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Eta invariant and holonomy: The even dimensional case

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ABSTRACT

In previous work, we introduced eta invariants for even dimensional manifolds. It plays the same role as the eta invariant of Atiyah–Patodi–Singer, which is for odd dimensional manifolds. It is associated to K^1 representatives on even dimensional manifolds and is closely related to the so called WZW theory in physics. In fact, it is an intrinsic interpretation of the Wess–Zumino term without passing to the bounding 3-manifold. Spectrally the eta invariant is defined on a finite cylinder, rather than on the manifold itself. Thus it is an interesting question to find an intrinsic spectral interpretation of this new invariant. We address this issue here using adiabatic limit technique. The general formulation relates the (mod $\mathbb{Z}$ reduction of) eta invariant for even dimensional manifolds with the holonomy of the determinant line bundle of a natural family of Dirac type operators. In this sense our result might be thought of as an even dimensional analogue of Witten's holonomy theorem proved by Bismut–Freed and Cheeger independently.

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1. Introduction

The η-invariant is introduced by Atiyah–Patodi–Singer in their seminal series of papers [1–3] as the correction term from the boundary for the index formula on a manifold with boundary. It is a spectral invariant associated to the natural geometric operator on the (boundary) manifold and it vanishes for even dimensional manifolds (in this case the corresponding manifold with boundary will have odd dimension). In our previous work [14], we introduced an invariant of eta type for even dimensional manifolds. It plays the same role as the eta invariant of Atiyah–Patodi–Singer.

Any elliptic differential operator on an odd dimensional closed manifold will have index zero. In this case, the appropriate index to consider is that of Toeplitz operators. This also fits perfectly with the interpretation of the index of Dirac operator on even dimensional manifolds as a pairing between the even K-group and K-homology. Thus in the odd dimensional case one considers the odd K-group and odd K-homology. For a closed manifold M, an element of $K^{-1}(M)$ can be represented by a differentiable map from M into the unitary group

$$g:\ M \longrightarrow U(N), \tag{1.1}$$

where N is a positive integer. As we mentioned the appropriate index pairing between the odd K-group and K-homology is given by that of the Toeplitz operator, defined as follows.

Consider $L^2(S(TM)\otimes E)$, the space of L^2 spinor fields[3] twisted by an auxilliary vector bundle E. It decomposes into an orthogonal direct sum

$$L^2(S(TM)\otimes E) = \bigoplus_{\lambda\in \mathrm{Spec}(D^E)} E_\lambda,$$

according to the eigenvalues λ of the Dirac operator D^E. The "Hardy space" will be

$$L^2_{\geq 0}(S(TM)\otimes E) = \bigoplus_{\lambda\geq 0} E_\lambda.$$

The corresponding orthogonal projection from $L^2(S(TM)\otimes E)$ to $L^2_{\geq 0}(S(TM)\otimes E)$ will be denoted by $P^E_{\geq 0}$.

The Toeplitz operator T^E_g is then defined as

$$T^E_g = P^E_{\geq 0} g P^E_{\geq 0} : L^2_{\geq 0}\left(S(TM)\otimes E\otimes \mathbf{C}^N\right) \longrightarrow L^2_{\geq 0}\left(S(TM)\otimes E\otimes \mathbf{C}^N\right). \tag{1.2}$$

[3] In this paper, for simplicity, we will generally assume that our manifolds are spin, although our discussion extends trivially to the case of Dirac type operators.

This is a Fredholm operator whose index is given by

$$\operatorname{ind} T_g^E = -\left\langle \widehat{A}(TM)\operatorname{ch}(E)\operatorname{ch}(g), [M] \right\rangle, \tag{1.3}$$

where $\operatorname{ch}(g)$ is the odd Chern character associated to g [4]. It is represented by the differential form (cf. [21, Chap. 1])

$$\operatorname{ch}(g) = \sum_{n=0}^{\frac{\dim M-1}{2}} \frac{n!}{(2n+1)!} \operatorname{Tr}\left[\left(g^{-1}dg\right)^{2n+1}\right].$$

In [14] we established an index theorem which generalizes (1.3) to the case where M is an odd dimensional spin manifold with boundary ∂M. The definition of the Toeplitz operator now uses Atiyah–Patodi–Singer boundary conditions on ∂M. The self-adjoint Atiyah–Patodi–Singer boundary conditions depend on choices of Lagrangian subspaces $L \subset \ker D^E_{\partial M}$. We will denote the corresponding boundary condition by $P_{\partial M}(L)$. The resulting Toeplitz operator will then be denoted by $T_g^E(L)$.

We recall the main result in [14] as follows.

Theorem 1.1. *The Toeplitz operator $T_g^E(L)$ is Fredholm with index given by*

$$\begin{aligned}\operatorname{ind} T_g^E(L) = &-\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \operatorname{ch}(g) \\ &- \bar{\eta}(\partial M, E, g) + \tau_\mu\left(gP_{\partial M}(L)g^{-1}, P_{\partial M}(L), \mathcal{P}_M\right).\end{aligned} \tag{1.4}$$

Here $\bar{\eta}(\partial M, E, g)$ denotes the invariant of η-type for even dimensional manifold ∂M and the K^1 representative g. The third term is an interesting new *integer* term, a triple Maslov index introduced in [15]. See [14] for details.

