L^2-cohomology of Spaces with Non-isolated Conical Singularities and Non-multiplicativity of the Signature

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Abstract

We study from a mostly topological standpoint, the L^2-signature of certain spaces with non-isolated conical singularities. The contribution from the singularities is identified with a topological invariant of the link fibration of the singularities. This invariant measures the failure of the signature to behave multiplicatively for fibrations for which the boundary of the fibre in nonempty. The result extends easily to cusp singularities and can be used to compute the L^2 cohomology of certain noncompact hyper-kähler manifolds which admit geometrically fibered end structures.

1 Introduction

In this paper, we study the L^2-cohomology and L^2-signature for certain spaces with non-isolated conical singularities. We call these generalized Thom spaces. Appropriately formulated, our results extend easily to cusp singularities as well. Our main theorem identifies the contribution to the L^2-signature from a singular stratum with a topological invariant of the link fibration of the stratum. As an immediate application, we get a proof of the adiabatic limit formula of [13], in the case of odd dimensional fibre, without resorting to the quite nontrivial analytical results of [24]. This was actually one of original motivations. A second motivation was to study certain spaces with singularities which can be viewed as generalizations of Thom spaces.[1]

The L^2-cohomology of spaces with conical singularities has been studied extensively in [9, 10]; see also [9, 12] for the relation with intersection cohomology, and [12, 26] for the Cheeger-Goresky-MacPherson conjecture. For case of the cusp singularities, see [31, 32] and for Zucker's conjecture, see [22], [28]. For hyperbolic manifolds, see [23, 25].

The singular spaces that we consider can described as follows. Recall that a compact Riemannian manifold M with finite isolated conical singularities is modeled on the finite cone. That is, M is a compact topological space such that there are finite many points $p_1, \cdots, p_k$, so that $M \setminus \{p_1, \cdots, p_k\}$ is a smooth Riemannian manifold and a neighborhood of each singular point p_i is isomorphic to a finite metric cone, $C_{[0,a]}(Z_i)$, on a closed Riemannian

[1]This part of our work was done more than fifteen years ago but remained unpublished. We learned of the more recent connections with the Sen's conjecture from conversations with Tamas Hausel; see the end of this section.

manifold Z_i. In addition to isolated conical singularities, we also allow finitely many closed singular strata of positive dimension, whose normal fibres are of metric conical type. A prominent feature of these spaces is that the link fibration of a singular stratum need not be trivial. The discussion in this generality is necessary since we wish to identify the contribution to the L^2-signature from the singular strata in terms of global topological invariants of the link fibration.

A neighborhood of a singular stratum of positive dimension can be described as follows. Let

$$Z^n \to M^m \xrightarrow{\pi} B^l \tag{1.1}$$

denote a fibration of closed oriented smooth manifolds. Denote by $C_\pi M$ the mapping cylinder of π. This is obtained by attaching a cone to each of the fibres. Indeed, we have

$$C_{[0,a]}(Z) \to C_\pi M \to B\,. \tag{1.2}$$

The space $C_\pi M$ also comes with a natural quasi-isometry class of metrics. A metric can be obtained by choosing a submersion metric on M:

$$g_M = \pi^* g_B + g_Z.$$

Then, on the nonsingular part of $C_\pi M$, we take the metric,

$$g_1 = dr^2 + \pi^* g_B + r^2 g_Z\,, \tag{1.3}$$

and complete it.

The general class of spaces with non-isolated conical singularities as above can be described as follows. A space X in the class will be of the form

$$X = X_0 \cup X_1 \cup \cdots \cup X_k, \tag{1.4}$$

where X_0 is a compact smooth manifold with boundary, and each X_i (for $i = 1, \ldots, k$) is the associated mapping cylinder, $C_{\pi_i} M_i$, for some fibration, (M_i, π_i), as above. We require that the restriction of the metric to X_i is quasi-isometric to one of the form (1.3). Spaces with more complicated singularities can be obtained by iterating this construction, namely, by allowing the base and fibre of the fibration, (1.1), to be closed manifolds with non-isolated conical singularities.

Consider again the space, $C_\pi M$, in (1.2). By coning off the boundary, ∂M, we obtain what we call a generalized Thom space T. Thus, $T = C_\pi M \cup_M C(M)$ is a stratified space with two singular strata, namely, B and a single point.

The metric on T is constructed as follows. Equip $C(M)$ with the conical metric

$$g_2 = dr^2 + r^2 g_M\,.$$

Perturb g_1, g_2 near $r = 1$ so that they can be glued together so as to obtain to a smooth metric, g, on T. We will call (T, g) a *generalized Thom space*; see the example below. Clearly, a different choice of g_M will give rise to a metric quasi-isometric to g.

Example 1 Let $\xi \xrightarrow{\pi} B$ be a vector bundle of rank k. Then we have the associated sphere bundle:

$$S^{k-1} \to S(\xi) \xrightarrow{\pi} B.$$

The generalized Thom space constructed out of this fibration coincides with the usual Thom space equipped with a natural metric.

