t\Delta_k(u)}$ the heat kernel and let $E_k = E_k^{ex} + E_k^{ce} + E_k^{h}$ denote the Hodge decomposition of E_k into its exact, coexact and harmonic parts. We have the following result regarding the variation of $\ln T_p(Y,\rho)$.

Theorem 1.2 *The variation of $\ln T_p(Y,\rho)$ is given by*

$$\begin{aligned} \frac{d}{du}\ln T_p(Y,\rho) &= \frac{1}{2}\sum_{k=0}^{m-p-1} (-1)^{k+1}\mathrm{Tr}(P_{H^k}\alpha) + \frac{1}{2}\sum_{k=0}^{m-p-1} (-1)^{k+1}\mathrm{LIM}_{t\to 0}\mathrm{Tr}(e^{-t\Delta_k}\alpha) \\ &+ (-1)^{m-p+1}\frac{1}{2}\mathrm{LIM}_{t\to 0}\mathrm{Tr}(E_{m-p}^{ex}(t)\alpha), \end{aligned}$$

where P_{H^k} denote the projection onto the cohomology H^k and $\mathrm{LIM}_{t\to 0}\mathrm{Tr}(E_{m-p}^{ex}(t)\alpha)$ denotes the constant term in the asymptotic expansion of $\mathrm{Tr}(E_{m-p}^{ex}(t)\alpha)$.

Finally, we examine the R-torsion of the Mayer-Vietoris sequence.

Theorem 1.3 *Assume that the Witt condition $H^{\frac{m}{2}}(Y) = 0$ holds. Then the R-torsion of the Mayer-Vietoris sequence in intersection cohomology*

$$\cdots \longrightarrow IH^q_{(2)}(Y) \longrightarrow IH^{q+1}_{(2)}(X) \longrightarrow IH^{q+1}_{(2)}(M) \oplus IH^{q+1}_{(2)}(C(Y)) \longrightarrow IH^{q+1}_{(2)}(Y) \longrightarrow \cdots$$

is equal to the R-torsion of the truncated exact sequence of the pair (M,Y)

$$0 \longrightarrow H^{\frac{m}{2}+1}(M,Y) \longrightarrow H^{\frac{m}{2}+1}(M) \longrightarrow H^{\frac{m}{2}+1}(Y) \longrightarrow H^{\frac{m}{2}+2}(M,Y) \longrightarrow \cdots$$

2 The definition of Intersection R-torsion

We briefly recall the definition and characteristic properties of R-torsion for short. Roughly speaking, the R-torsion measures to what extent the boundary map of a chain complex can be made to preserve a preferred volume element. Let C be a real vector space of dimension n and let $b = (b_1, \cdots, b_n)$, $c = (c_1, \cdots, c_n)$ be two different bases for C, Then $c_i = a_{ij} b_j$ and $(a_{ij}) \in GL(n, R)$. We denote $det(a_{ij})$ by $[c/b]$.

Let $(C, \partial) : 0 \to C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} \cdots C_1 \overset{\partial_1}{\to} C_0 \to 0$ be a chain complex of real vector spaces. Let c_i be a preferred basis for C and h a preferred basis for the homology group $H^*(C)$. Denote B_i the image of the boundary map $\partial_{i+1} :\ C_{i+1} \to C_i$ and Z_i its kernel. We choose a basis b_i for B_i, which lifts to linearly independent set $\tilde{b}_i \in C_{i+1}$, i.e. $\partial \tilde{b}_i = b_i$. Using the inclusions $0 \subset B_i \subset Z_i \subset C_i$ where $Z_i/B_i \equiv H_i$, $C_i/Z_i \equiv B_{i-1}$ we see that $b_i, h_i, \tilde{b}_{i-1}$ combine to give a new basis for C_i. The R-torsion of the chain complex is the real number $\tau(c, h)$ defined by

$$\ln \tau(c, h) = \sum_{i=0}^{n} (-1)^i \ln |[b_i h_i \tilde{b}_{i-1}/c_i]|. \tag{2.1}$$

The R-torsion $\tau(c, h)$ does not depend on the choice of $b_i, \tilde{b}_{i-1}$, but it depends on the preferred bases c_i, h_i. In fact, it depends only on the volume elements determined by these preferred bases.

When the preferred basis of the homology is chosen according to the preferred basis of the chain complex, there is an elegant representation of the R-torsion in terms of the combinatorial Laplacian. The choice of a preferred basis for each C_i represents $\partial_i :\ C_i \to C_{i-1}$ as a real matrix. Let $\partial_i^* :\ C_{i-1} \to C_i$ be the transpose matrix. The combinatorial Laplacian is $\Delta_i^{(c)} = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i :\ C_i \to C_i$. By the finite dimensional Hodge theory, $\ker \Delta_i^{(c)} \cong H_i(C, \partial)$. If we choose the preferred basis h on $H_i(C, \partial)$ to correspond to an orthonormal basis of $\ker \Delta_i^{(c)}$, then,

$$\ln \tau(c, h) = \frac{1}{2} \sum_{i=0}^{n} (-1)^{i+1} i \log \det \Delta_i^{(c)}. \tag{2.2}$$

Now if K be a finite CW complex, consider $\tilde{K}$ the universal covering complex of K. The fundamental group π of K acts on $\tilde{K}$ as the group of covering transformations. This action makes $C(\tilde{K})$, the cellular chain complex associated with $\tilde{K}$, a free $\mathbb{R}\pi$-module generated by the cells e_i of the complex K. We pick a preferred basis for $C_i(\tilde{K})$ coming from the i-cells of K, denoted $(e_i^1, e_i^2, \cdots, e_i^{k_i})$.