Remark. Our index formula is closely related to the so called WZW theory in physics [18]. When $\partial M = S^2$ or a compact Riemann surface and E is trivial, the local term in Theorem 1.1 is precisely the Wess–Zumino term, which allows an integer ambiguity, in the WZW theory. Thus, our eta invariant $\bar{\eta}(\partial M, g)$ gives an intrinsic interpretation of the Wess–Zumino term without passing to the bounding 3-manifold. In fact, for $\partial M = S^2$, it can be further reduced to a local term on S^2 by using Bott's periodicity, see [14, Remark 5.9].

The eta invariant $\bar{\eta}(\partial M, E, g)$ is defined on a finite cylinder $[0,1]\times\partial M$, rather than on ∂M itself. Thus it is an interesting question to find an intrinsic spectral interpretation of this new invariant. In this paper we answer this question by using the adiabatic limit technique. First, under invertibility assumptions, we give an explicit formula for our eta invariant in terms of a natural family of Dirac type operators on the manifold.

This family arises from the original Dirac operator by a perturbation involving the K^1 representative. The general formulation relates (the mod $\mathbb{Z}$ reduction of) the eta invariant for even dimensional manifolds with the holonomy of the determinant line bundle of this natural family of Dirac type operators. The work of [10] on the adiabatic limit of eta invariants for manifolds with boundary and that of [13] on Witten's holonomy theorem play an important role here.

This paper is organized as follows. In Section 2, we review the definition of the eta invariant for an even dimensional closed manifold introduced in [14]. In Section 3, we give an intrinsic spectral interpretation of the eta invariant under certain invertibility assumption. Section 4 deals with the general case. And we end with a conjecture and a few remarks in the last section.

Some of the results in this paper have been described in [12].

2. An invariant of η type for even dimensional manifolds

For an even dimensional closed manifold X (which may or may not be the boundary of an odd dimensional manifold) and a K^1 representative $g : X \to U(N)$, the eta invariant will be defined in terms of an eta invariant on the cylinder $[0,1] \times X$ with appropriate APS boundary conditions.

In general, for a compact manifold M with boundary ∂M with the product structure near the boundary, the Dirac operator D^E twisted by a Hermitian vector bundle $E \otimes \mathbf{C}^N$ decomposes near the boundary as

$$D^E = c\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x} + D^E_{\partial M}\right). \tag{2.1}$$

The APS projection $P_{\partial M}$ is an elliptic global boundary condition for D^E. However, for self-adjoint boundary conditions, we need to modify it by a Lagrangian subspace of $\ker D^E_{\partial M}$, namely, a subspace L of $\ker D^E_{\partial M}$ such that $c(\frac{\partial}{\partial x})L = L^\perp \cap (\ker D^E_{\partial M})$. Since ∂M bounds M, by the cobordism invariance of the index, such Lagrangian subspaces always exist.

The modified APS projection is then obtained by reducing the kernel part of the projection to the projection onto the Lagrangian subspace. More precisely, denote

$$L^2_+((S(TM) \otimes E \otimes \mathbf{C}^N)|_{\partial M}) = \bigoplus_{\lambda>0} E_\lambda(D^{E\otimes\mathbf{C}^N}_{\partial M}),$$

where λ runs over the positive eigenvalues of $D^{E\otimes\mathbf{C}^N}_{\partial M}$. Denote by $P_{\partial M}$ the orthogonal projection from $L^2((S(TM) \otimes E \otimes \mathbf{C}^N)|_{\partial M})$ to $L^2_+((S(TM) \otimes E \otimes \mathbf{C}^N)|_{\partial M})$. Let $P_{\partial M}(L)$ denote the orthogonal projection operator from $L^2((S(TM) \otimes E \otimes \mathbf{C}^N)|_{\partial M})$ to $L^2_+((S(TM) \otimes E \otimes \mathbf{C}^N)|_{\partial M}) \oplus L$:

$$P_{\partial M}(L) = P_{\partial M} + P_L, \tag{2.2}$$

where P_L denotes the orthogonal projection from $L^2((S(TM) \otimes E \otimes \mathbf{C}^N)|_{\partial M})$ to L.

The pair $(D^E, P^E_{\partial M}(L))$ forms a self-adjoint elliptic boundary problem, and $P_{\partial M}(L)$ is called an Atiyah–Patodi–Singer boundary condition associated to L. We will denote the corresponding elliptic self-adjoint operator by $D^E_{P_{\partial M}(L)}$.

In [14], we originally intend to consider the conjugated elliptic boundary value problem $D^E_{gP_{\partial M}(L)g^{-1}}$ (cf. [22]). However, the analysis turns out to be surprisingly subtle and difficult. To circumvent this difficulty, a perturbation of the original problem was constructed.

Let $\psi = \psi(x)$ be a cut off function which is identically 1 in the ϵ-tubular neighborhood of ∂M ($\epsilon > 0$ sufficiently small) and vanishes outside the 2ϵ-tubular neighborhood of ∂M. Consider the Dirac type operator

$$D^\psi = (1-\psi)D^E + \psi g D^E g^{-1}.$$

The motivation for considering this perturbation is that, near the boundary, the operator D^ψ is actually given by the conjugation of D^E, and therefore, the elliptic boundary problem $(D^\psi, gP_{\partial M}(L)g^{-1})$ is now the conjugation of the APS boundary problem $(D^E, P_{\partial M}(L))$, i.e., this is now effectively standard APS situation and we have a self-adjoint boundary value problem $(D^\psi, gP_{\partial M}(L)g^{-1})$ together with its associated self-adjoint elliptic operator $D^\psi_{gP_{\partial M}(L)g^{-1}}$.