We now introduce the topological invariant which gives the contribution to the L^2-signature for each singular strata. In, [13], in studying adiabatic limits of eta invariants, the second author introduced a global topological invariant associated with a fibration. (For adiabatic limits of eta invariants, see also [5, 6, 11, 30].) Let (E_r, d_r) be the E_r-term with differential, d_r, of the Leray spectral sequence of (1.1). Define a pairing

$$\begin{array}{lcl} E_r \otimes E_r & \to & \mathbf{R} \\ \phi \otimes \psi & \mapsto & \langle \phi \cdot d_r \psi, \xi_r \rangle \end{array},$$

where ξ_r is a basis for E_r^m naturally constructed from the orientation. In case $m = 4k-1$, when restricted to $E_r^{\frac{m-1}{2}}$, this pairing becomes symmetric. We define τ_r to be the signature of this symmetric pairing and put

$$\tau = \sum_{r \geq 2} \tau_r .$$

It is shown in [13] that, unlike the case of fibrations whose fibres are closed manifolds, when the fibres have nonempty boundary, the signature does not always behave multiplicatively, even in a generalized sense; compare [27], [1]. The failure of such multiplicative behavior is intrinsically measured by the τ invariant of the associated boundary fibration; see [13].

Bismut and Cheeger studied related questions by introducing spaces with conical singularities as a technical tool; see [3], [4] . They showed that if one closes up the fibration of manifolds with boundary by attaching cones to the boundary of each fibre then for the corresponding fibration of manifolds with singularities, then the L^2-signature does in fact behave multiplicatively. One reason for studying generalized Thom spaces is to understand the difference between the approaches of [13] and [3], [4].

In this paper, we restrict attention to the case in which the fibre Z of (1.1) is either odd dimensional or its middle dimensional L^2-cohomology vanishes. Furthermore, we make the same assumptions for the links of the isolated conical singular points of the base and the fibre. (The general case requires the introduction of an "ideal boundary condition" as in [10, 11].) The result of [9] shows that $H^*_{(2)}(T)$, the L^2-cohomology of T, is finite dimensional and the Strong Hodge theorem holds. In fact, $H^*_{(2)}(T)$ agrees with the middle intersection cohomology of Goresky and MacPherson [14, 15]. Consequently, the L^2-signature of T is a topological invariant. Here, in defining the signature, we take the natural orientation on $C_\pi M$ and glue to $C(M)$ with the reverse of its natural orientation, in order to obtain the orientation on T.

Theorem 1.1. *The L^2-signature of the generalized Thom space T is equal to $-\tau$:*

$$\operatorname{sign}_{(2)}(T) = -\tau.$$

Let $\tau(X_i)$ denote the τ invariant for the fibration associated with X_i. Theorem 1.1 combined with Novikov additivity of the signature yields the following. (Note the reversing of the orientation.)

Corollary 1.2. *For the space X of the form (1.4) the L^2-signature is given by*

$$\operatorname{sign}_{(2)}(X) = \operatorname{sign}(X_0) + \sum_{i=1}^{k} \tau(X_i)\,.$$

Example 2. Consider again the sphere bundle of a vector bundle,

$$S^{k-1} \to S(\xi) \xrightarrow{\pi} B\,.$$

Let Φ denote the Thom class and χ the Euler class. Then the Thom isomorphism gives

$$\begin{array}{ccccc} H^*(D(\xi),S(\xi)) & \otimes & H^*(D(\xi),S(\xi)) & \to & \mathbf{R} \\ \uparrow \pi^*(\cdot)\cup\Phi & & \uparrow \pi^*(\cdot)\cup\Phi & & \\ H^*(B) & \otimes & H^*(B) & \to & \mathbf{R} \\ \phi & & \psi & \to & [\phi\cup\psi\cup\chi][B]\,. \end{array}$$

Thus, $\operatorname{sign}(D(\xi)) = -\operatorname{sign}_{(2)}(T)$, is the signature of this bilinear form on $H^*(B)$. Since in this case, the spectral sequence degenerates at E_2, and $d_2\psi = \psi \cup \chi$, it follows that the invariant, $\operatorname{sign}(D(\xi))$, agrees with τ. According to Theorem 1.1, the same result is still true even if the sphere bundle does not arise from a vector bundle.

In spirit, our proof of Theorem 1.1 follows Example 2. Thus, we first establish an analog of Thom's isomorphism theorem in the context of generalized Thom spaces. In part, this consists of identifying the L^2-cohomology of (T,g) in terms of the spectral sequence of the original fibration. The Mayer-Vietoris argument as in [9] shows that

$$H^i_{(2)}(T) = \begin{cases} H^i_{(2)}(C_\pi M, M), & i > \frac{m+1}{2} \\ \operatorname{Im}\,(H^i_{(2)}(C_\pi M, M) \to H^i_{(2)}(C_\pi M)), & i = \frac{m+1}{2} \\ H^i_{(2)}(C_\pi M), & i < \frac{m+1}{2} \end{cases}.$$

(Recall that $m = \dim M$ is the dimension of the total space of the fibration and $n = \dim Z$ is the dimension of the fibre.)

Let $E_r(M) = \oplus E_r^{p,q}(M)$, $d_r^{p,q} : E_r^{p,q}(M) \to E_r^{p+r,q-r+1}(M)$, denote the Leray spectral sequence of the fibration (1.1). (For some of the notation in the following theorem, we refer to Section 4.)