Let $\epsilon :\ \pi \longrightarrow O(n)$ be an orthogonal representation of the fundamental group. Then one can construct a chain complex of real vector spaces by setting $C_i(K, \epsilon) = C_i(\tilde{K}) \otimes_{\mathbb{R}\pi} R^n$. We have a preferred choice of basis for each vector space $C_i(K, \epsilon)$ given by $e_i^j \otimes x_k$ where x_k is an orthonormal basis for R^n. With a choice of preferred basis h in homology, the torsion of the complex of real vector spaces $C_i(K, \epsilon)$ is a real number and will be denoted $\tau(K, \epsilon, h)$.

The R-torsion is a combinatorial invariant i.e. invariant under subdivision of K. It is a topological invariant when the chain complex is acyclic.

The R-torsion of a closed manifold M is the R-torsion of the cell complex determined by a cell structure of M. In this case, the preferred base for the homology is obtained via Hodge theory through an orthonormal basis of the harmonic forms. With this choice of preferred basis in homology it was shown in celebrated work of Cheeger [4] and Müller [20] that $\tau(M, \epsilon)$ equals the so called analytic torsion (Ray-Singer conjecture).

The intersection R-torsion is defined for pseudomanifolds by Dar [12] using the intersection homology theory of Goresky-MacPherson. We recall the basic facts of Intersection Homology Theory.

A pseudomanifold X of dimension n is a compact PL space for which there exists a closed subspace Z with dimension $Z \leq n-2$ such that $X - Z$ is an n-dimensional oriented manifold which is dense in X. A stratification of a pseudomanifold is a filtration by closed subspaces

$$X = X_n = X_{n-1} = Z \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0$$

such that for each point $p \in X_i - X_{i-1}$ there is a filtered space $V = V_n \supset V_{n-1} \supset \cdots V_i =$a point and a mapping $V \times B^i \to X$ which takes $V_j \times B^i$ (PL) homeomorphically to a neighborhood of p in X_j. $X_i - X_{i-1}$ is an i-dimensional manifold called the i-dimensional stratum. Every pseudomanifold admits a stratification.

The space of geometric chains $C_*(X)$ is the collection of all simplicial chains with respect to some triangulation where one identifies the two chains if their images coincide under some common subdivision. The intersection homology theory is obtained by restricting to only *allowable* chains, described by the so called perversity.

A perversity is a sequence of integers $\bar{p} = (p_2, p_3, \cdots, p_n)$ such that $p_2 = 0$ and $p_{k+l} = p_k$ or p_{k+1}. If i is an integer and $\bar{p}$ is a perversity, a subspace $Y \subset X$ is $(\bar{p}, i)$ allowable if $\dim(Y) \leq i$ and $\dim(Y \cap X_{n-k}) \leq i - k + p_k$ for $k \geq 2$. In other words, p_k describes how much X is allowed to deviate from intersecting the stratum X_{n-k} transversally. The intersection chains $IC_i^{\bar{p}}(X)$ is the subspace of $C_i(X)$ consisting of those chains ξ such that $|\xi|$ is $(\bar{p}, i)$ allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$ allowable. The i-th Intersection Homology Group of perversity $\bar{p}$, $IH_i^{\bar{p}}(X)$ is the i-th homology group of the chain complex $IC_*^{\bar{p}}(X)$.

The intersection chain complex as we defined is not finitely generated. In order to define the Intersection R-torsion we need to work with finitely generated chain groups. To do this one uses the basic sets $R_i^{\bar{p}}$.

Let X be a pseudomanifold with a fixed stratification. Let T be a triangulation of X subordinate to the stratification i.e. such that each X_k is a subcomplex of T. Define $R_i^{\bar{p}}$ be the subcomplex of T', the first barycentric subdivision of T, consisting of all simplices which are $(\bar{p}, i)$ allowable.

Let $\mathcal{R}^{\bar{p}}(X)$ be the chain complex whose i-th chain group consists of simplicial chains e_i such that $|e_i| \in R_i^{\bar{p}}$ and $|\partial e_i| \in R_{i-1}^{\bar{p}}$. It is a free abelian group generated by finitely many chains $\{e_i^j\}$. The homology group $H_i(\mathcal{R}^{\bar{p}}(X))$ is canonically isomorphic to $IH_i^{\bar{p}}(X)$.

Let $\tilde{X}$ be the universal covering complex of X. Then the chain complex $\mathcal{R}^{\bar{p}}(\tilde{X})$ is a free $\mathbb{R}\pi$-module generated by the lifts of the chains $\{e_i^j\}$. If $\epsilon : \pi \to O(n)$ be an orthogonal representation one obtain a chain complex of real vector spaces $\mathcal{R}^{\bar{p}}(X, \epsilon) = \mathcal{R}^{\bar{p}}(\tilde{X}) \otimes_{\mathbb{R}\pi} \mathbb{R}^n$ with a preferred basis given by $\{e_i^j \otimes x_k\}$ where x_k is an orthonormal basis for $\mathbb{R}^n$.

The intersection R-torsion of X is then defined to be the torsion of the chain complex $\mathcal{R}^{\bar{p}}(X, \epsilon)$, provided a preferred basis in homology is chosen. Dar [12] proved that the intersection R-torsion is a combinatorial invariant and independent of the stratification.

3 Intersection R-torsion of finite cone

In this section we restrict ourself to the finite cone. Let $X = C(Y)$ be a finite cone with $dim X = n$, where the cross section Y is a closed manifold. We will also write $X = w * Y$ with w the cone tip.

