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# ETA INVARIANTS FOR MANIFOLD WITH BOUNDARY

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Dedicated to Krzysztof P. Wojciechowski on his 50th birthday

For a compact manifold with boundary, $M$, there are well known local boundary conditions that make the de Rham operator $A = d+\delta$ elliptic, namely the absolute and relative boundary conditions. We study the eta invariants of such elliptic boundary value problems under the metric deformation

$$g_\epsilon = \frac{dx^2}{x^2+\epsilon^2} + g,$$

where $x \in C^\infty(M)$ is, near the boundary, the geodesic distance to the boundary, and $g$ is a Riemannian metric on $M$ which is of product type near the boundary. Under certain acyclicity condition we show that when $M$ is odd dimensional

$$\eta(A_a) = \eta(A_r) = \eta_b(A_0),$$

where the subscript $a$ ($r$) indicates the absolute (relative) boundary condition, and $\eta_b(A_0)$ is the $b$-eta invariant of the limiting operator $A_0$. If $M$ is even dimensional then

$$\eta(A_a) = -\eta(A_r) = \frac{1}{2}\eta(A_{\partial M}).$$

Most of the analysis extends to analytic torsion, yielding

$$\log T_\epsilon(M,\rho) = \log {}^bT(\bar{M},\rho) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon$$

when $\dim M$ is odd, and

$$\log T_\epsilon(M,\rho) = \pm\frac{1}{2}\log T(\partial M,\rho) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon$$

when $\dim M$ is even. Here the sign $\pm$ depends on the choice of the boundary condition and $r_1, r_2$ vanishes at $\epsilon = 0$.

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## 1. Introduction

The eta invariant for a closed manifold is introduced by Atiyah-Patodi-Singer [1] as the boundary correction term in the index formula for manifold with boundary. It has found many significant applications in diverse fields of mathematics and physics. There are now various works generalizing it to manifolds with boundary. Using his cone method, Cheeger [5] introduced an eta invariant in the context of manifolds with conical singularity. In [9] Gilkey and Smith considered eta invariants for local boundary conditions. On the other hand, Douglas and Wojciechowski defined and studied eta invariants for generalized APS boundary conditions [8] (see also Bunke [4], Lesch-Wojciechowski [12], Müller [19]). Also, in the context of manifolds with asymptotically cylindrical end Melrose introduced a regularized eta invariant, the $b$-eta invariant [18]. Meanwhile Müller [19] introduced an $L^2$-eta invariant for manifolds with cylindrical end, which turns out to be the same as the $b$-eta invariant. We also note that in his work on Casson invariant [23], Taubes used the local boundary condition, while in the subsequent work by others it is the APS boundary condition that is used, see, for example, Yoshida [25]. Thus it is a natural and interesting question to clarify the relationships among the various generalizations.

In the very interesting work [19] Müller considered the relationship between the eta invariants for generalized APS boundary conditions and the $L^2$-eta (or the $b$-eta) invariants. Using scattering theory he showed that they are essentially the same. Earlier Douglas and Wojciechowski [8] have considered the situation where the boundary operator is invertible.

In this work we consider the relationship between the eta invariants for local boundary conditions and the $b$-eta (or $L^2$-eta) invariants for the (twisted) de Rham operator $A = d + \delta$. Under certain acyclicity condition we show that they are the same. Thus, at least for de Rham operators, the three generalizations of eta invariant to manifolds with boundary, using local boundary condition, generalized APS boundary condition, or $L^2$ condition, all coincide.

**Theorem 1.1.** *Let $M$ be a compact manifold with boundary and $\xi$ a flat unitary bundle over $M$ such that $H^*(\partial M, \xi) = 0$ and Im$(H^*(M, \partial M; \xi) \to H^*(M; \xi)) = 0$. Then if $\dim M$ is odd we have*

$$\eta(A_a) = \eta(A_r) = \eta_b(A_0),$$

*where subscript 'a' ('r') denotes the absolute (relative) boundary condition, and $A_0$ is the de Rham operator on the complete manifold obtained from*

*M by attaching an infinite half cylinder. On the other hand, if* $\dim M$ *is even, then*

$$\eta(A_a) = -\eta(A_r) = \frac{1}{2}\eta(A_{\partial M}).$$

The theorem is proved by considering the behavior of the eta invariant on the manifold with boundary under a metric degeneration in which the boundary is being 'pushed' to infinity. This is motivated by the work [14] of R. Mazzeo and R. Melrose who studied the behavior of eta invariant on a closed manifold under the metric deformation

$$g_\epsilon = \frac{dx^2}{x^2+\epsilon^2} + h, \tag{1.1}$$

where $x$ is a defining function for an embedded hypersurface. The limiting metric $g_0$ for (1.1) is an exact $b$-metric on the compact manifold with boundary obtained by cutting along the hypersurface. (An exact $b$-metric gives the manifold with boundary asymptotically cylindrical ends.) Under the assumption that the induced Dirac operator on (a double cover of) the hypersurface is invertible, Mazzeo and Melrose showed that

$$\eta(D_\epsilon) = \eta_b(D_0) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon + \tilde{\eta}(\epsilon), \tag{1.2}$$

where $D_\epsilon$ is the Dirac operator associated to the metric $g_\epsilon$, and $\eta_b(D_0)$ is the $b$-eta invariant of the ($b$-)Dirac operator $D_0$ associated to the metric $g_0$. Also, $r_1$, $r_2$ are smooth functions vanishing at $\epsilon = 0$. Finally, $\tilde{\eta}(\epsilon)$ is the signature of the small eigenvalues of $D_\epsilon$. This analysis is extended to analytic torsion by Hassell in [10].

We consider the corresponding case for manifold with boundary and let the boundary play the role of the hypersurface in [14]. We study the eta invariants of elliptic boundary value problems under the metric deformation (1.1). In this case a formula similar to (1.2) holds. We also show that the eta invariant does not change under this deformation.

Another source of inspiration comes from a paper of I. M. Singer, [22], and the subsequent work of Klimek-Wojciechowski [11]. Singer considers the difference of two eta invariants of Dirac operators with local boundary conditions and shows that the limit of the difference under stretching is the log determinant. The result is viewed as an analog of the identity that the difference of the indexes of the two elliptic boundary value problems for Dirac operators is given by the index of the Dirac operator on the boundary. This is given full mathematical treatment and generalized in [11].

The consideration in [22] is motivated by E. Witten's 'adiabatic limit'. For this and related topics we refer to Witten [24], Bismut-Freed [3], Bismut-Cheeger [2], Cheeger [5], Dai [6], Mazzeo-Melrose [13] and Singer [22].

The idea of studying the behavior of eta invariant under singular degeneration probably goes back to [5] where the particular case of conical degeneration is briefly discussed. Conical degeneration has been discussed to greater extent by R. Seeley and Singer, see Seeley [20] and Seeley-Singer [21].

Finally, let us mention that the same analysis applies to analytic torsions as well, see §3 for the statement of the result (Theorem 3.3).

## 2. Elliptic boundary value problem and eta invariant

Let $M$ be a compact manifold with boundary and $V$ a vector bundle over $M$. Let

$$P: \ C^\infty(M,V) \to C^\infty(M,V)$$

be a differential operator of order $d$ and $B$ a boundary condition. By $P_B$ we denote the realization of the boundary value problem $(P,B)$; namely, $P_B$ is the operator $P$ acting on the space of smooth sections verifying $B(\phi|_{\partial M}) = 0$. Let

$$\mathcal{C} = \{z : \ |\mathrm{Re}z| \le |\mathrm{Im}z|\}$$

be the closed $45^\circ$ cone about the imaginery axis in the complex plane. According to [9], when $(P,B)$ is elliptic with respect to $\mathcal{C}$, $P_B$ has discrete spectrum with finite multiplicity, all except finite of which lie inside $\mathcal{C}$. Let $\{\lambda_i\}$ denote the spectrum of $P_B$ where each spectral value is repeated according to its multiplicity. Gilkey-Smith defined

$$\eta(s,P,B) = \sum_{\mathrm{Re}\lambda_i>0} \lambda_i^{-s} - \sum_{\mathrm{Re}\lambda_i<0} (-\lambda_i)^{-s}$$

for Re$s \gg 0$ and showed that $\eta$ has a meromorphic extension to the whole complex plane with isolated simple poles. Unlike the case when $M$ is boundaryless, $s=0$ may be a simple pole here. However, the residue being a local homotopy invariant, one defines the eta invariant

$$\eta(P,B) = \text{finite part of } \eta(s,P,B) \text{ at } 0 = (s\eta(s,P,B))'|_{s=0}.$$

When $P_B$ has no eigenvalues lying inside $\mathcal{C}$, $\eta(s,P,B)$ can be expressed in terms of heat kernel as is the case when $\partial M = \emptyset$:

$$\eta(s,P,B) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{Tr}(P_B e^{-tP_B^2}) dt. \tag{2.3}$$

(When $P_B$ does have eigenvalues lying inside $\mathcal{C}$, one just have to treat them separately.) Here $P_B e^{-tP_B^2}$ is defined via functional calculus

$$P_B e^{-tP_B^2} = \frac{-1}{2\pi i} \int_\Gamma (P_B - \lambda)^{-1} \lambda e^{-t\lambda^2} d\lambda$$

with $\Gamma$ an appropriate contour.

Thus defined, this invariant behaves much like the usual eta for manifold without boundary. For example, one has the following variation formula [9]:

**Theorem 2.1.** *Let $(P_u, B)$ be a smooth one-parameter family which is elliptic with respect to $\mathcal{C}$. Then*

$$\frac{d}{du}\{Res_{s=0}\eta(s, P_u, B)\} = 0.$$

*Further, if no eigenvalues lie inside $\mathcal{C}$, then the variation of eta itself is given by a local formula*

$$\frac{d}{du}\eta(P_u, B) = \int_M a(y, P_u', P_u) dvol(y) + \int_{\partial M} a(x, P_u', P_u, B) dvol(x),$$

*where the $a(y, P_u', P_u)$ and $a(x, P_u', P_u, B)$ are the coefficients of $t^{-1/2}$ in the asymptotic expansion for $tr(P_u' e^{-tP_{u,B}^2})$.*

We now specialize to the de Rham operator. Let $M$ be an odd dimensional compact manifold with boundary and $g$ be a Riemannian metric on $M$ which is of product type near the boundary

$$g = dx^2 + g_{\partial M},$$

where $x$ is the geodesic distance to the boundary. Let $\xi \to M$ be the flat bundle associated to a representation $\rho : \pi_1(M) \to O(k)$. By de Rham operator we mean

$$A = d + \delta : C^\infty(M; \Lambda(M) \otimes \xi) \to C^\infty(M; \Lambda(M) \otimes \xi). \tag{2.4}$$

At the boundary we have the splitting

$$\Lambda(M) \otimes \xi|_{\partial M} = \Lambda(\partial M) \otimes \xi \oplus \Lambda(\partial M) \otimes \xi \tag{2.5}$$

corresponding to the decomposition for a form $\theta \in C^\infty(M; \Lambda(M) \otimes \xi)$:

$$\theta = \theta_1 + dx \wedge \theta_2, \ \theta_1, \, \theta_2 \in C^\infty(M; \Lambda(\partial M) \otimes \xi)$$

near the boundary. Define a linear map $\sigma$:

$$\sigma(\theta) = \theta_1 - dx \wedge \theta_2.$$

Then $\sigma$ is self adjoint and $\sigma^2 = 1$. Moreover the splitting (2.5) corresponds to the decomposition into the $\pm 1$-eigenspace of $\sigma$.