Theorem 1.3. *The following are isomorphisms.*

$$H^k(C_\pi(M), M) \cong \oplus_{p+q=k, q\geq (n+3)/2} [\mathrm{Im}\,(d^{p,q-1}_{q-(n-1)/2})^* \oplus \\ \oplus \mathrm{Im}\,(d^{p,q-1}_{q-(n-1)/2+1})^*) \oplus \cdots \oplus E^{p,q-1}_\infty(M)]\,,$$

and

$$H^k(C_\pi(M)) \cong \oplus_{p+q=k, q\leq (n-1)/2} [\mathrm{Im}\, d^{k-(n+3)/2,(n+1)/2}_{(n+3)/2-q} \oplus \\ \oplus \mathrm{Im}\, d^{k-(n+5)/2,(n+3)/2}_{(n+5)/2-q} \oplus \cdots \oplus E^{p,q}_\infty(M)]\,.$$

Moreover, in terms of these identifications, the map,

$$H^k(C_\pi(M), M) \to H^k(C_\pi(M))\,,$$

is given by $\oplus d_r$.

Remark. It can be shown that Theorem 1.1, Corollary 1.2 and Theorem 1.3 have extensions to the case of iterated conical singularities.

From the standpoint of index theory, the L^2-signature is of particular interest. For the case of fibrations with smooth fibers, it was considered in [4].

Let Z and B of (1.1) be closed smooth manifolds. Let A_M denote the signature operator on M with respect to the metric g_M. In addition, let $A_{M,\epsilon}$ denote the signature operator on M with respect to the metric $g_{M,\epsilon}$, where

$$g_{M,\epsilon} = \epsilon^{-1}\pi^* g_B + g_Z.$$

Define the $\tilde{\eta}$-form as in [3, 5]. Let the modified, L-form, $\mathcal{L}$, be defined as in [4]. Let R^B denote the curvature of B. According to [4] the following holds.

Theorem 1.4 (Bismut-Cheeger). *If the fibre of (1.1) is odd dimensional, then*

$$\mathrm{sign}_{(2)}(T) = -\lim_{\epsilon\to 0} \eta(A_{M,\epsilon}) + \int_B \mathcal{L}(\frac{R^B}{2\pi}) \wedge \tilde{\eta}\,.$$

Remark. Since the smooth part of T is diffeomorphic to $(0,1)\times M$, its contribution to the index formula vanishes. In the above formula, the first term arises from the isolated conical point, while the second term arises from the singular stratum B.

Combining Theorem 1.4 with our result on the L^2-signature, Theorem 1.1, we recover the following adiabatic limit formula of [13]; see also [5, 6, 10, 30].

Corollary 1.5. *With the same assumptions as in Theorem 1.4,*

$$\lim_{\epsilon\to 0} \eta(A_{M,\epsilon}) = \int_B \mathcal{L}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \tau.$$

Note that if (1.1) actually bounds a fibration of manifolds with boundary, then the above adiabatic limit formula is a consequence of the signature theorem of Atiyah-Patodi-Singer [2] and the Families Index Theorem for manifolds with boundary of Bismut-Cheeger; [3].[2] On the other hand, it seems difficult to decide whether every fibration (with odd dimensional fibres) actually bounds and it is generally believed that this is not the case. The method of attaching cones enables one to avoid this issue.

In the case in which (1.1) consists of closed smooth manifolds, in place of a cone, one can attach a cusp to each fibre. The metric at infinity of a locally symmetric space of rank one is of this type. Essentially because the Poincaré lemma for metric cusps gives the same calculation as for metric cones, similar results hold in this case.

The study of the L^2-cohomology of the type of spaces with conical singularities discussed here turns out to be related to work on the L^2-cohomology of noncompact hyper-kähler manifolds which is motivated by Sen's conjecture; see e.g [17], [16] . Hyper-kähler manifolds often arise as moduli spaces of (gravitational) instantons and monopoles, and so-called S-duality predicts the dimension of the L^2-cohomology of these moduli spaces (Sen's conjecture). Many of these spaces can be compactified to given a space with non-isolated conical singularities. In such cases, our results can be applied. We would also like to refer the reader to the work of Hausel-Hunsicker-Mazzeo, [16], which studies the L^2-cohomology and L^2-harmonic forms of noncompact spaces with fibered geometric ends and their relation to the intersection cohomology of the compactification. Various applications related to Sen's conjecture are also considered there.

In the general case i.e. with no the dimension restriction on the fibre, the L^2-signature for generalized Thom spaces is discussed in [19]. In particular, Theorem 1.1 is proved for the general case in [19]. However, one of ingredients there is the adiabatic limit formula of [13], rather than the direct topological approach taken here. As mentioned earlier, one of our original motivations was to give a simple topological proof of the adiabatic limit formula. In [18], the methods and techniques introduced in our old unpublished work are used in the more general situation to derive a very interesting topological interpretation for the invariant τ_r. This circumstance provided additional motivation for us to write up this work for publication.