The same thing can be said about the conjugation of D^ψ:

$$D^{\psi,g} = g^{-1}D^\psi g = D^E + (1-\psi)g^{-1}[D^E, g]. \tag{2.3}$$

We will in fact use $D^{\psi,g}$.

We are now ready to construct the eta invariant for even dimensional manifolds. Given an even dimensional closed spin manifold X, we consider the cylinder $[0,1] \times X$ with the product metric. Let $g: \ X \to U(N)$ be a map from X into the unitary group which extends trivially to the cylinder. Similarly, $E \to X$ is a Hermitian vector bundle which is also extended trivially to the cylinder. We assume that $\operatorname{ind} D^E_+ = 0$ on X which guarantees the existence of the Lagrangian subspaces L.

Consider the analog of $D^{\psi,g}$ as defined in (2.3), but now on the cylinder $[0,1] \times X$ and denote it by $D^{\psi,g}_{[0,1]}$. Here $\psi = \psi(x)$ is a cut off function on $[0,1]$ which is identically 1 for $0 \le x \le \epsilon$ ($\epsilon > 0$ sufficiently small) and vanishes when $1 - 2\epsilon \le x \le 1$. We equip it with the boundary condition $P_X(L)$ on one of the boundary components $\{0\} \times X$ and the boundary condition $\mathrm{Id} - g^{-1}P_X(L)g$ on the other boundary component $\{1\} \times X$ (note that the Lagrangian subspace L exists by our assumption of vanishing index). Then $(D^{\psi,g}_{[0,1]}, P_X(L), \mathrm{Id} - g^{-1}P_X(L)g)$ forms a self-adjoint elliptic boundary problem. For simplicity, we will still denote the corresponding elliptic self-adjoint operator by $D^{\psi,g}_{[0,1]}$.

Let $\eta(D^{\psi,g}_{[0,1]}, s)$ be the η-function of $D^{\psi,g}_{[0,1]}$ which, when $\mathrm{Re}(s) >> 0$, is defined by

$$\eta(D^{\psi,g}_{[0,1]}, s) = \sum_{\lambda\neq 0} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}, \tag{2.4}$$

where λ runs through the nonzero eigenvalues of $D^{\psi,g}_{[0,1]}$.

By [17,13], one knows that the η-function $\eta(D^{\psi,g}_{[0,1]}, s)$ admits a meromorphic extension to $\mathbf{C}$ with $s = 0$ a regular point (and it has only simple poles). One then defines, as in [1], the η-invariant of $D^{\psi,g}_{[0,1]}$ by $\eta(D^{\psi,g}_{[0,1]}) = \eta(D^{\psi,g}_{[0,1]}, 0)$, and the reduced η-invariant by

$$\bar{\eta}\left(D^{\psi,g}_{[0,1]}\right) = \frac{\dim\ker D^{\psi,g}_{[0,1]} + \eta\left(D^{\psi,g}_{[0,1]}\right)}{2}. \tag{2.5}$$

Definition 2.1. We define an invariant of η type for the Hermitian vector bundle E on the even dimensional manifold X (with vanishing index) and the K^1 representative g by

$$\bar{\eta}(X, E, g) = \bar{\eta}\left(D^{\psi,g}_{[0,1]}\right) - \mathrm{sf}\left\{D^{\psi,g}_{[0,1]}(s); 0 \leq s \leq 1\right\}, \tag{2.6}$$

where $D^{\psi,g}_{[0,1]}(s)$ is a path connecting $g^{-1}D^E g$ with $D^{\psi,g}_{[0,1]}$ defined by

$$D^{\psi,g}(s) = D^E + (1 - s\psi)g^{-1}[D^E, g]$$

on $[0,1]\times X$, with the boundary condition $P_X(L)$ on $\{0\}\times X$ and the boundary condition $\mathrm{Id} - g^{-1}P_X(L)g$ at $\{1\}\times X$.

It was shown in [14] that $\bar{\eta}(X, E, g)$ does not depend on the cut off function ψ.

3. An intrinsic spectral interpretation, the invertible case

The usefulness of the eta invariant of Atiyah–Patodi–Singer comes, at least partially, from the spectral nature of the invariant, i.e. that it is defined via the spectral data of the Dirac operator on the (odd dimensional) manifold. Our eta invariant for even dimensional manifold is defined via the eta invariant on the corresponding odd dimensional cylinder by imposing APS boundary conditions. Thus, it will be desirable to have a direct spectral interpretation in terms of the spectral data of the original manifold (and the K^1 representative). In this section we give such an interpretation under certain invertibility assumption. This invertibility condition will be removed in the next section.

The crucial point here is the following observation. As in the previous section, we can also consider the invariant $\bar{\eta}(D^{\psi,g}_{[0,a]})$, similarly constructed on a cylinder $[0,a]\times X$ of radial size $a > 0$. In what follows, we will use subscript X to indicate the Dirac operator on X. Namely D_X^E is the Dirac operator on X while D^E denotes the Dirac operator on the cylinder $[0,1]\times X$. Note that $[D_X^E, g] = c(dg)$ where dg is a $U(N)$-valued 1-form on X and $c(dg)$ denotes the Clifford multiplication by the one forms on the spinor part and natural unitary action of $U(N)$ on $\mathbf{C}^N$.