From the splitting we define two projections

$$P_a,\ P_r :\ C^\infty(\partial M; \Lambda(M) \otimes \xi|_{\partial M}) \to C^\infty(\partial M; \Lambda(\partial M) \otimes \xi),$$
$$P_a(\theta) = \theta_2|_{\partial M}; \qquad P_r(\theta) = \theta_1|_{\partial M}.$$

I.e., $P_a$ is the orthogonal projection onto the $-1$-eigenspace of $\sigma$ and $P_r$ the orthogonal projection onto the $+1$-eigenspace. Let $A_a$ (resp. $A_r$) be the de Rham operator equipped with the boundary condition $P_a$ (resp. $P_r$). Then $A_a$, $A_r$ are elliptic boundary value problems; in fact they are also self adjoint. Hence $\eta(A_a)$ and $\eta(A_r)$ can be defined and moreover, because of the self-adjointness, the eta functions are actually regular at 0.

## 3. Deforming eta invariant

Now, for $\epsilon$ a positive parameter, consider the family of metrics

$$g_\epsilon = \frac{dx^2}{x^2 + \epsilon^2} + g. \tag{3.6}$$

The limiting metric $g_0$ is an exact $b$-metric on $M$, in the terminology of Melrose [18]. Let $A_{\epsilon,a}$ ($A_{\epsilon,r}$) be the associated elliptic boundary value problems. We note in the passing that the metric deformation (3.6) leaves invariant the projections $P_a$ ($P_r$), hence the boundary conditions. Let us also denote by $A_0$ the $b$-de Rham operator associated with $g_0$ (see [18]).

**Theorem 3.1.** *Assume that* $H^*(\partial M, \xi) = 0$ *and* $Im(H^*(M, \partial M; \xi) \to H^*(M; \xi)) = 0$. *Then if* $\dim M$ *is odd*

$$\eta(A_{\epsilon,a}) = \eta_b(A_0) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon, \tag{3.7}$$
$$\eta(A_{\epsilon,r}) = \eta_b(A_0) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon. \tag{3.8}$$

*And if* $\dim M$ *is even,*

$$\eta(A_{\epsilon,a}) = \frac{1}{2}\eta(A_{\partial M}) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon, \tag{3.9}$$
$$\eta(A_{\epsilon,r}) = -\frac{1}{2}\eta(A_{\partial M}) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon. \tag{3.10}$$

*As before,* $r_1$, $r_2$ *are smooth functions of* $\epsilon$ *vanishing at* 0.

*Remark.* 1. Without the assumption that $H^*(\partial M, \xi) = 0$ and $\mathrm{Im}(H^*(M, \partial M; \xi) \to H^*(M; \xi)) = 0$, the analysis of the small eigenvalues is much more complicated. In [19] this is dealt with via the scattering theory. Similar idea should apply here, which will be treated elsewhere.

*Remark.* 2. Intuitively the formula can be seen as follows. As $\epsilon \to 0$ the boundary is pushed to the infinity and in the heat kernel the interior contribution and boundary contribution separate. So in the end one is left with a manifold with cylindrical end and an infinite half-cylinder. The $b$-eta invariant comes from the former, and, depending on the parity of dimension, the contribution from the half-cylinder is either zero or the eta invariant of the boundary. In our proof this intuitive picture is realized geometrically by method of boundary-fibration structure of Melrose [17], [16].

When $(M, g)$ is of product type near the boundary the eta invariant can actually be shown to be invariant under this deformation. Thus we have

**Theorem 3.2.** *Assume additionally that $(M, g)$ is a product near the boundary. Then $\eta(A_{\epsilon,a}) \equiv \eta(A_a)$ is a constant independent of $\epsilon$. The same is true for $\eta(A_{\epsilon,r})$.*

**Proof.** Since $(M, g)$ is a product near the boundary we can assume that near the boundary

$$g_\epsilon = \frac{dx^2}{x^2+\epsilon^2} + g_{\partial M},$$

where $g_{\partial M}$ is a metric on the boundary independent of both $x$ and $\epsilon$. Put $y = \int_0^x \frac{dx}{\sqrt{x^2+\epsilon^2}}$. Then

$$g_\epsilon = dy^2 + g_{\partial M},$$

with $y \in [0, R(\epsilon)]$, $R(\epsilon) = \int_0^1 \frac{dx}{\sqrt{x^2+\epsilon^2}} \to \infty$ as $\epsilon \to 0$. Now choose a diffeomorphism $\varphi_\epsilon : [0,1] \to [0, R(\epsilon)]$ such that $\varphi_\epsilon(t) = t,\ t \in [0, 1/4]$ and $\varphi_\epsilon(t) = t + R(\epsilon) - 1,\ t \in [3/4, 1]$, and $\varphi'_\epsilon(t)$ symmetric with respect to $t = 1/2$. Then

$$g_\epsilon = (\varphi'_\epsilon(t))^2 dt^2 + g_{\partial M}.$$

By Theorem 2.1 the variation of $\eta(A_{\epsilon,a})$ is the same as that of $\eta(A_\epsilon)$, where $A_\epsilon$ is the corresponding operator on $\partial M \times S^1$ with the metric $(\varphi'_\epsilon(t))^2 dt^2 + g_{\partial M}$. By the symmetry of $\varphi'_\epsilon(t)$, we have $\eta(A_\epsilon) \equiv 0$. Therefore

$$\frac{d}{d\epsilon}\eta(A_{\epsilon,a}) \equiv 0. \qquad \square$$

Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2.

The same analysis (except the invariance under the deformation) also applies to the analytic torsion. Thus let $T^a_\epsilon(M, \rho)$ ($T^r_\epsilon(M, \rho)$ resp.) denote the analytic torsion associated to the representation $\rho: \pi_1(M) \to O(k)$ and the absolute (relative resp.) boundary condition on $M$ with the metric (1.1).

**Theorem 3.3.** *Assume that* $H^*(\partial M, \xi) = 0$ *and* $Im(H^*(M, \partial M; \xi) \to H^*(M; \xi)) = 0$*. Then if* $\dim M$ *is odd*

$$\log T_\epsilon(M, \rho) = \log {}^bT(\bar{M}, \rho) + r_1(\epsilon) + r_2(\epsilon) \log \epsilon, \tag{3.11}$$

*where* ${}^bT(\bar{M}, \rho)$ *is the analytic torsion for manifold with cylindrical end (the b-torsion* [18]*). Here the analytic torsion on* $M$ *is with respect to either of the boundary conditions. If* $\dim M$ *is even*

$$\log T^a_\epsilon(M, \rho) = \frac{1}{2} \log T(\partial M, \rho) + r_1(\epsilon) + r_2(\epsilon) \log \epsilon, \tag{3.12}$$

*and*

$$\log T^r_\epsilon(M, \rho) = -\frac{1}{2} \log T(\partial M, \rho) + r_1(\epsilon) + r_2(\epsilon) \log \epsilon. \tag{3.13}$$

The proof of Theorem 3.1 will be deferred to the last section, after the study of the uniform structure of the heat kernels involved. The rest of the paper is organized as follows. After the model case of the half infinite cylinder is discussed, we first show that for $\epsilon$ sufficiently small, the spectrum of $A_{\epsilon,a}$ falls uniformly outside a small neighborhood of the origin. This gives us sufficient control over the large time behavior of the heat kernel. Then for the finite time behavior the uniform structure of the heat kernel is examined by constructing the heat surgery 0-calculus, which is then exploited via Laplace transform to also extract information about the resolvent. The large time behavior of the heat kernel follows. Finally combining all these analyses we prove Theorem 3.1.

## 4. Computation on the half-cylinder

Our heat operator is defined via functional calculus:

$$e^{-tA_a^2} = \frac{i}{2\pi} \int_\Gamma (A_a - \lambda)^{-1} e^{-t\lambda^2} d\lambda.$$

Clearly, it satisfies the heat equation

$$(\partial_t + A_a^2) e^{-tA_a^2} = 0$$

with the correct initial condition:

$$e^{-tA_a^2}|_{t=0} = \mathrm{Id}.$$

From its definition, and the fact that

$$A_a e^{-tA_a^2} = \frac{i}{2\pi}\int_\Gamma (A_a - \lambda)^{-1}\lambda e^{-t\lambda^2} d\lambda,$$

it also satisfies the following boundary conditions:

$$\begin{cases} P_a e^{-tA_a^2}|_{x=0} &= 0 \\ P_a A e^{-tA_a^2}|_{x=0} &= 0. \end{cases} \tag{4.14}$$

For our purpose it is easier to deal with heat kernels satisfying such boundary conditions. As we are going to show later that the heat kernels satisfying such boundary conditions are unique, they are the same as defined via functional calculus.