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2 Review of L^2-cohomology

We begin by reviewing the basic properties of L^2-cohomology; for details, see [9]. Let (Y, g) denote an open (possibly incomplete) Riemannian manifold. We denote by $[g]$ the quasi-isometry class of g; i.e., the collection of Riemannian metrics g' on M such that for some

[2]In this case, the invariant, τ, enters because it measures the non-multiplicativity of the signature as in [13].

positive constant c,

$$\frac{1}{c} g \le g' \le cg.$$

Let $\Omega^i = \Omega^i(Y)$ denote the space of C^∞ i-forms on Y and $L^2 = L^2(Y)$ the L^2 completion of Ω^i with respect to the L^2-metric induced by g. Define d to be the exterior differential with the domain

$$\operatorname{dom} d = \{\alpha \in \Omega^i(Y) \cap L^2(Y);\ d\alpha \in L^2(Y)\}.$$

Put $\Omega^i_{(2)}(Y) = \Omega^i(Y) \cap L^2(Y)$. As usual, let δ denote the formal adjoint of d. In terms of a choice of local orientation for Y, we have $\delta = \pm * d *$, where $*$ is the Hodge star operator. We define the domain of δ by

$$\operatorname{dom} \delta = \{\alpha \in \Omega^i(Y) \cap L^2(Y);\ \delta\alpha \in L^2(Y)\}\,.$$

Note that d, δ have well defined strong closures $\bar{d}$, $\bar{\delta}$. That is, $\alpha \in \operatorname{dom} \bar{d}$ and $\bar{d}\alpha = \eta$ if there is a sequence $\alpha_j \in \operatorname{dom} d$ such that $\alpha_j \to \alpha$ and $d\alpha_j \to \eta$ in L^2.

Usually, the L^2-cohomology of Y is defined by

$$H^i_{(2)}(Y) = \ker d_i / \operatorname{Im} d_{i-1}\,.$$

One can also define the L^2-cohomology using the closure $\bar{d}$. Put

$$H^i_{(2),\#}(Y) = \ker \bar{d}_i / \operatorname{Im} \bar{d}_{i-1}\,.$$

In fact, the the natural map,

$$\iota_{(2)} :\ H^i_{(2)}(Y) \longrightarrow H^i_{(2),\#}(Y)\,,$$

is always an isomorphism.

In general, the image of $\bar{d}$ need not be closed. The reduced L^2-cohomology is defined by

$$\bar{H}^i_{(2)}(Y) = \ker \bar{d}_i / \overline{\operatorname{Im} \bar{d}_{i-1}}\,.$$

The space of L^2-harmonic i-forms $\mathcal{H}^i_{(2)}(Y)$ is the space,

$$\mathcal{H}^i_{(2)}(Y) = \{\theta \in \Omega^i \cap L^2; d\theta = \delta\theta = 0\}.$$

When Y is oriented, the Hodge star operator induces the Poincaré duality isomorphism

$$* \ : \mathcal{H}^i_{(2)}(Y) \to \mathcal{H}^{n-i}_{(2)}(Y). \tag{2.5}$$

Remark. Some authors define the space of harmonic forms differently, using the Hilbert spaces adjoint of $\bar{d}$; see for example, [29]. In this case, the Hodge star operator does not necessarily leave invariant the space of harmonic forms.

Clearly, there is a natural map

$$\mathcal{H}^i_{(2)}(Y) \to H^i_{(2)}(Y)\,. \tag{2.6}$$

The question of when this map is an isomorphism is of crucial interest. The most basic result here is the Kodaira decomposition,

$$L^2 = \mathcal{H}^i_{(2)} \oplus \overline{d\Lambda^{i-1}_0} \oplus \overline{\delta\Lambda^{i+1}_0},$$

which leaves invariant the subspaces of smooth forms. It follows then that

$$\ker \bar{d}_i = \mathcal{H}^i_{(2)} \oplus \overline{d\Lambda^{i-1}_0}.$$

Adopting the terminology of [9], we will say that the Strong Hodge Theorem holds if the natural map (2.6) is an isomorphism. By the above discussion, if Im $\bar{d}$ is closed, then the map in (2.6) is surjective. In particular, this holds if the L^2-cohomology is finite dimensional.

On the other hand, if we assume that Stokes' theorem holds for Y in the L^2 sense, i.e.,

$$\langle \bar{d}\alpha,\ \beta\rangle = \langle \alpha,\ \bar{\delta}\beta\rangle \tag{2.7}$$

for all $\alpha \in \operatorname{dom}\bar{d}$, $\beta \in \operatorname{dom}\bar{\delta}$, or equivalently, for all $\alpha \in \operatorname{dom} d$, $\beta \in \operatorname{dom}\delta$, then one has

$$\mathcal{H}^i_{(2)}(Y) \perp \operatorname{Im}\bar{d}_{i-1},$$

and hence,

$$\mathcal{H}^i_{(2)}(Y) \perp \overline{\operatorname{Im}\bar{d}_{i-1}}\,.$$

Thus, (2.6) is injective in this case. Moreover,

$$\mathcal{H}^i_{(2)}(Y) \cong \bar{H}^i_{(2)}(Y), \tag{2.8}$$

and

$$H^i_{(2)}(Y) = \bar{H}^i_{(2)}(Y) \oplus \overline{\operatorname{Im}\bar{d}_{i-1}}/\operatorname{Im}\bar{d}_{i-1}. \tag{2.9}$$

Here, by the closed graph theorem, the last summand is either 0 or infinite dimensional.

To summarize, if the L^2-cohomology of Y is finite dimensional and Stokes' Theorem holds on Y in the L^2-sense, then the L^2-cohomology of Y is isomorphic to the space of L^2-harmonic forms and therefore, when Y is orientable, Poincáre duality holds as well. Consequently, the L^2 signature of Y is well-defined in this case.

Next, we recall the relative de Rham theory [7] and relative L^2-cohomology. Let $f : S \to Y$ denote a map between manifolds. Define a complex, $(\Omega^*(f), d)$, by

$$\Omega^p(f) = \Omega^p(Y) \oplus \Omega^{p-1}(S), \quad d(\omega,\theta) = (d\omega, f^*(\omega) - d\theta)\,. \tag{2.10}$$

Clearly, $d^2 = 0$, and hence, the corresponding cohomology $H^*(f)$ is well defined.