Lemma 3.1. *Assuming that* $\ker[D_X^E + s\, c(g^{-1}dg)] = 0, \quad \forall\, 0 \le s \le 1$, *then* $\bar\eta(D^{\psi,g}_{[0,a]})$, *and hence* $\bar\eta(X, E, g)$, *is independent of* a. *Without the invertibility assumption, the mod* $\mathbb{Z}$ *reduction of* $\bar\eta(X, E, g)$ *is independent of* a.

Proof. The basic idea comes from [14]; compare also with [17, Proposition 2.16] and [11, Theorem 3.2].

First of all, without the invertibility assumption, we work with mod $\mathbb{Z}$ and hence can disregard the spectral flow term. Thus, it suffices to show that $\bar\eta\left(D^{\psi,g}_{[0,a]}\right)$ mod $\mathbb{Z}$ is independent of $a > 0$. Now choose a diffeomorphism $\varphi_a : \ [0,1] \to [0,a]$ such that $\varphi_a(x) = x$, $x \in [0,\epsilon]$ and $\varphi_a(x) = x + a - 1$, $x \in [1-\epsilon, 1]$. Then φ_a extends trivially to a diffeomorphism from $[0,1]\times X$ to $[0,a]\times X$.

Such a φ_a defines an isomorphism φ_a^* between spinors fields (with values in E) on $[0,a]\times X$ and spinors fields (with values in E) on $[0,1]\times X$. Clearly

$$\bar\eta(D^{\psi,g}_{[0,a]}) = \bar\eta(\varphi_a^* D^{\psi,g}_{[0,a]} (\varphi_a^*)^{-1}).$$

But $D^{\psi,g,a}_{[0,1]} = \varphi_a^* D^{\psi,g}_{[0,a]}(\varphi_a^*)^{-1} = \widetilde{D}^E + (1-\psi\circ\varphi_a)c(g^{-1}dg)$, where $\widetilde{D}^E$ is the Dirac operator on $[0,1]\times X$, but with the metric $(\varphi_a'(x)dx)^2 + g_X$. By the product structure near the boundaries, the variation of $\bar\eta(D^{\psi,g,a}_{[0,1]})$ (mod $\mathbb{Z}$) is given by a local formula independent of the boundary conditions. More precisely, it is given by the integration over $[0,1]\times X$ of

$$\mathrm{LIM}_{t\to 0}\, t^{1/2}\, \mathrm{tr}\left(\frac{\partial}{\partial a} D^{\psi,g,a} e^{-t(D^{\psi,g,a})^2}\right),$$

where we have dropped the subscript $[0,1]$ on the operator to indicate that we take the heat kernel, say, over $\mathbb{R}\times X$, and $\mathrm{LIM}_{t\to 0}$ means taking the constant term in the asymptotic expansion.

This can be computed by now standard local index theory technique. Indeed, we introduce an auxilliary Grassman variable z. Then

$$t^{1/2}\, \mathrm{tr}\left(\frac{\partial}{\partial a} D^{\psi,g,a} e^{-t(D^{\psi,g,a})^2}\right) = \mathrm{tr}_z\left(e^{-t(D^{\psi,g,a})^2 + z t^{1/2}\frac{\partial}{\partial a} D^{\psi,g,a}}\right).$$

Here the subscript "z" means taking the z-part of the trace. Now

$$-t(D^{\psi,g,a})^2 + z t^{1/2}\frac{\partial}{\partial a} D^{\psi,g,a}$$

$$= -t\left[c(dx)\varphi_a'(x)\frac{\partial}{\partial x} + D_X + (1-\psi\circ\varphi_a(x))c(g^{-1}dg)\right]^2 + zt^{1/2}\frac{\partial}{\partial a}\varphi_a'(x)c(dx)\frac{\partial}{\partial x}$$
$$- zt^{1/2}\frac{\partial}{\partial a}(\psi\circ\varphi_a)(x)c(g^{-1}dg)$$
$$= t\left[\varphi_a'(x)\frac{\partial}{\partial x} + \frac{1}{2}zt^{-1/2}\frac{\partial}{\partial a}\ln\varphi_a'(x)c(dx)\right]^2 - tD_X^2 - t(1-\psi\circ\varphi_a(x))[D_X, c(g^{-1}dg)]$$
$$- t(1-\psi\circ\varphi_a(x))^2(c(g^{-1}dg))^2 + t\varphi_a'(x)(\psi\circ\varphi_a)'(x)c(dx)c(g^{-1}dg)$$
$$- zt^{1/2}\frac{\partial}{\partial a}(\psi\circ\varphi_a)(x)c(g^{-1}dg) - \frac{1}{2}zt^{1/2}\varphi_a'(x)\frac{\partial^2}{\partial x\partial a}\ln\varphi_a'(x)c(dx).$$

The singular term in the first term of the last equation can be cancelled out by conjugating with the exponential transform

$$e^{\frac{1}{2}zt^{-1/2}xc(dx)(\varphi_a'(x))^{-1}\frac{\partial}{\partial a}\ln\varphi_a'(x)} = 1 + \frac{1}{2}zt^{-1/2}xc(dx)(\varphi_a'(x))^{-1}\frac{\partial}{\partial a}\ln\varphi_a'(x).$$