For later purpose, and also to get a flavor of the boundary condition, we now consider the situation on the infinite half-cylinder:

$$H = \partial M \times [0, \infty). \tag{4.15}$$

In this case we have the global decomposition

$$\Lambda^*(H) = \Lambda^*(\partial M) \oplus \Lambda^*(\partial M). \tag{4.16}$$

With respect to this decomposition $\theta_1 + du \wedge \theta_2$ corresponds to $(\theta_1, \theta_2)$ (where we now use $u$ to denote the variable in $[0, \infty)$. Therefore

$$d = \begin{pmatrix} d_{\partial M} & 0 \\ \partial_u & -d_{\partial M} \end{pmatrix}.$$

Hence

$$A = \gamma \partial_u + \sigma A_{\partial M}, \tag{4.17}$$

where

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider only $A_a^2$, the other being similar. Its heat kernel $E$ satisfies (4.14). Write $E$ in terms of the decomposition (4.16):

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}.$$

Then the equation (4.14) becomes four parabolic boundary value problems:

$$\begin{cases} (\partial_t - \partial_u^2 + A_{\partial M}^2) E_{11} = 0, \\ E_{11}|_{t=0} = \mathrm{Id}, \\ (\partial_u E_{11} - A_{\partial M} E_{21})|_{u=0} = 0. \end{cases} \tag{4.18}$$

$$\begin{cases} (\partial_t - \partial_u^2 + A_{\partial M}^2)E_{12} = 0, \\ E_{12}|_{t=0} = 0, \\ (\partial_u E_{12} - A_{\partial M} E_{22})|_{u=0} = 0. \end{cases} \tag{4.19}$$

$$\begin{cases} (\partial_t - \partial_u^2 + A_{\partial M}^2)E_{21} = 0, \\ E_{21}|_{t=0} = 0, \\ E_{21}|_{u=0} = 0. \end{cases} \tag{4.20}$$

$$\begin{cases} (\partial_t - \partial_u^2 + A_{\partial M}^2)E_{22} = 0, \\ E_{22}|_{t=0} = \mathrm{Id}, \\ E_{22}|_{u=0} = 0. \end{cases} \tag{4.21}$$

The same discussion applies to the heat kernel on our manifold with boundary, restricted to the cylindrical part, since everything is local. From here we have the uniqueness of the heat kernel.

**Proposition 4.1.** *Let $M$ be a compact Riemannian manifold with boundary, with product metric near the boundary. Let $A_a$ be the de Rham operator equipped with the absolute boundary condition defined above. The heat kernel $E$ satisfying*

$$\begin{cases} (\partial_t + A_a^2)E = 0 \\ \quad E|_{t=0} \quad = Id \\ \ P_a E|_{u=0} \ \ = 0, \ \ P_a A E|_{u=0} = 0 \end{cases}$$

*is unique.*

**Proof.** If $E$ and $E'$ are two heat kernels satisfying the above equations, then $\widetilde{E} = E - E'$ satisfies the same set of equations except the initial condition, which should be replaced by $\widetilde{E}|_{t=0} = 0$. We first look at $\widetilde{E}$ near the boundary where it decomposes into $\widetilde{E}_{11}, \cdots, \widetilde{E}_{22}$ satisfying, respectively, (4.18) – (4.21), but once again with initial conditions replaced by zero ones. Now (4.20), (4.21) are heat equations with Dirichlet boundary condition, therefore by the energy estimate, we have $\widetilde{E}_{21} \equiv 0$, $\widetilde{E}_{22} \equiv 0$ (on the cylindrical part). From this, we find that (4.18), (4.19) reduce to heat equations with Neumann boundary condition. Hence again by the energy estimate we have $\widetilde{E}_{11} \equiv 0$, $\widetilde{E}_{12} \equiv 0$ on the cylindrical part. Now this implies that $\widetilde{E}$ satisfies a heat equation on the whole manifold with completely decoupled Dirichlet and Neumann boundary conditions. Therefore again we invoke the energy estimate to deduce that $\widetilde{E} \equiv 0$ on $M$. □

We now return to the half-cylinder. The last equation is a Dirichlet problem and can be solved explicitly in terms of the heat kernel on $\partial M$:

$$E_{22} = f_D(t,u,v)e^{-tA_{\partial M}^2}, \tag{4.22}$$

where

$$f_D(t,u,v) = \frac{1}{\sqrt{4\pi t}}(e^{-(u-v)^2/4t} - e^{-(u+v)^2/4t}).$$

The third equation has the trivial solution $E_{21} = 0$. Hence the first equation (4.18) becomes a Neumann problem while the second also gives the trivial solution:

$$E_{11} = f_N(t,u,v)e^{-tA_{\partial M}^2}, \tag{4.23}$$

where

$$f_N(t,u,v) = \frac{1}{\sqrt{4\pi t}}(e^{-(u-v)^2/4t} + e^{-(u+v)^2/4t}).$$

It follows that

$$e^{-tA_a^2} = e^{-tA_{\partial M}^2}\begin{pmatrix} f_N & 0 \\ 0 & f_D \end{pmatrix}. \tag{4.24}$$

Similarly

$$e^{-tA_r^2} = e^{-tA_{\partial M}^2}\begin{pmatrix} f_D & 0 \\ 0 & f_N \end{pmatrix}. \tag{4.25}$$

We now compute the pointwise trace $tr(A_a e^{-tA_a^2})$. Using (4.17) and (4.24) we find

$$tr(A_a e^{-tA_a^2}) = \frac{1}{\sqrt{\pi t}}e^{-u^2/t}tr(A_{\partial M}e^{-tA_{\partial M}^2}). \tag{4.26}$$

Integrating (4.26) gives

$$Tr(A_a e^{-tA_a^2}) = \frac{1}{2}Tr(A_{\partial M}e^{-tA_{\partial M}^2}). \tag{4.27}$$

Consequently we deduce

**Proposition 4.2.** *For the infinite half cylinder,*

$$\eta(A_a) = -\eta(A_r) = \frac{1}{2}\eta(A_{\partial M}). \tag{4.28}$$

## 5. Surgery 0-calculus

The proof of Theorem 3.1 depends essentially on the analysis of the uniform structure of the heat kernels for the elliptic boundary value problems. As in [14] this will be examined from the point of view of boundary-fibration structure (see [17]). That is, a calculus of pseudo-differential operators will be constructed, quite geometrically in the sense that the Schwartz kernels of these operators are to live on a space obtained from the usual space by blowing up certain submanifolds. The blowup resolves, analytically and geometrically, the singularities of the Schwartz kernels of these pseodo-differential operators. The construction in our case, loosely speaking, incorporates the $\mathcal{V}_0$-calculus (see Mazzeo-Melrose [15], [17] and the references therein) into the calculus of [14].

In this section the elliptic part of the calculus will be discussed, leading to the construction of the uniform resolvent and the uniform structure of the spectrum.

### 5.1. *Single surgery space*

The single surgery space is a natural compactification of the geometric degeneration, and the structure algebra defined on it captures the degeneration of the geometric operator, the de Rham operator in our case here. The space is defined as (Cf. [14] for the blowup notation):

$$X_{s0} = [M \times [0,\ 1]; \partial M \times \{0\}].$$

Here $[0, 1]$ is the parameter space for $\epsilon$.

This is a manifold with corner, with the "trivial" extension boundary at $\epsilon = 1$. The more interesting boundary hypersurfaces are: $B_{ss}$ resulting from the blow up; $B_{bb}$ from the lift of $\{\epsilon = 0\}$; and $B_{0b}$ from the lift of $\partial M \times [0,\ 1]$.

The boundary face $B_{bb}$ is diffeomorphic to $M$ while the interior of $B_{ss}$ is diffeomorphic to the normal bundle of $\partial M$ in $M$. The two intersect at the corner $\partial M$. On the other hand, the boundary face $B_{0b}$ is diffeomorphic to $\partial M \times [0,\ 1]$.

Let

$$\beta_{s0} :\ X_{s0} \to X = M \times [0,\ 1]$$

be the blow-down map. Composed with the projection

$$\pi_\epsilon : X \to [0,\ 1]$$

we get a $b$-fibration map

$$\tilde{\pi}_\epsilon = \pi_\epsilon \circ \beta_{s0} : X_{s0} \longrightarrow [0,\ 1].$$

Note that for $\epsilon > 0$, the fibers of $\tilde{\pi}_\epsilon$ are diffeomorphic to $M$ while at $\epsilon = 0$, $\tilde{\pi}_\epsilon^{-1}(0) = B_{ss} \cup B_{bb}$. This $b$-fibration captures the metric degeneration. In this picture, the geometric degeneration appears as the creasing of $M$ into $M$ together with the normal bundle of $\partial M$.

The structure algebra $\mathcal{V}_{s0}(X_{s0})$ is defined as

$$\mathcal{V}_{s0}(X_{s0}) = \{V \in \mathcal{V}_b(X_{s0}); (\tilde{\pi}_\epsilon)_*(V) = 0, \text{ and } V|_{B_{0b}} = 0\}.$$

This determines the structure bundle ${}^{s0}TX_{s0}$ by the equation

$$\mathcal{V}_{s0}(X_{s0}) = C^\infty(X_{s0};\ {}^{s0}TX_{s0}). \tag{5.29}$$

That ${}^{s0}TX_{s0}$ is a well-defined smooth vector bundle over $X_{s0}$ follows from a general statement in [7] (see also [14]). In fact, over the part of $X_{s0}$ where $\epsilon > 0$, ${}^{s0}TX_{s0}$ is simply the pull-back of ${}^0TM$ while restricted to $B_{bb}$ it is canonically isomorphic to the $b$-tangent bundle of this compact manifold with boundary. When restricted to $B_{ss}$ it is canonically isomorphic to the $b$-tangent bundle of this manifold near the boundary that meets $B_{bb}$ and the 0-tangent bundle near the boundary that meets $B_{0b}$.

The structure algebra $\mathcal{V}_{s0}(X_{s0})$ is a Lie algebra of vector fields which degenerates in the same manner as the de Rham operator in this geometric degeneration (except at the boundary where the degeneration is *created* for treating the boundary problem). To analyze the de Rham operator via microlocal analysis we first construct from it the space of $s0$-differential operators $\mathrm{Diff}^*_{s0}(M; E, F)$ ($E$, $F$ vector bundles on $X_{s0}$) in the usual way. Indeed, the space $\mathrm{Diff}^k_{s0}(M; E, F)$ consists of those differential operators from $C^\infty(X_{s0}; E)$ to $C^\infty(X_{s0}; F)$ which are given, with respect to local basis of $E$ and $F$, by sums of up to $k$-fold products of elements of $\mathcal{V}_{s0}(X_{s0})$.