If $\sigma = [a_0, \cdots, a_p]$ is an oriented simplex of Y, then $[w, \sigma] = [w, a_0, \cdots, a_p] = w * \sigma$ is an oriented simplex of $w * Y$. Similarly, if $\eta = \sum n_i \sigma_i$ is a p-chain of Y, then $[w, \eta] = \sum n_i [w, \sigma_i]$, and

$$\partial[w, \eta] = \begin{cases} \eta - \omega & \dim \eta = 0 \\ \eta - [\omega, \partial\eta] & \dim \eta > 0 \end{cases}$$

We have the following results:

Lemma 3.1 *If Z_p is a p-cycle of X for $p \geq 1$, then $Z_p = \partial[w, C_p]$ for some p-chain C_p of Y.*

Proof: Write $Z_p = C_p + [w, D_{p-1}]$, where C_p and D_{p-1} are both carried by Y. Then by the above observation and using that Z_p is closed, we have $Z_p = \partial[w, C_p]$. ■

Of course, this is compatible with the well known fact that

$$H_p(X) = \begin{cases} 0 & p \geq 1 \\ Z & p = 0 \end{cases}$$

Now let $\bar{p}$ be a perversity. Since X has only strata of dimension n and 0, the intersection chains and homology will only depend on p_n. In fact, we have

Lemma 3.2 *The intersection chains of X are given by*

$$IC_i^{\bar{p}}(X) = \begin{cases} C_i(Y), & i < n - p_n, \\ \{\ \xi \in C_i(X) \mid \partial\xi \in C_{i-1}(Y)\ \}, & i = n - p_n, \\ C_i(X), & i > n - p_n. \end{cases}$$

From now on we will suppress the superscript $\bar{p}$ here.

Theorem 3.3 *Let $\Delta^{(c)}$ denote the combinatorial Laplacian of the cross section Y. Then*

$$\ln I\tau^{\bar{p}}(X) = \sum_{p=0}^{n-p_n-1} (-1)^{p+1} p \ln\det \Delta_p^{(c)} + (n - p_n) \sum_{p=n-p_n}^{n-1} (-1)^{p+1} \ln\det \Delta_p^{(c)}.$$

Proof: The intersection R-torsion is defined in terms of the chain complex

$$\cdots \longrightarrow IC_{p+1}(X) \longrightarrow IC_p(X) \longrightarrow IC_{p-1}(X) \longrightarrow \cdots . \tag{3.3}$$

We examine the terms of this complex according to their degrees.

Case 0. $p = n$

In this case, $IC_n(X) = C_n(X)$, so we only need to consider the normal chains of X.

Let $c_{n-1}(Y) = \{\sigma_1^{n-1}(Y), \cdots, \sigma_{i_{n-1}}^{n-1}(Y)\}$ be the preferred basis of $(n-1)$-chains of Y. Then $\{[w, c_{n-1}(Y)]\}$ is the preferred basis of $C_n(X)$. Choose a basis $b_p(Y) = \{b_1^p(Y), \cdots, b_{k_p}^p(Y)\}$ for $B_p(Y)$, and their lifts $\tilde{b}_p(Y) = \{\tilde{b}_1^p(Y), \cdots, \tilde{b}_{k_p}^p(Y)\}$, and $h_p(Y) = \{h_1^p(Y), \cdots, h_{j_p}^p(Y)\}$ the basis for $H_p(Y)$. Then by the fact that $B_n(X) = Z_n(X) = 0$ and $C_n(X) = \partial[w, C_{n-1}(X)]$, we can choose basis for $\tilde{b}_{n-1}(X)$ as follows:

$$\tilde{b}_{n-1}(X) = \{[w, b_{n-1}(Y)], [w, h_{n-1}(Y)], [w, \tilde{b}_{n-2}(Y)]\}$$

So the transition matrix D_n is:

$$D_n = \Big[\frac{[w, b_{n-1}(Y)], [w, h_{n-1}(Y)], [w, \tilde{b}_{n-2}(Y)]}{[\omega, C_{n-1}(Y)]}\Big]$$

which is $D_n = A_{n-1}$ where A_{n-1} denotes the corresponding transition matrix for Y.

Case 1. $n - p_n < p < n$

In this case, $IC_p(X) = C_p(X)$, so we only need to consider the normal chains of X.

Let $c_p(Y) = \{\sigma_1^p(Y), \cdots, \sigma_{i_p}^p(Y)\}$ be the preferred basis of p-chains of Y. Then $\{c_p(Y), [w, c_p(Y)]\}$ is the preferred basis of $C_p(X)$. Choose a basis $b_p(Y) = \{b_1^p(Y), \cdots, b_{k_p}^p(Y)\}$ for $B_p(Y)$, and their lifts $\tilde{b}_p(Y) = \{\tilde{b}_1^p(Y), \cdots, \tilde{b}_{k_p}^p(Y)\}$, and $h_p(Y) = \{h_1^p(Y), \cdots, h_{j_p}^p(Y)\}$ the basis for $H_p(Y)$. Then by the fact that $B_p(X) = Z_p(X) = \partial[w, C_p(X)]$, we can choose a basis for $B_p(X)$ as follows:

$$\begin{aligned} b_p(X) &= \{\partial[w, b_p(Y)], \partial[w, \tilde{b}_{p-1}(Y)], \partial[w, h_p(Y)]\} \\ &= \{b_p(Y), h_p(Y), \tilde{b}_{p-1}(Y) - [w, b_{p-1}(Y)]\} \end{aligned}$$

and a basis for $\tilde{b}_{p-1}(X)$:

$$\tilde{b}_{p-1}(X) = \{[w, b_{p-1}(Y)], [w, h_{p-1}(Y)], [w, \tilde{b}_{p-2}(Y)]\}$$

So the transition matrix D_p is:

$$D_p = \Big[\frac{b_p(Y), h_p(Y), \tilde{b}_{p-1}(Y), [\omega, b_{p-1}(Y)], [\omega, h_{p-1}(Y)], [\omega, \tilde{b}_{p-2}(Y)]}{C_p(Y), [\omega, C_{p-1}(Y)]}\Big]$$

which is

$$D_p = \begin{bmatrix} A_p & 0 \\ 0 & A_{p-1} \end{bmatrix} \tag{3.4}$$

Case 2. $p = n - p_n$.