Put $\alpha(\theta) = (0, \theta)$, $\beta(\omega, \theta) = \omega$, and consider the short exact sequence,

$$0 \to \Omega^{p-1}(S) \xrightarrow{\alpha} \Omega^p(f) \xrightarrow{\beta} \Omega^p(Y) \to 0, \tag{2.11}$$

where the differential of the complex $\Omega^{*-1}(S)$ is $-d$. There is an induced a long exact sequence on the cohomology,

$$\cdots \to H^p(f) \xrightarrow{\beta^*} H^p(Y) \xrightarrow{f^*} H^p(S) \xrightarrow{\alpha^*} H^{p+1}(f) \to \cdots . \tag{2.12}$$

If S is a submanifold of M and $i: S \to Y$ is the inclusion map, we define the relative cohomology, $H^*(Y, S)$, to be $H^*(i)$. To define the relative L^2-cohomology $H^*_{(2)}(Y, S)$, we assume further that S has trivial normal bundle in Y and that the metric in a neighborhood of S is quasi-isometric to the product metric. Then $H^*_{(2)}(Y, S)$ can be defined as the cohomology of the complex,

$$\Omega^*_{(2)}(Y, S) = \Omega^*_{(2)}(Y) \oplus \Omega^{*-1}_{(2)}(S)\,,$$

with $\mathrm{dom}(d) = \{(\omega, \theta) \mid d\omega \in L^2, \ \ d\theta \in L^2.\}$. It follows that the long exact sequence for the pair (Y, S) is also valid in L^2-cohomology.

We note that L^2-cohomology is quasi-isometry invariant, and conformally invariant in the middle dimension. Also Künneth formula holds for the L^2-cohomology. Furthermore, given an open cover $\{U_\alpha\}_\alpha$, the Mayer-Vietoris principle holds for the L^2-cohomology, provided there is a constant, C, such that there is a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$, such that $|df_\alpha| \le C$, for all α . Hence, the Leray spectral sequence in L^2-cohomology is valid for a fibration, if such a partition of unity, subordinate to a trivializing open cover, can be found on the base.[3] Clearly, this holds for the fibrations considered here.

3 L^2-cohomology of generalized Thom spaces

In this section, we begin to specialize to the case of generalized Thom spaces. Thus, we retain the notation of (1.1)–(1.3). The results in this section are special cases, and in some instances, refinements, of those which hold for more general stratified pseudomanifolds; see [9], and also [29]. For completeness and later purposes we provide a somewhat detailed account.

Let N be a Riemannian manifold, possibly incomplete. For simplicity, we assume that $m = \dim N$ is odd. Further, we assume that for N, the L^2-cohomology is finite dimensional and that Stokes' theorem holds in the L^2 sense. Then for the finite cone, $C_{[0,1]}(N)$, over N, we have the following facts from [9].

1) The L^2-cohomology is finite dimensional and Stokes' theorem holds in the L^2 sense. Consequently, the Strong Hodge Theorem holds for the cone.

[3] This can be shown by the usual double complex construction as in [7]

2) For $i \leq m/2$, the restriction map induces an isomorphism,

$$H^i_{(2)}(C_{[0,1]}(N)) \cong H^i_{(2)}(N)\,,$$

and for $i \geq (m+1)/2$,

$$H^i_{(2)}(C_{[0,1]}(N)) = 0\,.$$

The above results are consequences of the following lemmas; for proofs, see [9].

Lemma 3.1. *Let θ be an i-form on N that is in L^2. Let $\tilde{\theta}$ denote the extension of θ to $C_{[0,1]}(N)$ so that $\tilde{\theta}$ is radially constant. Then $\tilde{\theta} \in L^2(C_{[0,1]}(N))$ if and only if $i < (m+1)/2$.*

When there is no danger of confusion, we will just write θ for $\tilde{\theta}$.

For some $a \in (0,1)$, define the homotopy operator K^0 as follows. If $\alpha = \phi + dr \wedge \omega$ is an i-form and $i < (m+1)/2$, then

$$K^0\alpha = \int_a^r \omega\,.$$

If $i \geq (m+1)/2$, then

$$K^0\alpha = \int_0^r \omega.$$

Lemma 3.2 (Poincaré lemma)**.** *For $\alpha \in$ dom $\bar{d}$,*

$$(\bar{d}K^0 + K^0\bar{d})\alpha = \alpha - \alpha(a),\ \ i < (m+1)/2$$

and

$$(\bar{d}K^0 + K^0\bar{d})\alpha = \alpha,\ \ i \geq (m+1)/2.$$

We now turn to the case of fibrations whose fibres are cones.

Theorem 3.3. *The L^2-cohomology of $C_\pi M$ is finite dimensional and Stokes' theorem holds in the L^2 sense. Hence the Strong Hodge Theorem holds.*

Proof. By the Mayer-Vietoris principle, verifying finite dimensionality reduces to verifying finite dimensionality for a product fibration. Since the L^2-cohmology is a quasi-isometry invariant, we can use the product metric. Then the Künneth formula yields the desired result.