A $s0$-differential operator can be analyzed by its symbol plus the so-called normal homomorphisms. The symbol sort of measures its "interior strength", and is defined as follows. By (5.29) and the natural isomorphism between a vector space and its double dual, a vector field in $\mathcal{V}_{s0}(X_{s0})$ can be naturally identified with a $C^\infty$ function on ${}^{s0}T^*X_{s0}$ that is linear along the fiber. This gives rise to the symbol map

$${}^{s0}\sigma : \mathrm{Diff}^k_{s0}(M; E, F) \to S^k({}^{s0}T^*X_{s0}; \hom(E, F)).$$

The normal homomorphisms, on the other hand, capture the leading terms in the degeneration. These are defined by restriction. The restriction of the Lie algebra $\mathcal{V}_{s0}(X_{s0})$ to the boundary hypersurface $B_{bb}$ gives the full algebra $\mathcal{V}_b(B_{bb})$, the space of vector fields on $B_{bb}$ tangent to the boundary of $B_{bb}$ and its restriction to $B_{ss}$ gives the algebra $\mathcal{V}_{0b}(B_{ss})$, the space of vector fields on $B_{ss}$ tangent to one boundary component, $B_{ss}\cap B_{bb}$, and vanishing at the other, $B_{ss}\cap B_{0b}$. As a consequnce, the space $\mathrm{Diff}^*_{s0}(M;E,F)$ comes equipped with the normal homomorphisms

$$\begin{aligned} N_b &: \mathrm{Diff}^k_{s0}(M;E,F) \to \ \mathrm{Diff}^k_b(M;E,F), \\ N_s &: \mathrm{Diff}^k_{s0}(M;E,F) \to \mathrm{Diff}^k_{0b}(B_{ss};E,F). \end{aligned}$$

Here the image space $\mathrm{Diff}^k_b$ has a normal homomorphism itself, called the indicial homomorphism:

$$I: \ \mathrm{Diff}^k_b(M;E,F) \to \mathrm{Diff}^k_{I,b}(\partial M\times[0,\,\infty);E,F),$$

where the space with the subscript $I$ denotes the subspace of $\mathbf{R}^+$-invariant operators. Similarly the indicial operator of an element of $\mathrm{Diff}^k_{0b}(B_{ss};E,F)$ at the $b$-boundary $\partial M$ is also an element of $\mathrm{Diff}^k_{I,b}(\partial M\times[0,\,\infty);E,F)$. The compatibility condition between the normal operators is just

$$N_{\partial M}(P) \overset{\mathrm{def}}{=} I(N_b(P)) = I(N_s(P)),\ P\in \mathrm{Diff}^k_{s0}(M;E,F),$$

which is a consequence of (5.29).

If we choose local coordinates $(x,y)$ on $M$ near the boundary, where $y$ is a local coordinate on $\partial M$ and $x$ the geodesic distance to the boundary, one obtains defining functions for the various boundary hypersurfaces:

$$\rho_{ss}=\sqrt{x^2+\epsilon^2},\quad \rho_{bb}=\frac{\epsilon}{\sqrt{x^2+\epsilon^2}},\quad \rho_{0b}=\frac{x}{\sqrt{x^2+\epsilon^2}}.$$

From (4.17) we have for the de Rham operator $A_\epsilon$

$$A_\epsilon=\gamma\sqrt{x^2+\epsilon^2}\partial_x+\sigma A_{\partial M}=\gamma\rho_{ss}\partial_x+\sigma A_{\partial M}. \tag{5.30}$$

This is not yet a $s0$-differential operator. However

$$\rho_{0b}A_\epsilon\in \mathrm{Diff}^1_{s0}(M;F),$$

and

$$N_b(\rho_{0b}A_\epsilon)=\rho_{0b}A_0\in \mathrm{Diff}^1_b(M;F). \tag{5.31}$$

$$N_s(\rho_{0b}A_\epsilon)=\rho_{0b}A_{B_{ss}}\in \mathrm{Diff}^1_{0b}(B_{ss};F). \tag{5.32}$$

Moreover the restriction at the corner $B_{ss}\cap B_{0b}=\partial M$ is given by

$$R_{\partial M}(\rho_{0b}A_\epsilon)=\rho_{0b}A_{\partial M}\in \mathrm{Diff}^1(\partial M;F). \tag{5.33}$$

### 5.2. *Double surgery space*

We analyze the degenerating de Rham operator by looking at the resolvent and the singularity of its Schwartz kernel. This is done by constructing a pseudo-differential calculus in which lies the resolvent of the degenerating de Rham operator. This pseudo-differential calculus comes from microlocalizing $s0$-differential operators.

To microlocalize the Lie algebra of vector fields $\mathcal{V}_{s0}(X_{s0})$ we now define the double surgery 0-space, on which live the kernels of surgery 0-operators (or $s0$-operators):

$$\begin{aligned} X^2_{s0,f} = [M^2 \times [0,\ 1];\, (\partial M)^2 \times \{0\};\, \partial M \times M \times \{0\};\\ M \times \partial M \times \{0\};\, \Delta(\partial M) \times [0,1]], \end{aligned}$$

where the subscript $f$ indicates that this is a full blown-up version of the double surgery 0-space. The blow down map will be denoted by $\beta^2_{s0}$.

There are seven boundary hypersurfaces besides the trivial extension face $\{\epsilon = 1\}$, which we will ignore. We have $B_{ds}$ from the first blow up; $B_{ls}$, $B_{rs}$ from the second and third respectively; and $B_{0s}$ from the last blow up. Finally the original boundary hypersurfaces $\{\epsilon = 0\}$, $\partial M \times M \times [0,1]$, and $M \times \partial M \times [0,1]$ lift to boundary hypersurfaces $B_{db}$, $B_{lb}$ and $B_{rb}$ respectively. Also the diagonal $\Delta(M) \times [0,1]$ lifts to an embedded submanifold $\Delta_{s0}$ meeting only $B_{ds}$, $B_{db}$, $B_{0s}$ and does so transversally.

Let $\pi_L,\ \pi_R$ denote the projections of $X^2 \overset{\text{def}}{=} M^2 \times [0,1]$ onto $X$ by omitting the right and left factors respectively. These lift to $b$-fibrations

$$\begin{aligned} \pi_{s0,L} : X^2_{s0,f} \to X_{s0},\\ \pi_{s0,R} : X^2_{s0,f} \to X_{s0}. \end{aligned}$$

Both restrict to $\Delta_{s0}$ to a diffeomorphism: $\Delta_{s0} \cong X_{s0}$. Moreover, by analyzing the lifting properties of $\mathcal{V}_{s0}(X_{s0})$, it is not hard to see that there is a natural isomorphism:

$$N(\Delta_{s0}) \cong\ {}^{s0}TX_{s0}. \tag{5.34}$$

Let $\rho_{0s}$ be a defining function of $B_{0s}$. Define the kernel density bundle $KD$ so that

$$C^\infty(X^2_{s0,f}, KD) = \rho_{0s}^{-n/2} C^\infty(X^2_{s0,f}, \Omega^{1/2}((X^2_{s0,f})).$$

The small surgery 0-calculus is

$$\Psi^m_{s0}(M; E, F) = \rho^\infty_{ls}\rho^\infty_{rs}\rho^\infty_{lb}\rho^\infty_{rb} I^{m-1/4}(X^2_{s0,f}, \Delta_{s0}; Hom(F,E) \otimes KD).$$

This is a microlocalization for $\mathcal{V}_{s0}(X_{s0})$ since $\mathrm{Diff}^*_{s0}(M) \subset \Psi^*_{s0}(M)$. However this calculus is too small to contain the inverses of its elliptic elements. Thus one has to enlarge the calculus to include boundary terms. Let us denote by $\mathcal{A}(X^2_{s0,f}; Hom(F,E) \otimes KD)$ the space of all sections of $Hom(F,E) \otimes KD$ smooth in the interior and conormal to all boundary faces. For a positive number $\tau$ define

$$\begin{aligned} &\mathcal{A}^\tau_-(X^2_{s0,f}; Hom(F,E) \otimes KD) \\ &= \bigcap_{\delta>0} \rho_{ds}^{\tau-\delta} \rho_{db}^{\tau-\delta} \rho_{ls}^{\tau-\delta} \rho_{rs}^{\tau-\delta} \mathcal{A}(X^2_{s0,f}; Hom(F,E) \otimes KD) \\ &= \bigcap_{\delta>0} \epsilon^{\tau-\delta} \mathcal{A}(X^2_{s0,f}; Hom(F,E) \otimes KD). \end{aligned}$$

We call $\tau$ the conormal bound for the conormal sections in $\mathcal{A}^\tau_-$.

Using this notation the residual calculus is defined as

$$\Psi^\tau_{s0,res}(M; E, F) = \mathcal{A}^\tau_-(X^2_{s0,f}; Hom(F,E) \otimes KD) \tag{5.35}$$

This is the space of 'good' error terms in the sense that they vanish at a positive rate at $\epsilon = 0$.

The space of boundary terms is defined as (using the notation of [14])

$$\Psi^{-\infty,\tau}_{s0}(M; E, F) = \mathcal{B}_{dB} \mathcal{A}^\tau_-(X^2_{s0,f}; Hom(F,E) \otimes KD), \tag{5.36}$$

where $dB = \{ds, db, 0s\}$ and $\tau$ is a positive number. Roughly speaking $\Psi^{-\infty,\tau}_{s0}$ consists of all sections smooth in the interior and conormal to the boundary faces (with conormal bound 0) and vanish at rate $\tau$ at the boundary faces $B_{ls}$, $B_{rs}$ and have some partial smoothness up to $B_{ds}$, $B_{db}$, $B_{0s}$.

Now the 'calculus with (conormal) bounds' is defined as

$$\Psi^{m,\tau}_{s0}(M; E, F) = \Psi^m_{s0}(M; E, F) + \Psi^{-\infty,\tau}_{s0}(M; E, F). \tag{5.37}$$

Since

$$\Psi^m_{s0}(M; E, F) \cap \Psi^{-\infty,\tau}_{s0}(M; E, F) = \Psi^{-\infty}_{s0}(M; E, F),$$

the first thing to note here is that the symbol map for conormal distributions

$${}^{s0}\sigma_m : \Psi^m_{s0}(M; E, F) \to S^m({}^{s0}TX^*_{s0}; E, F) \tag{5.38}$$

extends to the whole calculus.

The symbol map alone is not enough to invert the elliptic elements modulo compact errors. The utility of the calculus constructed above lies largely in the existence of additional, non-commutative 'symbols'. These are obtained by restricting the elements to each of the boundary faces $B_{ds}$,

$B_{db}$, $B_{0s}$, $B_{ls}$, $B_{rs}$. Since an element of $\Psi_{s0}^{m,\tau}$ is required to vanish at a positive rate at the boundary faces $B_{ls}$, $B_{rs}$, the restrictions will be trivial there and will be ignored. The only nontrivial ones are at $B_{db}$, $B_{ds}$, $B_{0s}$, called the $b$-normal homomorphism, the surgery normal homomorphism, and the 0-normal homomorphism respectively.