In this case, we still have $IH_p(X) = H_p(X) = 0$. By Lemma ??, $IC_p(X) = \{\ \eta \in C_i(X) \mid \partial\eta \in C_{i-1}(Y)\ \}$.

For $\eta \in IC_p(X)$, write $\eta = C_p(Y) + [w, D_{p-1}(Y)]$. Then $\partial\eta = \partial C_p(Y) + D_{p-1}(Y) - [w, \partial D_{p-1}(Y)]$. Thus we must have $\partial D_{p-1}(Y) = 0$ for $\partial\eta \in C_{i-1}(Y)$, which implies that $D_{p-1}(Y) \in B_{p-1}(Y) \oplus H_{p-1}(Y)$.

Thus,

$$\begin{aligned} \partial(IC_{\frac{m+1}{2}+1)}(X) &= \partial(C_p(Y)) \oplus \partial[w, B_{p-1}(Y) \oplus H_{p-1}(Y)] \\ &= \partial(C_p(Y)) \oplus B_{p-1}(Y) \oplus H_{p-1}(Y) \\ &= B_{p-1}(Y) \oplus H_{p-1}(Y) \end{aligned}$$

Hence we can take $\{[w, b_{p-1}(Y)], [w, h_{p-1}(Y)]\}$ as basis of $\tilde{B}_{p-1}(X)$.

The fact that $IH_p(X) = H_p(X) = 0$ implies

$$IC_p(X) = B_p(X) \oplus \tilde{B}_{p-1}(X).$$

Then the transition matrix D_p is

$$D_p = \Big[\frac{b_p(Y), h_p(Y), \tilde{b}_{p-1}(Y), [\omega, b_{p-1}(Y)], [\omega, h_{p-1}(Y)]}{C_p(Y), [\omega, b_{p-1}(Y)], [\omega, h_{p-1}(Y)]}\Big]$$

which is

$$D_p = \begin{bmatrix} A_p & 0 \\ 0 & I \end{bmatrix} \tag{3.5}$$

Case 3: $p = n - p_n - 1$

In this case

$$IC_p(X) = C_p(Y)$$
$$IH_p(X) = \mathrm{Im}\big(H_p(Y) \to H_p(X)\big) = 0$$

Consider the following sequence:

$$\cdots \xrightarrow{\partial} IC_{p+1}(X) \xrightarrow{\partial} IC_p(X) \xrightarrow{\partial} IC_{p-1}(X) \xrightarrow{\partial} \cdots .$$

Then as before,

$$\partial(IC_{p+1}(X)) = B_p(Y) \oplus H_p(Y),$$

and

$$\partial[IC_p(X)] = \partial[C_p(Y)] = B_{p-1}.$$

Thus the transition matrix is:

$$D_p = \Big[\frac{b_p(Y), H_p(Y), \tilde{b}_{p-1}(Y)}{C_p(Y)}\Big] = A_p \tag{3.6}$$

Case 4: $p < n - p_n$

In this case, it is easy to see that $D_p = A_p$

Combining the above results, we have:

$$\begin{aligned}
\tau(IC) &= \prod_{p=0}^{n} (D_p)^{(-1)^p} \\
&= \prod_{p=0}^{n-p_n} (A_p)^{(-1)^p} \cdot \prod_{p=n-p_n+1}^{n-1} (A_p \cdot A_{p-1})^{(-1)^p} \cdot (A_{n-1})^{(-1)^n} \\
&= \prod_{p=0}^{n-p_n-1} (A_p)^{(-1)^p}
\end{aligned} \tag{3.7}$$

Thus,

$$\begin{aligned}
\ln I\tau^{\bar{p}}(X) = \ln \tau(IC) &= \sum_{p=0}^{n-p_n-1} (-1)^p \ln A_p \\
&= \sum_{p=0}^{n-p_n-1} (-1)^{p+1} p \ln\det \Delta_p^{(c)} + (n - p_n) \sum_{p=n-p_n}^{n-1} (-1)^{p+1} \ln\det \Delta_p^{(c)}.
\end{aligned} \tag{3.8}$$

Here we have used the equation $\ln A_p = -\frac{1}{2} \sum_{k=p}^{n-1} (-1)^{k-p} \ln\det \Delta_k^{(c)}$ [22]. $\blacksquare$

4 An analytic analogue

Following Ray-Singer's idea of defining analytic torsion as a formal analog of the R-torsion on closed manifolds, we now study the formal analytic analog of the intersection R-torsion (1.2), which is intrinsic to the even dimensional cross section. That is, by replacing the combinational Laplacian by the Hodge Laplacian, we define an analytic invariant for an even dimensional closed manifold.

More precisely, let Y be an even dimensional closed manifold with $m = \dim Y$. Let p be an integer such that $0 \leq p \leq m-1$ (p corresponds to p_n which is determined by a given perversity). Given an orthogonal representation $\rho : \pi_1(Y) \longrightarrow O(N)$, one has an associated flat vector bundle E_ρ with compatible metric on Y. Let Δ_k be the Laplacian acting on differential k forms on Y with coefficients in E_ρ. Then we define

$$\ln T_p(Y,\rho) = \frac{1}{2}\left[\sum_{k=0}^{m-p}(-1)^{k+1}k\,\ln\det(\Delta_k) + (m-p)\sum_{k=m-p+1}^{m}(-1)^{k+1}\ln\det(\Delta_k)\right]. \tag{4.9}$$