Then we can apply the Getzler transform to see that the conjugation of $-t(D^{\psi,g,a})^2 + zt^{1/2}\frac{\partial}{\partial a}D^{\psi,g,a}$ by the exponential transform converges to

$$\left[\varphi_a'\frac{\partial}{\partial x} - \frac{z}{2}(\varphi_a'(x))^{-1}\frac{\partial^2}{\partial x\partial a}\ln\varphi_a'(x)xdx\right]^2 + \mathcal{H}_X - \psi\circ\varphi_a(1-\psi\circ\varphi_a)(g^{-1}dg)^2$$
$$+ \varphi_a'(\psi\circ\varphi_a)'dx\wedge g^{-1}dg - z\frac{\partial}{\partial a}(\psi\circ\varphi_a)g^{-1}dg - z\varphi_a'\frac{\partial^2}{\partial x\partial a}\ln\varphi_a'dx,$$

where $\mathcal{H}_X$ denotes the generalized harmonic oscillator on X. Computing the heat kernel and collecting the zdx terms, noticing that $\mathrm{tr}(g^{-1}dg)^{2k} = 0$ for all nonnegative integer k, we found that

$$\mathrm{LIM}_{t\to 0}\, t^{1/2}\,\mathrm{tr}\left(\frac{\partial}{\partial a}D^{\psi,g,a}e^{-t(D^{\psi,g,a})^2}\right) = 0.$$

This shows that $\bar{\eta}\left(D^{\psi,g}_{[0,a]}\right) \bmod \mathbb{Z}$ is independent of $a > 0$.

The invertibility assumption guarantees that the spectral flow term vanishes and the computation of the variation of the eta invariant is still the same. □

On the other hand, as we mentioned before,

Lemma 3.2. $\bar{\eta}(X, E, g)$, *is independent of the choice of the cut off function* ψ.

This is Proposition 5.1 of [14].

These two lemmas together show that

$$\bar{\eta}(X, E, g) = \lim_{a\to\infty}\bar{\eta}(D^{\psi,g}_{[0,a]}) \tag{3.1}$$

for any cut off function which may depend on a ((3.1) is to be interpreted as an equation mod $\mathbb{Z}$ without the invertibility assumption). This is exactly the adiabatic limit.

We now recall the setup and result from [10] on the adiabatic limit of eta invariant, which is an extension of [5] to manifolds with boundary. More precisely, let

$$Y \to M \xrightarrow{\pi} B \tag{3.2}$$

be a fibration where the fiber Y is closed but the base B may have nonempty boundary. Let g_B be a metric on B which is of the product type near the boundary ∂B. Now equip M with a submersion metric g,

$$g = \pi^* g_B + g_Y$$

so that g is also product near ∂M. This is equivalent to requiring g_Y to be independent of the normal variable near ∂B, given by the distance to ∂B.

The adiabatic metric g_x on M is given by

$$g_x = x^{-2}\pi^* g_B + g_Y, \tag{3.3}$$

where x is a positive parameter.

For simplicity we assume that M as well as the vertical tangent bundle $T^V M$ are spin. Associated to these data we have in particular the total Dirac operator D_x on M, the boundary Dirac operator $D_x^{\partial M}$ on ∂M, and the family of Dirac operators D_Y along the fibers. If the family D_Y is invertible, then, according to [5], the boundary Dirac operator $D_x^{\partial M}$ is also invertible for all small x, therefore the eta invariant of D_x with the APS boundary condition, $\eta(D_x)$, is well-defined. We have the following result from [10].

Theorem 3.3. *Consider the fibration $Y \to M \to B$ as above. Assume that the Dirac family along the fiber, D_Y, is invertible. Consider the total Dirac operator D_x on X with respect to the adiabatic metric g_x and let $\eta(D_x)$ denote the eta invariant of D_x with the APS boundary condition. Then the limit $\lim_{x\to 0} \bar{\eta}(D_x) = \lim_{x\to 0} \frac{1}{2}\eta(D_x)$ exists in $\mathbb{R}$ and*

$$\lim_{x\to 0} \bar{\eta}(D_x) = \int_B \hat{A}\left(\frac{R^B}{2\pi}\right) \wedge \tilde{\eta}, \tag{3.4}$$

where R^B is the curvature of g_B, $\hat{A}$ denotes the the $\hat{A}$-polynomial and $\tilde{\eta}$ is the η-form of Bismut and Cheeger [5].

Recall that the (unnormalized) η-form of Bismut–Cheeger, the $\hat{\eta}$ form, is defined as

$$\hat{\eta} = \begin{cases} \int_0^\infty \mathrm{tr}_s \left[\left(D_Y + \frac{c(T)}{4t} \right) e^{-B_t^2} \right] \frac{dt}{2t^{1/2}} & \text{if } \dim Y = 2l \\ \int_0^\infty \mathrm{tr}^{\mathrm{even}} \left[\left(D_Y + \frac{c(T)}{4t} \right) e^{-B_t^2} \right] \frac{dt}{2t^{1/2}} & \text{if } \dim Y = 2l-1 \end{cases}, \tag{3.5}$$

assuming that $\ker D_Y$ does define a vector bundle on B. Here B_t denotes the rescaled Bismut superconnection:

$$B_t = \tilde{\nabla}^u + t^{1/2} D_Y - \frac{c(T)}{4t^{1/2}}. \tag{3.6}$$

We normalize $\hat{\eta}$ by defining (note that from definition $\hat{\eta}$ is a differential form with only odd (even resp.) degrees when $\dim Y = 2l$ ($\dim Y = 2l-1$ resp.))