Clearly the $b$-normal homomorphism $N_b$ maps onto the $b$-calculus with conormal bounds on $M$:

$$N_b : \Psi_{s0}^{m,\tau}(M; E, F) \to \Psi_b^{m,\tau}(M; E, F). \tag{5.39}$$

The name homomorphism indicates that $N_b$ respects the composition (in the sense of operators acting on distributions, see Proposition 5.1). But only the weaker form $N_b(P \circ A) = N_b(P) \circ N_b(A),\ P \in \mathrm{Diff}_{s0}^*$ will be used here. This will be discussed below (Proposition 5.2).

Similarly the surgery normal homomorphism is a map

$$N_s : \Psi_{s0}^{m,\tau}(M; E, F) \to \Psi_{0b}^{m,\tau}(\bar{H}; E, F). \tag{5.40}$$

Here $\bar{H} = \partial M \times [0, 1]$ is the compactification of the half normal bundle of $\partial M$, or in other words the half infinite cylinder. And the image lies in the 0b-calculus which will be briefly discussed in the next section.

Finally for the 0-normal homomorphism note that $B_{0s}$ can be identified with a natural compactification of the half tangent bundle of $M$ at $\partial M$ lifted to $\partial M \times [0, 1]$. By definition then, one finds that $N_0$ maps onto the conormal distributions conormal to the section of the lifted normal bundle over $\partial M \times [0, 1]$ given by $(1, 0, \cdots, 0)$ and which are smooth up to the boundaries.

From definition it is not hard to see that, for an element in $\Psi_{s0}^{m,\tau}$ its various 'symbols' have to be compatible in the sense that restricted to the common corner or the intersection with the diagonal the resulting 'symbols' have to agree. Moreover these are the only obstructions for the existence of surgery 0-calculus with prescribed 'symbols'.

Although defined as distributions the surgery 0-operators can be made to act on distributions on $X_{s0}$, thus justifying the name. We state the mapping properties in the following

**Proposition 5.1.** *An element $A$ of $\Psi_{s0}^{m,\tau}(M; E, F)$ defines a bounded linear map*

$$A : \ C^{-\infty}(X_{s0}; E) \to C^{-\infty}(X_{s0}; F)$$

*which restricts to*

$$A : \ \mathcal{A}_-^r(X_{s0}; E) \to \mathcal{A}_-^r(X_{s0}; F),$$

*if $r < \tau$. Moreover, if $m \leq 0$, $\tau > 0$, then*

$$A : L^2(X_{s0}; E \otimes \Omega_{s0}^{1/2}) \to L^2(X_{s0}; F \otimes \Omega_{s0}^{1/2}). \tag{5.41}$$

*is also bounded. Here $\Omega_{s0}^{1/2}(X_{s0}) = \rho_{0s}^{-n/2}\Omega^{1/2}(X_{s0})$.*

**Proof.** Recall that the projections $\pi_L,\ \pi_R$ from $X^2 = M^2 \times [0,\ 1]$ to $X$, obtained by dropping the right and left $M$ factor in $X^2$ respectively, lift to $b$-fibrations

$$\tilde{\pi}_L,\ \tilde{\pi}_R : X^2_{s0,f} \to X_{s0}.$$

Similarly the projection onto the $\epsilon$ variable, $\pi_\epsilon :\ X^2 \to [0,\ 1]$, lifts to $b$-fibration

$$\tilde{\pi}_\epsilon : X^2_{s0,f} \to [0,\ 1].$$

Now the equation

$$Au = (\tilde{\pi}_L)_*[A \cdot (\tilde{\pi}_R)^*(u)(\tilde{\pi}_\epsilon)^*(|d\epsilon|^{-1/2})]$$

defines the action of $A \in \Psi_{s0}^{m,\tau}(M; E, F)$; the fact that it is well defined is a consequence of the calculus of wave front sets. This proves the first part. The second follows from the calculus of conormal functions (Cf. [14]).

To show the $L^2$ boundedness, it suffices to show that for $A \in \Psi_{s0}^{-\infty,\tau}$ (by Hörmander's lemma). We decompose $A$ into four pieces, $A = A_1 + A_2 + A_3 + A_4$, where $A_1$ is supported near $B_{0s}$; $A_2$ supported near $B_{lb}$, but away from $B_{0s}$; $A_3$ supported near $B_{rb}$, but away from $B_{0s}$; and the final piece $A_4$ supported away from $B_{lb} \cup B_{os} \cup B_{rb}$.

By its support property, the $L^2$-boundedness of $A_4$ is a consequence of [14]. For $A_2,\ A_3$, since the result of its action will always have support away from $B_{0s}$, the $L^2$-boundedness also follows similarly. The $L^2$-boundedness of $A_1$ is a uniform version of the result in [Ma] and can be shown in the same way. $\square$

We now turn to the composition with $s0$-differential operators.

**Proposition 5.2.** *If $P \in Diff^*_{s0}(M; E, F)$, $A \in \Psi_{s0}^{*;\tau}(M; E, F)$, then $P \circ A \in \Psi_{s0}^{*;\tau}(M; E, F)$. Further*

$$N_b(P \circ A) = N_b(P) \circ N_b(A) \tag{5.42}$$

*and similarly for the other homomorphisms.*

**Proof.** Clearly, if $P \in \mathrm{Diff}^k_{s0}$, $A \in \Psi^m_{s0}$, then $P \circ A \in \Psi^{m+k}_{s0}$. Also if $A \in \Psi^{-\infty,\tau}_{s0}$, then $P \circ A \in \Psi^{-\infty,\tau}_{s0}$. Now if $V \in \mathrm{Diff}^1_{s0}$, we have $N_b(V \circ A) = \pi^*_{s0,L} V(A)|_{B_{db}} = \pi^*_{s0,L} V|_{B_{db}}(A|_{B_{db}}) = N_b(V) \circ N_b(A)$. The general case follows. □

The residual space $\Psi^\tau_{s0,res}$ is the space of good error terms. Crucial to our construction is the 'semi-ideal property' that this space satisfies. Define $\mathcal{L}_C(M)$ to be the algebra of bounded operators on $L^2(X_{s0}; \Omega^{1/2}_{s0})$ which depends parametrically and conormally on $\epsilon$, i.e. an element of $\mathcal{L}_C(M)$ is a $B \in \mathcal{L}(L^2(X_{s0}; \Omega^{1/2}_{s0}))$ such that

$$[\epsilon, B] = 0 \text{ and } (\epsilon\frac{\partial}{\partial\epsilon})^k B \in \mathcal{L}(L^2(X_{s0}; \Omega^{1/2}_{s0})) \text{ for all } k \geq 0.$$

Clearly $\Psi^\tau_{s0,res}(M) \subset \mathcal{L}_C(M)$ is a subalgebra, but more is true.

**Proposition 5.3.** *If $\tau > 0$, then*

$$\Psi^\tau_{s0,res}(M) \cdot \mathcal{L}_C(M) \cdot \Psi^\tau_{s0,res}(M) \subset \Psi^\tau_{s0,res}(M).$$

**Proof.** Let $A$, $B \in \Psi^\tau_{s0,res}(M)$, and $K \in \mathcal{L}_C(M)$. We need to examine the kernel of $BKA$ and show that it has the required regularity. For this purpose, we apply the operator $BKA$ to certain weighted delta half-density. For $z \in M$, let $\delta_z \in C^{-\infty}(M; \Omega^{1/2})$ be a delta half-density at $z$. This gives a continuous map

$$M \ni z \mapsto \delta_z |d\epsilon|^{1/2} \in H^{-k}(X; \Omega^{1/2}), \quad \text{for } k > \frac{n}{2} + 1.$$

The continuity is a consequence of the Sobolev Embedding Theorem. Since this family of half-densities is $\epsilon$-independent, it follows that the lifts to $X_{s0}$ of the following weighted half-densities give rise to a continuous map

$$M \ni z \mapsto (x^2 + \epsilon^2)^{1/4} \epsilon^{\nu - \frac{1}{2}} \delta_z |d\epsilon|^{1/2} \in H^{-k}_b(X_{s0}; \Omega^{1/2}), \quad \forall \nu > 0.$$

By the assumption on $K$ and the mapping properties of $\Psi^\tau_{s0,res}(M)$, we obtain a continuous map

$$z \mapsto \epsilon^{-2\tau'} BKA((x^2 + \epsilon^2)^{1/4} \epsilon^{\nu - \frac{1}{2}} \delta_z |d\epsilon|^{1/2}) \in H^\infty_b(X_{s0}; \Omega^{1/2}), \quad \forall \tau' < \tau.$$

However the space $H^\infty_b(X_{s0}; \Omega^{1/2})$ consists of half-densities of the form $\epsilon^{-1/2} a\mu$ where $a$ is continuous on $X_s$ and $\mu$ is a non-vanishing smooth half-density on $X_{s0}$. This implies that the Schwartz kernel of $BKA$ is of the form

$$\epsilon^{2\tau'} b\mu \otimes \frac{v}{(x^2 + \epsilon^2)^{1/4}} \otimes |d\epsilon|^{-1/2},$$

where $b$ is continuous on $X_{s0}\times M$ and $\tau' < \tau$ arbitrary. Lifting to $X^2_{s0}$ shows that the kernel is the product of $\epsilon^{2\tau'}$ and a continous section of the kernel density bundle. Since this regularity is clearly stable under the repeated action of $\epsilon\partial_\epsilon$ and of $\mathcal{V}_b(X_{s0})$ lifted from either the left or the right it follows that

$$\Psi^{\tau}_{s0,res}(M)\cdot\mathcal{L}_C(M)\cdot\Psi^{\tau}_{s0,res}(M)\subset\Psi^{2\tau}_{s0,res}(M). \qquad \square$$

For its role in the heat surgery 0-calculus, the reduced double surgery space is defined to be

$$X^2_{s0}=[M^2\times[0,\ 1];\,(\partial M)^2\times\{0\};\,\partial M\times M\times\{0\};\,M\times\partial M\times\{0\}].$$

It can be obtained from $X^2_{s0,f}$ by blowing down the boundary face $B_{0s}$.