For $p = 0$, which corresponds to the minimum perversity,

$$\ln T_0(Y,\rho) = \frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1}k\,\ln\det(\Delta_k) = 0$$

is the usual analytic torsion which is trivial for even dimensional manifolds. On the other hand, for $p = m-1$ corresponding to the maximum perversity,

$$\ln T_{m-1}(Y,\rho) = \frac{1}{2}\sum_{k=1}^{m}(-1)^{k+1}\ln\det(\Delta_k).$$

The more interesting cases are given by $p = \frac{m}{2}-1$ and $p = \frac{m}{2}$ corresponding to the lower and upper middle perversity, respectively. In these cases, we have

$$\begin{aligned}
\ln T_{\frac{m}{2}-1}(Y,\rho) &= \frac{1}{2}\left[\sum_{k=0}^{\frac{m}{2}+1}(-1)^{k+1}k\ln\det(\Delta_k) + (\frac{m}{2}+1)\sum_{k=\frac{m}{2}+2}^{m}(-1)^{k+1}\ln\det(\Delta_k)\right]\\
&= \frac{1}{2}\left[\sum_{k=0}^{\frac{m}{2}}(-1)^{k+1}k\ln\det(\Delta_k) + (\frac{m}{2}+1)\sum_{k=\frac{m}{2}+1}^{m}(-1)^{k+1}\ln\det(\Delta_k)\right]
\end{aligned}$$

and

$$\ln T_{\frac{m}{2}}(Y,\rho) = \frac{1}{2}\left[\sum_{k=0}^{\frac{m}{2}}(-1)^{k+1}k\ln\det(\Delta_k) + \frac{m}{2}\sum_{k=\frac{m}{2}+1}^{m}(-1)^{k+1}\ln\det(\Delta_k)\right].$$

When Y is oriented, we can actually use Poincare duality to write it in terms of the Laplacians on half of the degrees. For example, for $p = \frac{m}{2}-1$ corresponding to the lower middle perversity, we have

$$\ln T_{\frac{m}{2}-1}(Y,\rho) = \frac{1}{2}\left[\sum_{k=0}^{\frac{m}{2}-1}(-1)^{k+1}(k+\frac{m}{2}+1)\,\ln\det(\Delta_k) + (-1)^{\frac{m}{2}+1}\frac{m}{2}\ln\det(\Delta_{\frac{m}{2}})\right]. \tag{4.10}$$

To investigate what kind of invariant $\ln T_p(Y,\rho)$ defines, we now look at its variation under metric change. Let $g(u)$ be a family of Riemannian metrics on Y and $\Delta_k(u)$ the corresponding

Laplacian (when there is no ambiguity we will often write Δ_k instead of $\Delta_k(u)$. Let $\dot\star = d\star/du$ and $\alpha = \star^{-1}\dot\star$. Denote by $E_k(t) = e^{-t\Delta_k(u)}$ the heat kernel and let $E_k = E_k^{ex} + E_k^{ce} + E_k^h$ denote the Hodge decomposition of E_k into its exact, coexact and harmonic parts. We have the following result regarding the variation of $\ln T_p(Y,\rho)$.

Theorem 4.1 *The variation of* $\ln T_p(Y,\rho)$ *is given by*

$$\begin{aligned}\frac{d}{du}\ln T_p(Y,\rho) &= \frac{1}{2}\sum_{k=0}^{m-p-1}(-1)^{k+1}\mathrm{Tr}(P_{H^k}\alpha) + \frac{1}{2}\sum_{k=0}^{m-p-1}(-1)^{k+1}\mathrm{LIM}_{t\to 0}\mathrm{Tr}(e^{-t\Delta_k}\alpha)\\ &+ (-1)^{m-p+1}\frac{1}{2}\mathrm{LIM}_{t\to 0}\mathrm{Tr}(E_{m-p}^{ex}(t)\alpha),\end{aligned}$$

where P_{H^k} *denote the projection onto the cohomology* H^k *and* $\mathrm{LIM}_{t\to 0}\mathrm{Tr}(E_{m-p}^{ex}(t)\alpha)$ *denotes the constant term in the asymptotic expansion of* $\mathrm{Tr}(E_{m-p}^{ex}(t)\alpha)$.

Before we give the proof of our theorem, we need the following result from [4] (compare also with [22]) concerning the variation of heat kernel.

Theorem 4.2 (Cheeger) *The variation of the trace of the heat kernel* E_k *is given by*

$$\begin{aligned}\frac{d}{du}\mathrm{tr}(E_k(t)) &= -t[\mathrm{tr}\left(\Delta_{k+1}E_{k+1}^{ex}\alpha\right) - \mathrm{tr}\left(\Delta_k E_k^{ce}\alpha\right) + \mathrm{tr}\left(\Delta_k E_k^{ex}\alpha\right) - \mathrm{tr}\left(\Delta_{k-1}E_{k-1}^{ce}\alpha\right)]\\ &= t\frac{d}{dt}[\mathrm{tr}\left(E_{k+1}^{ex}\alpha\right) - \mathrm{tr}\left(E_k^{ce}\alpha\right) + \mathrm{tr}\left(E_k^{ex}\alpha\right) - \mathrm{tr}\left(E_{k-1}^{ce}\alpha\right)].\end{aligned}$$

The following lemma is an immediate consequence of Cheeger's result.