$$\tilde{\eta} = \begin{cases} \sum_{j=1}^{l} \frac{1}{(2\pi i)^j} [\hat{\eta}]_{2j-1} & \text{if } \dim Y = 2l \\ \sum_{j=0}^{l-1} \frac{1}{(2\pi i)^j} [\hat{\eta}]_{2j} & \text{if } \dim Y = 2l-1 \end{cases} \tag{3.7}$$

where $[\hat{\eta}]_k$ denotes the degree k component of the eta form $\hat{\eta}$.

We now turn to the intrinsic spectral interpretation of our eta invariant.

Theorem 3.4. *Under the assumption that* $\ker[D_X + s\, c(g^{-1}dg)] = 0$, $\forall\, 0 \le s \le 1$,

$$\bar{\eta}(X, E, g) = \frac{i}{4\pi} \int_0^1 \int_0^\infty \mathrm{tr}_s \left[c\left(g^{-1}dg\right) \left(D_X + s\, c\left(g^{-1}dg\right) \right) e^{-t\left(D_X + s\, c\left(g^{-1}dg\right)\right)^2} \right] dt\, ds.$$

Proof. We apply Theorem 3.3 to our current situation where $M = [0,1] \times X$ fibers over $[0,1]$ with the fibre X. The operator

$$D^{\psi,g}_{[0,1]} = D^E + (1-\psi) g^{-1}[D^E, g] = D^E + (1-\psi) c(g^{-1}dg)$$

is of Dirac type, and of product type near the boundaries. Hence the result still applies.

By the invertibility assumption there is no spectral flow contribution and hence, by (3.1), $\bar{\eta}(X, E, g)$ is given by the adiabatic limit formula.

The Dirac family along the fiber is $D_X + (1-\psi(x)) c(g^{-1}dg)$. The curvature of the Bismut superconnection is given by

$$B_t^2 = t \left[D_X + (1-\psi(x)) c\left(g^{-1}dg\right) \right]^2 - t^{1/2} \psi'(x) dx\, c\left(g^{-1}dg\right).$$

Thus,

$$\hat{\eta} = \frac{\psi'(x)dx}{2} \int_0^\infty \mathrm{tr}_s \left[c\left(g^{-1}dg\right) \left(D_X + (1-\psi(x))c\left(g^{-1}dg\right)\right) e^{-t\left(D_X+(1-\psi(x))c(g^{-1}dg)\right)^2} \right] dt.$$

Since $\tilde{\eta} = \frac{1}{2\pi i}\hat{\eta}$ here, the adiabatic limit formula in Theorem 3.3 gives

$$\begin{aligned}
&\bar{\eta}(X, E, g) \\
&= \int_0^1 \frac{\psi'(x)}{4\pi i} \int_0^\infty \mathrm{tr}_s \left[c\left(g^{-1}dg\right) \left(D_X + (1-\psi(x))c\left(g^{-1}dg\right)\right) e^{-t\left(D_X+(1-\psi(x))c(g^{-1}dg)\right)^2} \right] dtdx \\
&= \frac{i}{4\pi} \int_0^1 \int_0^\infty \mathrm{tr}_s \left[c\left(g^{-1}dg\right) \left(D_X + s\, c\left(g^{-1}dg\right)\right) e^{-t\left(D_X+s\, c(g^{-1}dg)\right)^2} \right] dt\, ds
\end{aligned}$$

as claimed. □

4. The noninvertible case

For a fibration over the circle, Witten's holonomy theorem [19,6,9] says that the adiabatic limit of the eta invariant of the total space is related to the holonomy of the determinant line bundle of the family operators along the fibers. Indeed, in the invertible case, namely the family operators along the fibers are invertible, there is an explicit formula for the adiabatic limit of the eta invariant in terms of the family operators, [6, (3.166)], [9, (1.56)], which states

$$\lim_{x\to 0} \bar{\eta}(D_x) = \frac{i}{4\pi} \int_{S^1} \int_0^\infty \mathrm{tr}_s \left[\left(\tilde{\nabla}^u D_Y\right) D_Y e^{-tD_Y^2} \right] dt. \tag{4.1}$$

Of course, the integrand in the formula (4.1) is just the degree one term of the Bismut–Cheeger η-form.

If one applies (4.1) to the family $s \in [0,1] \longrightarrow D_X + s\ c(g^{-1}dg)$, we would obtain Theorem 3.4. However, the family here is not periodic. Nevertheless, it is almost periodic in the sense that the operators at the endpoints differ by a conjugation. This leads us to the generalization to the general noninvertible case.

To deal with the noninvertible case, we make use of the framework and result of [13]. We first recall the setup of [13].

Suppose M is a compact odd dimensional Riemannian manifold with nonempty boundary. For simplicity, we assume M is spin so that one can consider the Dirac operator D_M (the same consideration can be adapted to Dirac type operators). Further, assume that the metric is of product type near the boundary. In order to consider eta invariant, one needs to impose boundary conditions and the self-adjoint APS boundary condition amounts to a "trivialization" of the graded kernel of the boundary Dirac operator $D_{\partial M}$. Taking this into consideration, the result of [13] says that the exponentiated eta invariant of D_M actually defines an element of the inverse determinant line of the boundary Dirac operator $D_{\partial M}$.