The elements of $\Psi^{-\infty}_{s0}(M;E)$ are smoothing operators on $M$, hence trace class. By Lidsky's theorem the trace is the integral over the diagonal of the point wise trace of the kernel, which can be interpreted as a density:

$$Hom(E)\otimes\Omega^{1/2}(X^2_{s0})|_{\Delta_{s0}}\cong hom(E)\otimes\Omega(X_{s0}).$$

Thus the trace of $A\in\Psi^{-\infty}_{s0}(M;E)$ is, as a function, the push-forward to $[0,\ 1]$ of the density

$$(tr\,A)|_{\Delta_{s0}}\in C^\infty(X_{s0};\Omega).$$

The following lemma is from [14].

**Lemma 5.4.** *As a map*

$$Tr:\ \Psi^{-\infty}_{s0}(M;E)\to C^\infty([0,\ 1])+\log\epsilon\, C^\infty([0,\ 1]).$$

*I.e.*

$$Tr(A)=r_A(\epsilon)+\log\epsilon\,\tilde r_A(\epsilon),$$

*for $r_A$, $\tilde r_A$ smooth functions of $\epsilon$. Moreover for the leading terms*

$$\tilde r_A(0)=\int_{\partial M}(tr\,A)|_{\partial M}, \tag{5.43}$$

$$r_A(0)=b\text{-}Tr(N_s(A))+b\text{-}Tr(N_b(A)). \tag{5.44}$$

### 5.3. *The 0b-calculus*

To construct a good parametrix for an elliptic $s0$-operator we need to invert its various normal operators. The normal operator at $B_{0s}$ lands in the $0b$-calculus, which we discuss here in somewhat more detail.

Let $\bar{H} = \partial M \times [0,1]$ be the compactified normal bundle of $\partial M$. The structure algebra $\mathcal{V}_{0b}$ is defined to be the Lie algebra of all vector fields that vanish at $\partial M \times \{0\}$ and tangent to $\partial M \times \{1\}$. The structure bundle ${}^{0b}T\bar{H}$ is defined, as usual, via

$$C^\infty(\bar{H}, {}^{0b}T\bar{H}) = \mathcal{V}_{0b}.$$

From the structure algebra we construct the $0b$-differential operators in the usual way.

To define $0b$-pseudodifferential operators we construct the double $0b$-space

$$\bar{H}^2_{0b} = [\bar{H}^2; \Delta(\partial M) \times \{0\}; (\partial M)^2 \times \{1\}].$$

Denote by $\Delta_{0b}$ the lifted diagonal. There are six boundary hypersurfaces for $\bar{H}^2_{0b}$, namely $B_{d0}$, $B_{db}$ from the blow-up operations respectively; $B_{l0}$, $B_{r0}$, $B_{lb}$, $B_{rb}$ from the lift of the original boundary faces. The lifted diagonal intersects only $B_{d0}$ and $B_{db}$ and does so transversally.

Now the space of $0b$-pseudodifferential operators is defined to be ($\tau > 0$)

$$\Psi^{m,\tau}_{0b}(\bar{H}, \Omega^{1/2}) = \rho^\infty_{l0}\rho^\infty_{r0}\rho^\infty_{lb}\rho^\infty_{rb} I^m(\bar{H}^2_{0b}; \Delta_{0b}; KD) + \mathcal{A}^\tau_-(\bar{H}^2_{0b}; KD),$$

where kernel density bundle $KD$ is defined so that

$$C^\infty(\bar{H}^2_{0b}; KD) = \rho_{d0}^{-n/2} C^\infty(\bar{H}^2_{0b}; \Omega^{1/2}).$$

We will denote $\Psi^\tau(\bar{H}; \Omega^{1/2}) = \Psi^{-\infty,\tau}_{0b}(\bar{H}, \Omega^{1/2}) = \mathcal{A}^\tau_-(\bar{H}^2_{0b}; KD)$. Since this is just a mixture of the 0-calculus and the $b$-calculus, it is quite clear that their common properties carry over.

**Proposition 5.5.**

*(1) The $0b$-differential operators are $0b$-pseudo-differential operators.*

*(2) The symbol map is a homomorphism:*

$$\sigma_{0b}: \ \Psi^{m,\tau}_{0b}(\bar{H}, \Omega^{1/2}) \to S^m({}^{0b}T^*\bar{H}).$$

*We also have the $0$-normal and $b$-normal homomorphisms:*

$$N_0 : \Psi^{m,\tau}_{0b}(\bar{H}; \Omega^{1/2}) \to \Psi^{m,\tau}_{0}(\bar{H}; \Omega^{1/2}),$$
$$N_b : \Psi^{m,\tau}_{0b}(\bar{H}; \Omega^{1/2}) \to \Psi^{m,\tau}_{b,I}(\bar{H}; \Omega^{1/2}).$$

*(3) Elements of $\Psi^{m,\tau}_{0b}(\bar{H}, \Omega^{1/2})$ define continuous linear operators:*

$$C^{-\infty}(\bar{H}; \Omega^{1/2}) \to C^{-\infty}(\bar{H}; \Omega^{1/2}),$$
$$\mathcal{A}^r_-(\bar{H}; \Omega^{1/2}) \to \mathcal{A}^r_-(\bar{H}; \Omega^{1/2}). \ (r < \tau)$$

(4) *We also have $L^2$-continuity: $A \in \Psi_{0b}^{m,\tau}(\bar{H};\Omega^{1/2})$ defines a continuous linear map*

$$A : \rho^z H_{0b}^k(\bar{H};KD) \to \rho^z H_{0b}^{k-m}(\bar{H};KD), \quad KD = \rho_{d0}^{-n/2}\Omega^{1/2}.$$

Crucial to our discussion is the so-called semi-ideal property of the residual calculus. Let $\mathcal{L}_c(L^2(\bar{H};KD)) = \cap_z \mathcal{L}(\rho^z L^2(\bar{H},KD))$.

**Proposition 5.6.** *If $\tau > 0$, $\Psi^\tau(M)$ is a semi-ideal in $\mathcal{L}_c(L^2(\bar{H};KD))$, i.e. for any $K$ a continuous linear operator on $\rho^z L^2(\bar{H};KD)$ for all $z$ and any $A,\ B \in \Psi^\tau(M)$,*

$$BKA \in \Psi^{2\tau}(M).$$

**Proof.** To examine the Schwartz kernel of $BKA$, we apply it to the delta densities. For $z \in \bar{H}$, let $\delta_z \in C^{-\infty}(\bar{H};\Omega^{1/2})$ be the delta half density at $z$. As a map,

$$\bar{H} \ni z \mapsto \delta_z \in H_b^{-k}(\bar{H};\Omega^{1/2}),\ k > \frac{n}{2}$$

is continuous. It follows that

$$\bar{H} \ni z \mapsto A\delta_z \in H_b^\infty(\bar{H};\Omega^{1/2})$$

is also continuous. Therefore

$$\bar{H} \ni z \mapsto BKA\delta_z \in H_b^\infty(\bar{H};\Omega^{1/2})$$

is continuous as well. But an element in $H_b^\infty(\bar{H};\Omega^{1/2})$ can be written as a continuous section of the half-density bundle divided by the square root of a defining function to the boundaries. This shows that the kernel of $BKA$ can be lifted to $\bar{H}_{0b}$. $\square$

Recall that $A = d + \delta$ is the (twisted) de Rham operator. We use $A_{\bar{H}}$ to denote the de Rham operator on $\bar{H}$. Now we can show

**Proposition 5.7.** *The resolvent of $A_{\bar{H}}^2$ lies in the $0b$-calculus, i.e. $\exists \tau > 0$ such that*

$$(A_{\bar{H}}^2 - \lambda)^{-1} \in \Psi_{0b}^{-2,\tau}(\bar{H};\Omega^{1/2}).$$

**Proof.** First of all, by taking the Laplace transform of (4.24), we have

$$(A_{\bar{H}}^2 - \lambda)^{-1} \in \mathcal{L}_c(L^2(\bar{H};\Omega^{1/2})).$$

On the other hand, using the $0b$-calculus, one can easily construct left and right parametrices for $D^2_{\bar H} - \lambda$:

$$(A^2_{\bar H} - \lambda)G_1 = \mathrm{Id} + R_1,$$
$$G_2(A^2_{\bar H} - \lambda) = \mathrm{Id} + R_2,$$

with $G_1,\ G_2 \in \Psi^{-2,\tau}_{0b}(\bar H;\Omega^{1/2}),\ R_1,\ R_2 \in \Psi^{\tau}(\bar H;\Omega^{1/2})$. Applying $(A^2_{\bar H} - \lambda)^{-1}$ to both equations we obtain

$$\begin{aligned}(A^2_{\bar H} - \lambda)^{-1} &= G_1 - (A^2_{\bar H} - \lambda)^{-1}R_1\\ &= -G_2R_1 + R_2(A^2_{\bar H} - \lambda)^{-1}R_1 \in \Psi^{-2,\tau}_{0b}(\bar H;\Omega^{1/2})\end{aligned}$$

by Proposition 5.6. □

As before, all the constructions and the discussions apply to operators acting on sections of a vector bundle. From now on, we denote by $E$ the vector bundle

$$E = \Lambda(M) \otimes \xi.$$

## 5.4. *The uniform structure of the resolvent*

Let $A_0$ denote the (twisted) de Rham operator associated to the exact $b$-metric $g_0$ on $M$. Also, denote by $A_{\epsilon,a}$ ($A_{\epsilon,r}$ resp.) the (twisted) de Rham operator associated to $g_\epsilon$ with the absolute (relative resp.) boundary condition. With all the machinery developed so far we can now prove

**Proposition 5.8.** *Assume that $A_{\partial M}$ is invertible and also $0$ is not in the spectrum of $A_0$. If $\Omega \subset \mathbf{C}$ is an open bounded set with closure disjoint from the spectrum of $A_0^2$, then for some $\tau > 0$ and $\epsilon_0 > 0$ the resolvent of $A^2_{\epsilon,a}$ ($A^2_{\epsilon,r}$ resp.) is a holomorphic map*

$$\begin{aligned}\Omega &\to \Psi^{-2,\tau}_{s0}(M;E)\\ \lambda &\mapsto \quad R(\lambda).\end{aligned}$$

*In particular the spectrum of $A^2_{\epsilon,a}$ ($A^2_{\epsilon,r}$ resp.) falls outside a neighborhood of the imaginary axis.*

**Proof.** One tries to solve the equation

$$\begin{cases} (A^2_{\epsilon,a} - \lambda)R(\lambda) = \mathrm{Id} \\ R(\lambda) \text{ satisfies the boundary condition} \end{cases} \tag{5.45}$$

by solving the corresponding equations for the symbol map and the normal homomorphisms. The symbol for $R(\lambda)$ can be solved via

$$^{s0}\sigma_2(R(\lambda)) = |\xi|^{-2}\mathrm{Id},$$

as does the $b$-normal homomorphism,

$$N_b(R(\lambda)) = (A_0^2 - \lambda)^{-1} \in \Psi_b^{-2,\tau}(M; E).$$

For the surgery normal homomorphism we note that

$$N_s(R(\lambda)) = (A_{\bar{H}}^2 - \lambda)^{-1}$$

is the solution for the corresponding equation for the half infinite cylinder. By taking the Laplace transform of (4.24) we find

$$N_s(R(\lambda)) \in \Psi_{0b}^{-2,\tau}(\bar{H}; E).$$

Finally the 0-normal homomorphism of $R(\lambda)$ satisfies a family of Laplace equations on the half Euclidean space with the boundary condition. Hence it can be solved similarly as in the half cylinder case.