It was proved in [9] that if the L^2-Stokes theorem holds locally then it holds globally. Further, the validity of the L^2-Stokes theorem is a quasi-isometry invariant. Thus, once again we can reduce to a product situation and the validity of the L^2-Stokes theorem follows from its validity for $C_{[0,1]}(Z)$ and B. ■

Finally, we give a vanishing statement for L^2-cohomology which is a direct generalization of the one which holds in the case in which the base consists of a single point; [9]. Put $\dim B = \ell$, $\dim Z = n$.

Lemma 3.4.

$$H^i_{(2)}(C_\pi M) = 0 \qquad \text{for } i \geq l + \frac{n+1}{2}\,.$$

Consequently,

$$H^{m+1}_{(2)}(C_\pi M, M) \simeq H^m(M) \cong \mathbf{R}\,.$$

Proof. Assuming the first statement, the second follows from the long exact sequence

$$\to H^i_{(2)}(C_\pi M, M) \to H^i_{(2)}(C_\pi M) \to H^i_{(2)}(M) \to H^{i+1}_{(2)}(C_\pi M, M) \to .$$

To prove the first statement, we make use of K_0, the chain homotopy operator for cones. By defining K^0 fibrewise, we can naturally extend K^0 to a cone bundle. If $\alpha = \phi + dr \wedge \omega$ and $\deg \alpha \geq l + \frac{n+1}{2}$, put

$$K^0 \alpha = \int_0^r \omega\,.$$

Clearly, from what was shown for the case of a single cone, if $\alpha \in L^2(C_\pi M)$, then $K^0\alpha \in L^2(C_\pi M)$. Now we verify

$$(\bar{d}K^0 + K^0\bar{d})\alpha = \alpha.$$

Again, everything is local on B and quasi-isometric invariant, therefore, we can check it for a product fibration with product metric. In this case

$$\bar{d} = \bar{d}_{C_{[0,1]}(Z)} + d_B\,,$$

and $(\bar{d}_{C(Z)}K^0 + K^0\bar{d}_{C(Z)})\alpha = \alpha$. Hence it suffices to check

$$(d_B K^0 + K^0 d_B)\alpha = 0.$$

Note that α can be written as linear combinations of forms of type, $(\phi + dr \wedge \omega) \otimes \tau_B$, where $\phi + dr \wedge \omega$ lives on $C_{[0,1]}(Z)$. Then

$$d_B(K^0\alpha) = d_B\left(\int_0^r \omega \otimes \tau_B\right) = (-1)^{\deg \omega}\left(\int_0^r \omega\right) \otimes d\tau_B,$$

$$K^0 d_B \alpha = (-1)^{\deg \omega + 1}\left(\int_0^r \omega\right) \otimes d\tau_B\,.$$

These terms cancel one another. ■

4 Spectral Sequences

In this section, we recall some basics concerning spectral sequences; a general reference is [7].

A filtered differential complex (K, d) is a differential complex that comes with a filtration by subcomplexes

$$K = K_0 \supset K_1 \supset K_2 \supset \cdots\,. \tag{4.13}$$

A graded filtered complex, (K,d), has in addition, a grading $K=\oplus_{n\in\mathbb{Z}}K^n$. In this case, the filtration (4.13) induces a filtration on each K^n: $K_p^n=K^n\cap K_p$.

One has the following general result; see [7].

Theorem 4.1. *Let (K,d) denote a graded filtered differential complex such that the induced filtration on each K^n has finite length. Then the short exact sequence,*

$$0\to\oplus K_{p+1}\to\oplus K_p\to\oplus K_p/K_{p+1}\to 0\,,$$

induces a spectral sequence, $(E_r^{p,q},\ d_r)$, which converges to the cohomology group $H^*(K,d)$.

In fact, if $d:\ K^n\to K^{n+1}$ is of degree one, then

$$E_r^{p,q}=\frac{d^{-1}(K_{p+r}^{p+q+1})\cap K_p^{p+q}}{d^{-1}(K_{p+r}^{p+q+1})\cap K_{p+1}^{p+q}+K_p^{p+q}\cap d(K_{p-r+1}^{p+q-1})}\,, \tag{4.14}$$

where the differential, d_r, is naturally induced by d.

We now recall Serre's filtration and the Leray spectral sequence of the fibration (1.1). A p-form ω is in F^i if

$$\omega(U_1,U_2,\dots,U_p)=0,$$

whenever $p-i+1$ of the tangent vectors, U_j, are vertical. If the fibration is equipped with a connection, there is a splitting

$$TM=T^HM\oplus T^VM\cong\pi^*TB\oplus T^VM\,.$$

Thus, $\Lambda^*M=\pi^*\Lambda^*B\otimes\Lambda^*T^VM$. Formally, we can use $a(y,z)\,dy^\alpha\wedge dz^\beta$ with y the local coordinates of B, z local coordinates of Y to indicate such a splitting. With this convention Serre's filtration can be simply described as

$$F^i=\{a(y,z)\,dy^\alpha\wedge dz^\beta:|\alpha|\geq i\}.$$

The global definition shows that this filtration is independent of the particular choice of local coordinates.

Definition. The horizontal degree of ω is at least i, h-deg $\omega\geq i$, or equivalently, the vertical degree of ω is at most $p-i$ ($p=\deg\omega$), v-deg $\omega\leq p-i$, if $\omega\in F^i$.