More precisely, let $K^+_{\partial M} \oplus K^-_{\partial M}$ be the (graded) kernel of $D_{\partial M}$ and $\mathrm{Det}^{-1}_{\partial M}$ the inverse determinant line of $D_{\partial M}$:

$$\mathrm{Det}^{-1}_{\partial M} = \Lambda^{\max} K^+_{\partial M} \hat{\otimes} [\Lambda^{\max} K^-_{\partial M}]^{-1}. \tag{4.2}$$

Here inverse denotes the dual.

A self-adjoint APS boundary condition is determined by a choice of isometry

$$T \ : K^+_{\partial M} \longrightarrow K^-_{\partial M}. \tag{4.3}$$

Let $\bar{\eta}(D_M(T))$ denote the reduced eta invariant of D_M with the self-adjoint APS boundary condition determined by T (cf. [13]). A basic result of [13] says that

$$\tau_M = e^{2\pi i \bar{\eta}(D_M(T))} (\det T)^{-1} \in \mathrm{Det}^{-1}_{\partial M} \tag{4.4}$$

is independent of T (and satisfies the laws of TQFT as well as a variation formula).

Relevant to our discussion here is Witten's holonomy theorem as formulated in this framework. Let $\pi : Y \to Z$ be a fibration whose typical fiber is a closed even dimensional manifold, and as before we assume that both Y and $T^V Y$ are spin for simplicity. Let $L \to Z$ denote the corresponding inverse determinant line bundle. It comes equipped with a (Quillen) metric and a natural unitary (Bismut–Freed) connection ∇. The curvature of ∇ is [6, Theorem 1.21]

$$\Omega^L = -2\pi i \left[\int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)}.$$

Given $\gamma : [0,1] \to Z$ a smooth path, let $Y_\gamma = \gamma^* Y$ denote the pullback of $\pi : Y \to Z$ via γ; then $\pi_\gamma : Y_\gamma \to [0,1]$ is a fibration, the induced fibration. Let $g_{[0,1]}$ denote an arbitrary metric on the unit interval and $g_{Y/Z}$ the metric on the vertical tangent bundle $T^V Y$. Define a family of metrics on Y_γ by the formula

$$g_\epsilon = \frac{g_{[0,1]}}{\epsilon^2} \oplus g_{Y/Z}, \qquad \epsilon \neq 0.$$

(We assume that γ is constant near the two endpoints so that g_ϵ is of the product type near the boundary.)

The construction above gives rise to a linear map

$$\tau_{Y_\gamma}(\epsilon) : L_{\gamma(0)} \longrightarrow L_{\gamma(1)}. \tag{4.5}$$

Theorem 4.1 *(Dai–Freed). The adiabatic limit $\tau_\gamma = \lim_{\epsilon\to 0} \tau_{Y_\gamma}(\epsilon)$ exists and gives the holonomy along γ of the Bismut–Freed connection.*

Consider now the fibration $\pi : \mathbb{R} \times X \longrightarrow \mathbb{R}$ given by the projection, with the family of Dirac type operators

$$s \in \mathbb{R} \to D_s = D_X + sc(g^{-1}dg). \tag{4.6}$$

Let $L \to \mathbb{R}$ be the inverse determinant line bundle with the Quillen metric and the Bismut–Freed connection. Denote by L_s the fiber of L at $s \in \mathbb{R}$. Since $D_1 = D_X + c(g^{-1}dg) = g^{-1}D_X g = g^{-1}D_0 g$, there is an isomorphism

$$g^{-1} : L_0 \simeq L_1 \tag{4.7}$$

determined by the isomorphism g^{-1} between the graded kernels $\ker D_0$ and $\ker D_1$. On the other hand, since $\mathbb{R}$ is one dimensional, any two monotonic paths from 0 to 1 are reparametrizations of each other. Hence there is a unique holonomy map

$$\tau_{0,1} : L_0 \to L_1. \tag{4.8}$$

Composing with the isomorphism (4.7) gives rise to a map

$$L_0 \to L_1 \simeq L_0 \tag{4.9}$$

which can then be identified with a complex number $\tau \in \mathbb{C}$. In fact, since both the holonomy map (4.8) and the isomorphism (4.7) are unitary maps, τ has modulus one.

We can now state the main result of this paper as follows.