These solutions for the normal homomorphisms and symbol clearly satisfy the compatibility condition. Thus there exists a family of surgery 0-operators $E'(\lambda) \in \Psi_{s0}^{-2,\tau}(M; E)$ with the correct symbol and normal homomorphisms. This means that $E'(\lambda)$ is already a parametrix for the resolvent family.

To get a better parametrix, note that the interior singularity can be removed in the small calculus. It follows then that there is a correction term $G_0'(\lambda) \in \Psi_{s0}^{-2}(M; E)$ such that $E = E' - G_0'$ is a parametrix in the strong sense that

$$(A_{\epsilon,a}^2 - \lambda)E(\lambda) = \mathrm{Id} - G(\lambda),\ G(\lambda) \in \Psi_{s0,res}^{\tau}(M; E).$$

Now $G(\lambda)$ vanishes at a positive rate at $\epsilon = 0$. Hence where $\epsilon$ is small the Neumann series provides an inverse for $\mathrm{Id} - G(\lambda)$ and

$$R(\lambda) = E(\lambda)(\mathrm{Id} - G(\lambda))^{-1}. \qquad \square$$

## 6. Heat surgery 0-calculus

After the discussion of the elliptic calculus we now turn to the parabolic calculus and examine the uniform structure of the heat kernel.

### 6.1. *Heat surgery* 0*-operators*

To construct the heat surgery 0-space we note that $\Delta_{s0}$ intersects the boundaries of $X^2_{s0}$ at $B_{ds}$, $B_{db}$, and at the lift of $(\partial M)^2 \times [0,\,1]$ which is a corner. This means that in defining the heat surgery 0-space one needs to first blow up the intersection at the corner, which is $\Delta(\partial M) \times [0,\,1]$:

$$X^2_{hs0} = [X^2_{s0} \times [0,\infty);\, \Delta(\partial M) \times [0,\,1] \times \{0\}, S;\, \Delta_{s0} \times \{0\}, S],$$

where $S$ is the parabolic bundle $sp(dt)$ (see [14]). The blow down map is denoted by $\beta_h$.

For analyzing the normal homomorphisms we look at the structures of the boundary hypersurfaces of $X^2_{hs0}$. There are three of them lying above $\{t = 0\}$: $B_{ff}$ from the blow up of $\Delta(\partial M) \times [0,\,1] \times \{0\}$; $B_{tf}$ from the blow up of $\Delta_{s0} \times \{0\}$; and $B_{tb}$, the lift of $\{t = 0\}$. The first two are fibered over the submanifolds to be blown up. In fact $B_{ff}$ can be viewed as the natural compacitification of the lift to $\partial M \times [0,\,1]$ of the half tangent bundle of $M$ at $\partial M$ times $[0,\infty)$ and $B_{tf} \cong {}^{s0}TX_{s0}$.

The rest of the boundary hypersurfaces arise from the lift of those of $X^2_{s0}$. Precisely we have

$$\begin{aligned}
B_{ds}(X^2_{hs0}) &= [B_{ds}(X^2_{s0}) \times [0,\infty); \Delta(\partial M) \times \{0\}, S;\, \Delta_{ds} \times \{0\}, S],\\
B_{db}(X^2_{hs0}) &= [B_{db}(X^2_{s0}) \times [0,\infty); \Delta_{db} \times \{0\}, S],\\
B_{ls}(X^2_{hs0}) &= B_{ls}(X^2_{s0}) \times [0,\infty),\\
B_{rs}(X^2_{hs0}) &= B_{rs}(X^2_{s0}) \times [0,\infty),\\
B_{lb}(X^2_{hs0}) &= [B_{lb}(X^2_{s0}) \times [0,\infty);\, \Delta(\partial M) \times [0,\,1] \times \{0\}, S],\\
B_{rb}(X^2_{hs0}) &= [B_{rb}(X^2_{s0}) \times [0,\infty);\, \Delta(\partial M) \times [0,\,1] \times \{0\}, S],
\end{aligned}$$

Note that $B_{tf}$ only meets $B_{ff}$, $B_{tb}$, $B_{db}$ and $B_{ds}$.

The kernels of the heat surgery 0-operators are normalized with respect to the half-density

$$KD_{hs0} = \rho_{ff}^{-(n+2)/2} \rho_{tf}^{-(n+3)/2} \Omega^{1/2}(X^2_{hs0}).$$

Let $I$ denote the index set $\{k_1, k_2, k_3, k_4\}$. The space of the heat surgery 0-operators is defined to be

$$\Psi^I_{hs0} = \rho_{ff}^{k_1} \rho_{tf}^{k_2} \rho_{ds}^{k_3} \rho_{db}^{k_4} \rho_{ls}^{\infty} \rho_{rs}^{\infty} \rho_{tb}^{\infty} C^\infty(X^2_{hs0}; KD_{hs0}).$$

### 6.2. *Normal homomorphisms*

The normal homomorphism at $B_{ff}$ is defined by dividing by $\rho_{ff}^{k_1}$ and restricting to $B_{tf}$:

$$N_{hs0;f,k_1}:\ \Psi_{hs0}^I \to \rho_{tf}^{k_2}\rho_{ds}^{k_3}\rho_{tb}^{\infty}C^{\infty}(B_{ff}; KD_{hs0}|_{B_{ff}}).$$

By the previous discussion on the structure of $B_{ff}$ we see that it can also be thought as the front face of the heat 0-space of the half normal bundle of $\partial M$. Therefore the range of $N_{hs0;f,k_1}$ is also the range of the normal homomorphism of this heat 0-calculus at the front face.

The normal homomorphism at $B_{tf}$, or the heat homomorphism, is defined similarly.

$$N_{hs0;h,k_2}:\ \Psi_{hs0}^I \to \rho_{ff}^{k_1}\rho_{ds}^{k_3}\rho_{db}^{k_4}\rho_{tb}^{\infty}C^{\infty}(B_{tf}; KD_{hs0}|_{B_{tf}}).$$

Since $KD_{hs0}|_{B_{tf}}$ is canonically isomorphic to the fiber density bundle of ${}^{s0}TX_{s0}$, the heat homomorphism can be rewritten as

$$N_{hs0;h,k_2}:\ \Psi_{hs0}^I \to \rho_{ff}^{k_1}\rho_{ds}^{k_3}\rho_{db}^{k_4}\mathcal{S}({}^{s0}TX_{s0}; \Omega_{\text{fiber}}).$$

Restricting to $B_{db}$ gives us the surgery homomorphism:

$$N_{hs0;b}:\ \Psi_{hs0}(M) \to \Psi_{hb}(M),$$

while restricting to $B_{ds}$ gives a normal homomorphism which maps onto the heat $0b$-calculus of the compactified half normal bundle of $\partial M$:

$$N_{hs0;s}:\ \Psi_{hs0}(M) \to \Psi_{h0b}(N_+(\partial M)).$$

These normal homomorphisms are nontrivial only for $k_3 = 0$, $k_4 = 0$. Moreover if $N_{hs0;b}(A) = 0$ and $N_{hs0;s}(A) = 0$ for $A \in \Psi_{hs0}(M)$ then $A = \epsilon B$ for $B \in \Psi_{hs0}(M)$. This will be used in the construction of the heat kernels.

Individually, each normal homomorphism is surjective. However the normal operators for an element of $\Psi_{hs0}(M)$ have to agree at the common corners. These are the compatibility conditions. On the other hand, since essentially just smooth functions are involved, the compatibility conditions are the only obstructions to the existence of heat surgery 0-operator with given normal operators.

### 6.3. *Uniform structure of the heat kernel*

It suffices to consider the heat kernel for $A_{\epsilon,a}^2$, the other boundary condition being similar. The proceeding construction enables us to prove the following

**Theorem 6.1.** *There is a unique $H \in \Psi^{I}_{hs0}(M;E)$ where $I = \{-2,-2,0,0\}$ such that*

$$x^2(\partial_t + A_\epsilon^2)H = 0 \text{ in } \Psi^{I'}_{hs0}(M;E) \tag{6.46}$$

*for $I' = \{-2,0,-2,0\}$, and*

$$N_{hs0;h,-2}(H) = Id, \tag{6.47}$$

*and $H$ satisfies the boundary condition*

$$P_a H|_{B_{0b}} = 0, \quad P_a A_\epsilon H|_{B_{0b}} = 0.$$

**Proof.** Equation (6.46) and (6.47) translate to conditions on the four normal operators of $H$:

$$({}^{s0}\sigma_2(x^2A_\epsilon^2 - \frac{1}{2}(R+n))N_{hs0;h,-2}(H) = 0, \int_{\text{fiber}} N_{hs0;h,-2}(H) = \text{Id}, \tag{6.48}$$

$$x^2(\partial_t + N_b(A_\epsilon^2))N_{hs0;b}(H) = 0, \tag{6.49}$$

$$\rho_{0b}^2((\partial_t + N_s(A_\epsilon^2))N_{hs0;s}(H) = 0, \tag{6.50}$$

$$s'^2(\partial_{T'} + \Delta_E)N_{hs0;f}(H) = 0. \tag{6.51}$$

Finally the boundary condition translates to boundary conditions for (6.51) and (6.50).

The first equation is a fiber by fiber differential equation and can be solved uniquely subject to the integral condition. Furthermore, because of the compatibility condition, this fixes the integral conditions for (6.49) and (6.50). Thus the two normal operators $N_{hs0;b}(H)$, $N_{hs0;s}(H)$ are necessarily the heat kernels for the elliptic $b$-differential operator $N_b(A_\epsilon^2)$ and elliptic $0b$-differential operator $N_s(\rho_{0b}^2 A_\epsilon^2)$. As such they are unique and are elements of the corresponding small heat calculus. These two operators have the same indicial family, so using the existence part of the compatibility it follows that there is an element $H' \in \Psi^{I}_{hs0}(M;E)$ satisfying the symbolic conditions (6.48), (6.49), (6.50), (6.51).