Lemma 4.3 *For any integer* q, $0\le q\le m$, *we have*

$$\frac{\partial}{\partial u}\sum_{k=0}^{q}(-1)^k k\,\mathrm{tr}(E_k(t)) = t\frac{\partial}{\partial t}[\sum_{k=0}^{q}(-1)^k\mathrm{tr}(\underline{E}_k(t)\alpha)+(-1)^q q\,\mathrm{tr}(E_{q+1}^{ex}(t)\alpha)+(-1)^{q+1}(q+1)\mathrm{tr}(E_q^{ce}(t)\alpha)]. \tag{4.11}$$

Similarly, for any integer r, $0\le r\le m$,

$$\frac{\partial}{\partial u}\sum_{k=r}^{m}(-1)^k\mathrm{tr}(E_k(t)) = t\frac{\partial}{\partial t}[(-1)^r\mathrm{tr}(E_r^{ex}(t)\alpha) + (-1)^{r-1}\mathrm{tr}(E_{r-1}^{ce}(t)\alpha)]. \tag{4.12}$$

With these results at our disposal, we are now ready to prove the variational formula for our analytic invariant.

Proof of Theorem 4.1: Define for $\Re s$ sufficiently large

$$f(u,s) = \frac{1}{2}\big[\sum_{k=0}^{m-p}(-1)^k k\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t[\Delta_k+P_{H^k}]})\,dt + (m-p)\sum_{k=m-p+1}^{m}(-1)^k\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t[\Delta_k+P_{H^k}]})\,dt\big].$$

Then $f(u,s)$ has a meromorphic extension to the whole complex s-plane with a simple pole at $s=0$. Indeed, since

$$\mathrm{Tr}(e^{-t[\Delta_k+P_{H^k}]}) = \mathrm{Tr}(e^{-t\underline{\Delta}_k}) + e^{-t}\dim H^k,$$

we have

$$\mathrm{Res}_{s=0}f(u,s) = \frac{1}{2}\big[\sum_{k=0}^{m-p}(-1)^k k A_{m/2,k} + (m-p)\sum_{k=m-p+1}^{m}(-1)^k A_{m/2,k}\big],$$

where $A_{m/2,k}$ denotes the constant term in the asymptotic expansion of $\mathrm{Tr}(e^{-t\Delta_k})$. Now let

$$\tilde{f}(u,s) = f(u,s) - \Gamma(s)\mathrm{Res}_{s=0} f(u,s).$$

Then $\tilde{f}$ is holomorphic at $s=0$ and we have

$$\tilde{f}(u,0) = \ln T_p(Y,\rho).$$

Now, for $\Re s$ sufficiently large

$$\begin{aligned}
\frac{\partial}{\partial u}\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t[\Delta_k+P_{H^k}]})\,dt &= \int_0^\infty t^{s-1}\frac{\partial}{\partial u}\mathrm{Tr}(e^{-t[\Delta_k+P_{H^k}]})\,dt \\
&= \int_0^\infty t^{s-1}\frac{\partial}{\partial u}\mathrm{Tr}(e^{-t\Delta_k})\,dt
\end{aligned}$$

Hence, using (4.11), (4.12), we derive

$$\begin{aligned}
\frac{\partial}{\partial u} f(u,s) &= \frac{1}{2}\Big[\sum_{k=0}^{m-p}(-1)^k\int_0^\infty t^s\frac{\partial}{\partial t}\mathrm{Tr}(\underline{E}_k(t)\alpha)\,dt + (-1)^{m-p+1}\int_0^\infty t^s\frac{\partial}{\partial t}\mathrm{Tr}(E^{ce}_{m-p}(t)\alpha)\,dt\Big] \\
&= s\frac{1}{2}\Big[\sum_{k=0}^{m-p}(-1)^{k+1}\int_0^\infty t^{s-1}\mathrm{Tr}(\underline{E}_k(t)\alpha)\,dt + (-1)^{m-p}\int_0^\infty t^{s-1}\mathrm{Tr}(E^{ce}_{m-p}(t)\alpha)\,dt\Big] \\
&= s\frac{1}{2}\Big[\sum_{k=0}^{m-p-1}(-1)^{k+1}\int_0^\infty t^{s-1}\mathrm{Tr}(\underline{E}_k(t)\alpha)\,dt + (-1)^{m-p+1}\int_0^\infty t^{s-1}\mathrm{Tr}(E^{ex}_{m-p}(t)\alpha)\,dt\Big]
\end{aligned}$$

It follows then that

$$\begin{aligned}
\frac{\partial}{\partial u}\ln T_p(Y,\rho) &= \frac{1}{2}\sum_{k=0}^{m-p}(-1)^k\mathrm{Tr}(P_{H^k}\alpha) + \frac{1}{2}\sum_{k=0}^{m-p-1}(-1)^{k+1}\mathrm{LIM}_{t\to 0}\mathrm{Tr}(e^{-t\Delta_k}\alpha) \\
&+ (-1)^{m-p+1}\frac{1}{2}\mathrm{LIM}_{t\to 0}\mathrm{Tr}(E^{ex}_{m-p}(t)\alpha).
\end{aligned}$$

■

5 R-torsion of the Mayer-Vietoris sequences

Consider an $(m+1)$-dimensional Riemannian manifold X with isolated conical singularity. Thus, $X = C(Y)\cup M$, where M is a compact manifold with boundary and $\partial M = Y$. It is understood in this section that the collar neighborhoods of the boundaries of M and $C(Y)$ are extended so that they form an open cover of X. We assume that $m+1$ is odd.

As we mentioned, the general Mayer-Vietoris Principle reduces the torsion of X to that of $C(Y)$, M as well as the torsion of the Mayer-Vietoris sequence in the intersection cohomology. We now examine the torsion of the Mayer-Vietoris sequence.