Serre's filtration, together with the grading given by the degree of differential form, gives rise to the Leray spectral sequence of the fibration, (E_r,d_r), which converges to the cohomology of the total space M. Moreover, the E_2 term is given by

$$E_2=\oplus E_2^{p,q},\quad E_2^{p,q}=H^p(B,\mathcal{H}^q(Z))\,, \tag{4.15}$$

where $\mathcal{H}^q(Z)$ denotes the flat bundle over B, for which the fibre over $b\in B$ is the q-th L^2-cohomology of the fibre $\pi^{-1}(b)$.

Now assume that both the base B and the vertical tangent bundle T^VM are oriented. When the fibration is equipped with a submersion metric, we can identify $\mathcal{H}^q(Z)$ with the bundle of fibrewise L^2-harmonic q-forms. This induces a fiberwise metric on $\mathcal{H}^q(Z)$. Note that $\mathcal{H}^q(Z)$ is a flat vector bundle over B. Via Hodge theory for cohomology with coefficients in a flat bundle, we obtain an inner product on $E_2^{p,q} = H^p(B, \mathcal{H}^q(Z))$.

The E_3 term of the spectral sequence is given by the cohomology groups of the E_2 term with respect to the differential d_2. Therefore, finite dimensional Hodge theory gives us

$$E_2 = E_3 \oplus \operatorname{Im} d_2 \oplus \operatorname{Im}(d_2)^*, \tag{4.16}$$

when the adjoint is defined with respect to the inner product introduced above. This in turn, induces an inner product on the E_3 terms. By iterating this construction, we obtain inner products on each of the E_r, with the associated Hodge decomposition.

The above discussion extends to fibrations whose base and fibres are closed manifolds with conical singularities, if we replace the usual cohomology with the L^2-cohomology.

5 The Generalized Thom Isomorphism

This section is devoted to the proof of Theorem 1.3, which is a generalization of the Thom isomorphism theorem. As explained in the introduction, the statement consists of identifying the L^2-cohomology of $C_\pi M$ and $(C_\pi M, M)$ in terms of the Leray spectral sequence of M. Additionally, the map,

$$H^*_{(2)}(C_\pi M, M) \to H^*_{(2)}(C_\pi M)\,,$$

is identified in terms of the differential of the spectral sequence. We begin with $H^*_{(2)}(C_\pi M)$.

Note that since $C_\pi M$ is fibered over B,

$$C_{[0,1]}(Z) \to C_\pi M \to B\,,$$

we also have a spectral sequence, $(E_r^{p,q}(C_\pi M)\bar{d}_r^{p,q})$, converging to the L^2-cohomology of $C_\pi M$. On the other hand, the pullback of the inclusion $i:\ M \to C_\pi M$,

$$i^*:\ \Omega^*(C_\pi M) \to \Omega^*(M),$$

is filtration preserving. Hence, i^* induces homomorphism

$$i_r : E_r^{p,q}(C_\pi M) \to E_r^{p,q}(M)$$

which commutes with the differentials.

Now

$$E_2^{p,q}(C_\pi M) = H^p(B, \mathcal{H}^q(C(Z))) = \begin{cases} 0\,, & q > n/2 \\ H^p(B, \mathcal{H}^q(Z)) = E_2^{p,q}(M), & q < n/2 \end{cases}.$$

It follows that $E_r^{p,q}(C_\pi M) = 0$, for all $q > n/2$ and $r \geq 2$. Moreover, by the Poincaré Lemma 3.2, the identification here is given by i_2, i.e.

$$i_2^{p,q} = \begin{cases} 0, & q > n/2 \\ \text{Ident}, & q < n/2 \end{cases} .$$

It follows that

$$\bar{d}_2^{p,q} = \begin{cases} 0, & q > n/2 \\ d_2^{p,q}, & q < n/2 \end{cases} ,$$

and thus

$$E_3^{p,q}(C_\pi M) = E_3^{p,q}(M), \quad \text{if } q < (n-1)/2,$$

$$E_3^{p,(n-1)/2}(C_\pi M) = \ker\, d_2^{p,(n-1)/2} = E_3^{p,(n-1)/2}(M) \oplus \operatorname{Im} d_2^{p-2,(n+1)/2}.$$

This implies that for $q < (n-1)/2$, $i_3^{p,q} = \text{Ident}$.

Also since $i_3^{p,(n-1)/2}$ is induced by

$$\text{Ident} : E_2^{p,(n-1)/2}(C_\pi M) \to E_2^{p,(n-1)/2}(M)\,,$$

one sees that $i_3^{p,(n-1)/2}$ is the natural projection,

$$E_3^{p,(n-1)/2}(C_\pi M) = \ker\, d_2^{p,(n-1)/2} \to \frac{\ker\, d_2^{p,(n-1)/2}}{\operatorname{Im} d_2^{p-2,(n+1)/2}} = E_3^{p,(n-1)/2}(M)\,.$$

Hence, since i_r commutes with $\bar{d}_r$, d_r, we get

$$\bar{d}_3^{p,q} = d_3^{p,q}\,, \qquad \text{if } q < (n-1)/2\,,$$

and

$$\bar{d}_3^{p,(n-1)/2}\,|\,\operatorname{Im} d_2^{p-2,(n+1)/2} = 0\,,$$
$$\bar{d}_3^{p,(n-1)/2}\,|\,E_3^{p,(n-1)/2}(M) = d_3^{p,(n-1)/2}\,.$$