Theorem 4.2. *We have*

$$\tau = e^{2\pi i\bar{\eta}(X,E,g)}.$$

Proof. By taking the exponential we discount the contribution from the spectral flow to our eta invariant. Thus we are only concerned with $\bar{\eta}(D^{\psi,g}_{[0,1]})$. By definition, $\bar{\eta}(D^{\psi,g}_{[0,1]})$ is the reduced eta invariant of

$$D^{\psi,g} = D^E + (1-\psi)g^{-1}[D^E, g]$$

on the cylinder $[0,1] \times X$ with the boundary condition $P_X(L)$ on one of the boundary components $\{0\} \times X$ and the boundary condition $\mathrm{Id} - g^{-1}P_X(L)g$ on the other boundary component $\{1\} \times X$, where L is a Lagrangian subspace of $\ker D_X$. Let $\ker D_X = K_X^+ \oplus K_X^-$ be its $\mathbb{Z}_2$ grading. Then an isometry

$$T: \ K_X^+ \to K_X^-$$

gives rise to a Lagrangian subspace, namely the graph of T. A little linear algebra shows that the boundary condition $\mathrm{Id} - g^{-1}P_X(L)g$ corresponds to the isometry

$$g^{-1}T^{-1}g: \ g^{-1}K_X^- \to g^{-1}K_X^+.$$

Hence, we have by the previous theorem and the definition (using the notation a-lim to denote the adiabatic limit)

$$\begin{aligned}\tau_{0,1} &= a\text{-}\lim e^{2\pi i\bar{\eta}(D^{\psi,g}_{[0,1]})}(\det T)^{-1}\det g^{-1}Tg \\ &= \lim_{a\to\infty} e^{2\pi i\bar{\eta}(D^{\psi,g}_{[0,a]})}(\det T)^{-1}\det g^{-1}Tg \\ &= e^{2\pi i\bar{\eta}(D^{\psi,g}_{[0,1]})}(\det T)^{-1}\det g^{-1}Tg.\end{aligned}$$

Therefore

$$\tau = e^{2\pi i\bar{\eta}(X,E,g)}$$

using the identification. □

Remark 4.3. Recall that in [14, Remarks 2.5 and 5.9], the η-invariant is used to give an intrinsic analytic interpretation of the Wess–Zumino term in the WZW theory. Now by Theorem 4.2, this term is further interpreted by using holonomy.

Remark 4.4. Theorem 4.1 gives an adiabatic limit formula for (reduced) eta invariants without invertibility assumption for one dimensional manifolds with boundary, namely the interval. Theorem 3.3, on the other hand, is such a formula with invertibility assumption, but for any compact manifold with boundary as the base. It will be interesting to have a general result combining these two. This will be addressed elsewhere.

5. Final remarks

We end this paper by recalling a conjecture from [14], and also by some remarks.

As we mentioned before, the eta type invariant $\bar{\eta}(X,E,g)$, which we introduced using a cut off function, is in fact independent of the cut off function. This leads naturally to the question of whether $\bar{\eta}(X,E,g)$ can actually be defined directly. The following conjecture is stated in [14] and [22].

Let $D^{[0,1]}$ be the Dirac operator on $[0,1]\times X$. We impose the boundary condition $gP_X(L)g^{-1}$ at $\{0\}\times X$ and the boundary condition $\mathrm{Id} - P_X(L)$ at $\{1\}\times X$.

Then $(D^{[0,1]}, gP_X(L)g^{-1}, \mathrm{Id} - P_X(L))$ forms a self-adjoint elliptic boundary problem. We denote the corresponding elliptic self-adjoint operator by $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$.

Let $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ be the η-function of $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$. By [15, Theorem 3.1], one knows that the η-function $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ admits a meromorphic extension to $\mathbf{C}$ with poles of order at most 2. One then defines, as in [15, Definition 3.2], the η-invariant of $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$, denoted by $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)})$, to be the constant term in the Laurent expansion of $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ at $s = 0$.

Let $\bar{\eta}(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)})$ be the associated reduced η-invariant.

Conjecture.

$$\bar{\eta}(X, E, g) = \bar{\eta}\left(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}\right).$$

We would also like to say a few words about the technical assumption that $\operatorname{ind} D^E_+ = 0$ imposed in order to define the eta invariant $\bar{\eta}(X, E, g)$. The assumption guarantees the existence of the Lagrangian subspaces L which are used in the boundary conditions. In the Toeplitz index theorem, this assumption is automatically satisfied since $X = \partial M$ is a boundary. In general, of course, it may not. However, if one is willing to overlook the integer contribution (as one often does in applications), this technical issue can be overcome by using another eta invariant, this time on $S^1 \times X$, as follows. Note that we now have no boundary, hence no need for boundary conditions!

Consider $S^1 \times X = [0, 1] \times X/\sim$ where $\sim$ is the equivalence relation that identifies $0 \times X$ with $1 \times X$. Let $E_g \to S^1 \times X$ be the vector bundle which is $E \otimes \mathbb{C}^N$ over $(0, 1) \times X$ and the transition from $0 \times X$ to $1 \times X$ is given by $g : X \to U(N)$. Denote by D_{E_g} the Dirac operator on $S^1 \times X$ twisted by E_g.

Proposition 5.1. [4] *One has*

$$\bar{\eta}(X, E, g) \equiv \bar{\eta}(D_{E_g}) \mod \mathbb{Z}.$$

This is an easy consequence of the so called gluing law for the eta invariant, see [8,7, 13]. An analog of this result in the noncommutative setting plays an important role in [20], which also contains an odd dimensional analog of [16].

Remark 5.2. An application of the Witten holonomy theorem [19,6,9] to the right hand side of the above formula leads to an analogous result as Theorem 4.2. However the family of operators here is not as explicit as in Theorem 4.2.

Remark 5.3. It might be interesting to note the duality that $\bar{\eta}(X, E, g)$ is a spectral invariant associated to a K^1-representative on an *even* dimensional manifold, while the usual Atiyah–Patodi–Singer η-invariant [1] is a spectral invariant associated to a K^0-representative on an *odd* dimensional manifold.

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