We use the L^2-cohomology interpretation of the intersection cohomology in this setting [5]. The Mayer-Vietoris sequence goes

$$\cdots \longrightarrow H^q_{(2)}(Y) \xrightarrow{d^*} H^{q+1}_{(2)}(X) \longrightarrow H^{q+1}_{(2)}(M)\oplus H^{q+1}_{(2)}(C(Y)) \longrightarrow H^{q+1}_{(2)}(Y) \longrightarrow \cdots. \tag{5.13}$$

First, we have the following

Lemma 5.1 *For the Mayer-Vietoris long exact sequence in cohomology (5.13),*
a). its part for $q \le m/2$ splits into the following short exact sequences:

$$0 \longrightarrow H^q_{(2)}(X) \longrightarrow H^q_{(2)}(M) \oplus H^q_{(2)}(C(Y)) \longrightarrow H^q_{(2)}(Y) \longrightarrow 0 \tag{5.14}$$

b). further,

$$0 \longrightarrow H^q_{(2)}(X) \longrightarrow H^q_{(2)}(M) \oplus H^q_{(2)}(C(Y)) \longrightarrow H_{(2)}(Y) \longrightarrow 0 \tag{5.15}$$

is a split short exact sequence.
c). the part of the Mayer-Vietoris sequence for $q > m/2$ is naturally isomorphic to the truncated exact sequence for the pair (M, Y):

$$H^{m/2}(Y) \longrightarrow H^{m/2+1}(M, Y) \longrightarrow H^{m/2+1}(M) \longrightarrow H^{m/2+1}(Y) \longrightarrow \cdots \longrightarrow H^m(Y). \tag{5.16}$$

Proof: For a). we only need to show that, when $q \le m/2$, $Im(d^*) = 0$. Let ρ_1, ρ_2 be a partition of unity subordinate to the open cover of X by $M, C(Y)$. That is, $\rho_1, \rho_2 \in C^\infty(X)$, $0 \le \rho_1, \rho_2 \le 1$, $\rho_1 + \rho_2 = 1$ and $\operatorname{supp} \rho_1 \subset M$, $\operatorname{supp} \rho_2 \subset C(Y)$. Then, for a closed q-form on Y,

$$d^*[w] = \begin{cases} [-d(\rho_2 w)] & on \quad M, \\ [d(\rho_1 w)] & on \quad C(Y). \end{cases}$$

Here w is extended trivially along radial directions hence defines a q-form in a collared neighborhood of Y in X. In fact, $d^*[w]$ is supported in this collared neighborhood and, interpreted properly, either $[-d(\rho_2 w)]$ or $[d(\rho_1 w)]$ defines $d^*[w]$. Now, $d^*[w] = [-d(\rho_2 w)]$. By the result of [5], for $q \le m/2$, w defines an L^2 form on $C(Y)$. This shows that $d^*[w]$ is exact in L^2 cohomology. Hence $d^*[w] = 0$.

The statement b). is clear since these are short exact sequences of vector spaces. They can also be seen directly as follows. We show that the composition $p\, i^*$ in the follwoing diagram

$$\begin{array}{c} H^q_{(2)}(X) \xrightarrow{i^*} H^q(M) \oplus H^q_{(2)}(C(Y)) \longrightarrow H^q(Y) \\ \downarrow p \\ H^q(M) \end{array}$$

is an isomorphism. Here p is the projection onto the first factor. Indeed, for any $w \in H^q_{(2)}(X)$, $p\, i^* w = p\,(i^*_M w, i^*_{C(Y)} w) = i^*_M w$. If $i^*_M w$ is an exact form, $i^*_M w = d\eta_2$ then $i^*_Y i^*_M w = i^*_Y(d\eta_2) = d(i^*_Y \eta_2)$ is exact on Y. By [5], for $q \le m/2$, $i^*_Y \eta_2$ defines an L^2 form on $C(Y)$. Since the cohomology class of a closed form on $C(Y)$ is uniquely determined by its restriction on Y [5], we see that $i^*_{C(Y)}(w)$ is exact. It follows then that $i^*(w) = (i^*_M w, i^*_{C^*_{0,1}(N)} w)$ is exact. Namely $[i^* w] = 0$ on $H^*_2(M) \oplus H^*_2(C^*_{0,1}(N))$. So $[w] = 0$ on $H^q_{(2)}(X)$ by the injectivity of the short exact sequence. This shows that $p\, i^*$ is injective.

For the surjectivity, take $\eta \in H^q_{(2)}(M)$. Let $\xi = i^*_Y(\eta) \in H^q_{(2)}(Y)$. Then ξ extends to an L^2 form on $C(Y)$ which is cohomologous with the restriction of η in a collared neighborhood of Y. It follows that (η, ξ) is the image of some element of $H^q_{(2)}(X)$, say w. then $p\, i^*(w) = \eta$

Part c). follows from the natural isomorphisms $H^q_{(2)}(X) \cong H^q(M, Y)$, $H_{(2)}(C(Y)) \cong 0$ for $q > m/2$ [5].

∎

Lemma 5.2 *For a split short exact sequence*

$$0 \longrightarrow V_1 \xrightarrow{i} V_2 \xrightarrow{p} V_3 \longrightarrow 0$$

with preferred bases c_1, c_2, c_3, its R-torsion is determined by $i(c_1), j(c_3)$ and c_2, where j is an homomorphism from V_3 to V_2 such that $pj = \mathrm{id}$. In fact, the R-torsion is given by

$$|[i(c_1)j(c_3)/c_2]|$$

Proof: We choose $b_1 = 0$, $b_2 = i(c_1)$, and $b_3 = c_3$ and set $\tilde{b}_1 = c_1$, $\tilde{b}_2 = j(c_3)$ and $\tilde{b}_3 = 0$. The lemma follows. ■

A split short exact sequence can be written as

$$0 \longrightarrow V_1 \xrightarrow{i} V_1 \oplus V_3 \xrightarrow{p} V_3 \longrightarrow 0,$$

where i is not necessarily the natural inclusion, nor p the natural projection.