This first approximation therefore satisfies

$$x^2(\partial_t + A_\epsilon^2)H' = -\epsilon R_1, \ R_1 \in \Psi^{-3,-3,0,0}_{hs0}(M;E). \tag{6.52}$$

Now we proceed exactly as in [14]. $\square$

## 7. Large time behavior of heat kernel

The implication of the previous construction of the uniform heat kernel will be further exploited in this section following the ideas of [14].

### 7.1. *The resolvent near infinity*

The resolvent and the heat kernel are related by the Laplace transform

$$\begin{aligned}(A_{\epsilon,a}^2-\lambda)^{-1} &= \int_0^\infty e^{\lambda t} e^{-tA_{\epsilon,a}^2}dt,\\ e^{-tA_{\epsilon,a}^2} &= \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda}(A_{\epsilon,a}^2-\lambda)^{-1}d\lambda,\end{aligned} \tag{7.53}$$

where $\Gamma$ is a contour enclosing the spectrum of $A_{\epsilon,a}^2$. This makes it possible to obtain information about one from the other. In fact, the large spectral parameter behavior of the resolvent corresponds to the small time behavior of the heat kernel and the large time behavior of the heat kernel corresponds to the small spectral parameter of the resolvent.

To estimate the resolvent as the spectral parameter tends to infinity outside a sector containing the spectrum we use the discussion of the heat kernel in the last section. Choose $\phi \in C_c^\infty(\mathbf{R})$ with $\phi(t)=1$ in $|t|<1$ and $\phi(t)=0$ in $|t|>2$. Let

$$R_1(\lambda) = \int_0^\infty e^{\lambda t}\phi(t)e^{-tA_{\epsilon,a}^2}dt. \tag{7.54}$$

Then

$$(A_{\epsilon,a}^2-\lambda)R_1(\lambda) = \mathrm{Id} - E_1(\lambda), \tag{7.55}$$

where the error term

$$E_1(\lambda) = \int_0^\infty e^{\lambda t}\phi'(t)e^{-tA_{\epsilon,a}^2}dt \in \Psi_{s0}^{-\infty}(M), \tag{7.56}$$

is in the small calculus, and since $\phi'(t)$ has compact support in $(0,\infty)$, vanishes rapidly as $|\lambda|\to\infty$ in any closed sector in $Re\,\lambda<0$.

To improve on the parametrix we now solve the equation

$$(A_{\epsilon,a}^2-\lambda)R_2(\lambda) = E_1(\lambda) - \epsilon E_2(\lambda). \tag{7.57}$$

This reduces to solving for the resolvent of the normal operators. It follows that we can solve $R_2 \in \Psi_{s0}^{-\infty,\tau}(M)$ with the error $E_2(\lambda) \in \Psi_{s0}^{-\infty,\tau}(M)$. Therefore

$$(A_{\epsilon,a}^2-\lambda)(R_1(\lambda)+R_2(\lambda)) = \mathrm{Id} - \epsilon E_2(\lambda). \tag{7.58}$$

For small $\epsilon$, $\mathrm{Id} - \epsilon E_2(\lambda)$ can be inverted in $L^2$ by the Neumann series. Writing the inverse as $\mathrm{Id} - S(\lambda)$, one sees that the norm of $S(\lambda)$ is rapidly decreasing as $|\lambda| \to \infty$. Moreover, $S(\lambda)$ is conormal in $\epsilon$, and therefore, belongs to $\mathcal{L}_c$. Again from the Neumann series,

$$S(\lambda) = \epsilon E_2(\lambda) + \epsilon^2 E_2(\lambda) E_2(\lambda) + \epsilon^2 E_2(\lambda) S(\lambda) E_2(\lambda). \tag{7.59}$$

Thus, by the semi-ideal property, $S(\lambda)$ is also in the surgery calculus and rapidly decreasing in $\lambda$.

Hence we have

$$(A_{\epsilon,a}^2 - \lambda)^{-1} = R_1(\lambda) + R_1(\lambda) \tag{7.60}$$

with $R_1(\lambda) = R_2(\lambda)(\mathrm{Id} - S(\lambda)) \in \Psi_{s0}^{-\infty,\tau}(M)$ being holomorphic in $\lambda$ and rapidly decreasing as $|\lambda| \to \infty$.

## 7.2. *Large time behavior of heat kernel*

We can now determine the large time behavior of the heat kernel. By (7.53), (7.54), (7.60), one has

$$(1 - \phi(t)) e^{-tA_{\epsilon,a}^2} = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} R_1(\lambda) d\lambda. \tag{7.61}$$

By our assumption, the contour $\Gamma$ can be deformed to a contour lying in the right half plane but still below the spectrum. It follows then that $e^{-tA_{\epsilon,a}^2}$ is exponentially decreasing, with all $t$-derivatives, as $t \to \infty$ with values in $\Psi_{s0}^{-\infty,\tau}(M)$, where $\tau > 0$ is the largest $\tau$ for which the resolvent takes values in $\Psi_{s0}^{-\infty,\tau}(M)$ along the new contour.

## 7.3. *Proof of Theorem 3.1*

Finally we are in a position to prove Theorem 3.1.

**Proof.** Let $i: \Delta_{hs0} \to X_{hs0}^2$ be the embedding of the lifted diagonal. We have

$$\Delta_{hs0} \cong [X_{s0} \times [0, \infty); B_{0b} \times \{0\}, S] \tag{7.62}$$

which blows down to $X_{s0} \times [0, \infty)$. Denote the blow down map by $\beta$. On the other hand, the projection $\pi_\epsilon: M \times [0, 1] \to [0, 1]$ lifts to a $b$-fibration

$$\pi_{s0}: X_{s0} \to [0, 1].$$

Let us use the same notation to denote the induced $b$-fibration

$$\pi_{s0}: X_{s0} \times [0, \infty) \to [0, 1] \times [0, \infty).$$

Finally let $\pi_s = \pi_{s0} \circ \beta$ and $\pi_t$ be the projection $[0, 1] \times [0, \infty) \to [0, 1]$. Then we can rewrite the eta function as

$$\eta(A_{\epsilon,a}, s) = (\pi_t)_*(\pi_s)_*[i^* trF], \; F = \frac{t^{(s-1)/2}}{\Gamma((s+1)/2)} A_{\epsilon,a} e^{-tA^2_{\epsilon,a}}. \tag{7.63}$$

The polyhomogeneity of $i^* trF$ follows immediately from Theorem 6.1. Now for each $t > 0$ the computation of the pushforward $(\pi_s)_*$ falls into the realm of Lemma 5.4. To apply this result we must compute the three terms in (5.43). By (5.32) the leading term for the log term is the integral of $\mathrm{tr}(Ae^{-tA^2_a})|_{\partial M}$ where $A$ lives on the half infinite cylinder. By (4.24) and (4.17),

$$\mathrm{tr}(Ae^{-tA^2_a}) = \mathrm{tr}(\gamma \partial_x e^{-tA^2_a}) + \mathrm{tr}(\sigma A_{\partial M} e^{-tA^2_a}) = 0.$$

Here the second term is identically zero by the splitting (4.16). Thus, the leading log term vanishes. The leading coefficient for the other term is given by

$$b\text{-}Tr(N_b(i^*F)) + b\text{-}Tr(N_s(i^*F)).$$

It follows that, when $\dim M$ is odd,

$$\eta(A_{\epsilon,a}) = b\text{-}Tr(N_b(i^*F)) + b\text{-}Tr(N_s(i^*F)) + r_1(\epsilon) + r_2(\epsilon)\log\epsilon.$$

The first term is by definition $\eta_b(A_0)$, while the second one is computed in Proposition 4.2. Note that $\eta(A_{\partial M}) = 0$ in this case.

For the even dimensional case, the term $tr(A_{\partial M} e^{-t\Delta_{\partial M}})$ no longer vanishes and it gives rise to the eta invariant for $A_{\partial M}$, whereas the $b$-eta term vanishes because of the parity of the dimension. □

We now explain how the analysis extends to the analytic torsion. The analytic torsion is defined in terms of the zeta function

$$\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}_s(Ne^{-t\underline{\Delta}}) dt, \; \Re s >> 0,$$

where $\mathrm{Tr}_s$ is the supertrace associated to the usual $\mathbb{Z}_2$-grading via even/odd degree, and $N$ is the number operator acting as multiplication by $k$ on $k$-forms. Also, $\underline{\Delta}$ denotes the Laplacian restricted to the orthocomplement of its null space. In our situation, the acyclicity condition rules out the null space and so it is just the Laplacian. This zeta function extends to a meromorphic function on the entire complex plane with $s = 0$ a regular value. We define the analytic torsion of Ray and Singer by

$$\log T(M, \rho) = \zeta'_T(0).$$

For the half infinite cylinder, using (4.24), (4.25), one derives for $\Delta = A_a^2$

$$\begin{aligned}&\mathrm{tr}_s(Ne^{-t\Delta})\\&= \frac{1}{\sqrt{\pi t}} e^{-u^2/t} \mathrm{tr}_s^{\partial M}(N_{\partial M} e^{-t\Delta_{\partial M}}) - \frac{1}{\sqrt{4\pi t}}(1+e^{-u^2/t})\mathrm{tr}_s^{\partial M}(e^{-t\Delta_{\partial M}}).\end{aligned}$$

Hence

$$\mathrm{Tr}_s(Ne^{-t\Delta}) = \frac{1}{2}\mathrm{Tr}_s^{\partial M}(N_{\partial M}e^{-t\Delta_{\partial M}}).$$

Here we have used the fact that $\mathrm{Tr}_s^{\partial M}(e^{-t\Delta_{\partial M}}) = \chi(\partial M, \xi) = 0$ by our assumption. It follows then that for half infinite cylinder,

$$T_a(M,\rho) = \frac{1}{2}T(\partial M, \rho),$$

and similarly

$$T_r(M,\rho) = -\frac{1}{2}T(\partial M, \rho).$$

Even though the analytic torsion is defined in terms of analytic continuation, it has an explicit heat kernel representation involving the coefficients of the asymptotic expansion of $\mathrm{Tr}_s(Ne^{-t\Delta})$, see for example Dai-Melrose [7] where all negative powers except $t^{-1/2}$ are shown to vanish. Using this, one can proceed as before and derive the results for the analytic torsion.

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