Lemma 5.3 *For a split short exact sequence*

$$0 \longrightarrow V_1 \xrightarrow{i} V_1 \oplus V_3 \xrightarrow{p} V_3 \longrightarrow 0$$

with preferred bases $c_1, c_1 \oplus c_3, c_3$, consider the natural projection $p_1 : V_1 \oplus V_3 \longrightarrow V_1$ onto the first factor and the natural inclusion $i_2 : V_3 \longrightarrow V_1 \oplus V_3$ of the second factor. If $p_1 i : V_1 \longrightarrow V_1$ is an isometry with respect to the inner product induced by the preferred basis c_1 and $pi_2 = \mathrm{id} : V_3 \longrightarrow V_3$, then the R-torsion of the short exact sequence is trivial.

Proof: Using the lemma above we just need to compare the basis $i(c_1) \oplus c_3$ with $c_1 \oplus c_3$. Since pi_1 is an isometry, we might as well replace $c_1 \oplus c_3$ with $pi_2(c_1) \oplus c_3$. Then clearly, the transition matrix from $i(c_1) \oplus c_3$ to $pi_2(c_1) \oplus c_3$ is an upper triangular matrix with all diagonal entries one. The lemma follows. ■

Combining the above results, we obtain the main result of this section on the R-torsion of the Mayer-Vietoris sequence.

Theorem 5.4 *Assume that the Witt condition $H^{\frac{m}{2}}(Y) = 0$ holds. Then the R-torsion of the Mayer-Vietoris sequence in intersection cohomology*

$$\cdots \longrightarrow IH^q_{(2)}(Y) \longrightarrow IH^{q+1}_{(2)}(X) \longrightarrow IH^{q+1}_{(2)}(M) \oplus IH^{q+1}_{(2)}(C(Y)) \longrightarrow IH^{q+1}_{(2)}(Y) \longrightarrow \cdots$$

is equal to the R-torsion of the truncated exact sequence of the pair (M, Y)

$$0 \longrightarrow H^{\frac{m}{2}+1}(M, Y) \longrightarrow H^{\frac{m}{2}+1}(M) \longrightarrow H^{\frac{m}{2}+1}(Y) \longrightarrow H^{\frac{m}{2}+2}(M, Y) \longrightarrow \cdots$$

References

[1] J.-M. Bismut and W. Zhang. An extension of a theorem by Cheeger and Müller. *Astérisque*, 205, 1992.

[2] J. Brüning, X. Ma. An anomaly formula for Ray-Singer metrics on manifolds with boundary. *Geom. Funct. Anal.* 16 (2006), no. 4, 767–837.

[3] D. Burghelea, L. Friedlander, T. Kappeler, and P. McDonald. Analytic and Reidemeister Torsion for Representations in Finite Type Hilbert Modules. *Geom. Funct. Anal.*, 6 (1996), pp. 751-859.

[4] J. Cheeger. Analytic torsion and the heat equation. *Ann. Math.* 109 (1979) 259-322.

[5] J. Cheeger. Spectral geometry of singular Riemannian spaces. *J. Diff. Geom.* 18, 575-657 (1983).

[6] J. Cheeger. η-invariants, the adiabatic approximation and conical singularities. I. The adiabatic approximation. *J. Differential Geom.* 26 (1987), no. 1, 175–221.

[7] X. Dai. Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. *J. Amer. Math. Soc.*, 4:265–321, 1991.

[8] X. Dai. APS boundary conditions, eta invariants and adiabatic limits *Trans. AMS*, 354, pp. 107-122

[9] X. Dai, H. Fang. Analytic torsion and R-torsion for manifolds with boundary. *Asian J. Math.* 4 (2000), no. 3, 695–714.

[10] X. Dai, D. Freed. Eta invariants and determinant lines. *J. Math. Phys.*, 35(1994), 5155-5194.

[11] X. Dai, D. Freed. Invariants eta et droites determinants. *C. R. Acad. Sci., Paris, Series I*, 320(1995), 585-591.

[12] A. Dar. Intersection R-torsion and analytic torsion for pseudomanifolds. *Math. Z.* 194 (1987), no. 2, 193–216.

[13] W. Franz. Über die Torsion einer überdeckrung. *J. Reine Angew. Math.*, 173 (1935), pp. 245-254

[14] M. Goresky, R. MacPherson. Intersection homology theory. *Topology* 19 (1980), no. 2, 135–162.

[15] M. Goresky, R. MacPherson. Intersection homology. II. *Invent. Math.* 72 (1983), no. 1, 77–129.

[16] Frances Kirwan. *An Introduction to Intersection Homology Theory.* Pitman Reasearch Notes in Mathematics Series., 187.

[17] J. Lott and M. Rothenberg. Analytic Torsion for Group Actions. *J. Diff. Geom.*, 34 (1991), pp. 431-481.

[18] W. Lück. Analytic and Topological Torsion for Manifolds with Boundary and Symmetry. *J. Diff. Geom.*, 37 (1993), pp. 263-322.

[19] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.* 72 1966 358–426.

[20] W. Müller. Analytic Torsion and R-torison of Riemannian Manifolds. *Adv. in Math.*, 28 (1978), pp. 233-305.

[21] W. Müller. Analytic Torsion and R-torison for Unimoduler Representations. *J. Amer. Math. Soc.*, 6 (1993), pp. 31-34.

[22] D. B. Ray and I. M. Singer. R-Torsion and the Laplacian on Riemannian Manifolds. *Advances in Mathematics* 7., 145-210 (1971).

[23] K. Rediemeister. Homotopieringe und Linsenraüm. in *Hamburger Abhandl.*, 11, 1935, pp. 102-109

[24] Vladimir Turaev. *Introduction to Combinational Torsions.* Lectures in Mathematics., 2000.

[25] S. M. Vishik. Generalized Ray-Singer Conjecture I: a Manifold with a Smooth Boundary. *Comm. Math. Phys.*, 167 (1995), pp. 1-102.