It follows that

$$\begin{aligned}
E_4^{p,q}(C_\pi M) &= E_4^{p,q}(M) \qquad \text{if } q < (n-1)/2 - 1, \\
E_4^{p,(n-3)/2}(C_\pi M) &= \ker\, d_3^{p,(n-3)/2} \\
&= E_4^{p,(n-3)/2}(M) \oplus \operatorname{Im} d_3^{p-3,(n+1)/2}, \\
E_4^{p,(n-1)/2}(C_\pi M) &= \ker\, d_3^{p,(n-1)/2} \oplus \operatorname{Im} d_2^{p-2,(n+1)/2} \\
&= E_4^{p,(n-1)/2}(M) \oplus \operatorname{Im} d_3^{p-3,(n+3)/2} \oplus \operatorname{Im} d_2^{p-2,(n+1)/2}.
\end{aligned}$$

By an inductive argument, we obtain the following proposition, in which some of the $\operatorname{Im} d$ summands are obviously zero.

Proposition 5.1. *There are equalities,*

$$\begin{aligned}
E_\infty^{p,(n-1)/2}(C_\pi M) &= E_\infty^{p,(n-1)/2}(M) \oplus \operatorname{Im} d_2^{p-2,(n+1)/2} \oplus \operatorname{Im} d_3^{p-3,(n+3)/2} \oplus \cdots \\
&\quad \cdots \oplus \operatorname{Im} d_l^{p-l,(n+1)/2+l-2}, \\
E_\infty^{p,(n-3)/2}(C_\pi M) &= E_\infty^{p,(n-3)/2}(M) \oplus \operatorname{Im} d_3^{p-3,(n+1)/2} \oplus \operatorname{Im} d_4^{p-4,(n+3)/2} \oplus \cdots \\
&\quad \cdots \oplus \operatorname{Im} d_l^{p-l,(n+1)/2+l-3}, \\
&\vdots \\
E_\infty^{p,0}(C_\pi M) &= E_\infty^{p,0}(M) \oplus \operatorname{Im} d_{(n+3)/2}^{p-(n+3)/2,(n+1)/2} \oplus \operatorname{Im} d_{(n+5)/2}^{p-(n+5)/2,(n+3)/2} \oplus \cdots \\
&\quad \cdots \oplus \operatorname{Im} d_l^{p-l,l-1}.
\end{aligned}$$

Similarly, we have:

Proposition 5.2. *For the relative cohomology,*

$$\begin{aligned}
E_\infty^{p,(n+3)/2}(C_\pi M, M) &= E_\infty^{p,(n+1)/2}(M) \oplus \operatorname{Im}(d_2^{p,(n+1)/2})^* \oplus \operatorname{Im}(d_3^{p,(n+1)/2})^* \oplus \cdots \\
&\quad \cdots \oplus \operatorname{Im}(d_l^{p,(n+1)/2})^*, \\
E_\infty^{p,(n+5)/2}(C_\pi M, M) &= E_\infty^{p,(n+3)/2}(M) \oplus \operatorname{Im}(d_3^{p,(n+3)/2})^* \oplus \operatorname{Im}(d_4^{p,(n+3)/2})^* \oplus \cdots \\
&\quad \cdots \oplus \operatorname{Im}(d_l^{p,(n+3)/2})^*, \\
&\vdots \\
E_\infty^{p,n+1}(C_\pi M, M) &= E_\infty^{p,n}(M) \oplus \operatorname{Im}(d_{(n+3)/2}^{p,n})^* \oplus \operatorname{Im}(d_{(n+5)/2}^{p,n})^* \oplus \cdots \\
&\quad \cdots \oplus \operatorname{Im}(d_l^{p,n})^*.
\end{aligned}$$

Proof. Recall that

$$\Omega^*_{(2)}(C_\pi M, M) = \Omega^*_{(2)}(C_\pi M) \oplus \Omega^{*-1}_{(2)}(M)\,.$$

Clearly, the map,

$$\begin{array}{ccc} \Omega^{*-1}_{(2)}(M) & \to & \Omega^*_{(2)}(C_\pi M, M) \\ \theta & \to & (0,\theta)\,, \end{array}$$

is filtration preserving and commuting with differentials. Hence, this map induces homomorphisms

$$j_r:\ (E_r^{p,q-1}(M), d_r) \to (E_r^{p,q}(C_\pi M, M), \tilde{d}_r)\,.$$

Moreover,

$$E_2^{p,q}(C_\pi M, M) = \begin{cases} H^p(B, \mathcal{H}^{q-1}(Z)) = E_2^{p,q-1}(M), & q > n/2+1 \\ 0, & q < n/2+1 \end{cases}.$$

In terms of this identification,

$$j_2^{p,q} = \begin{cases} \mathrm{id} & q > n/2+1 \\ 0 & q < n/2+1 \end{cases}.$$

It follows that

$$\tilde{d}_2^{p,q} = \begin{cases} d_2^{p,q-1} & q > (n+3)/2 \\ 0 & q \leq (n+3)/2 \end{cases},$$

and thus

$$E_3^{p,q}(C_\pi M, M) = E_3^{p,q-1}(M), \text{ if } q > (n+3)/2\,,$$

$$E_3^{p,(n+3)/2}(C_\pi M, M) = \frac{E_2^{p,(n+1)/2}(M)}{\operatorname{Im} d_2} = E_3^{p,(n+1)/2}(M) \oplus \operatorname{Im}(d_2^{p,(n+1)/2})^*\,.$$

This implies that $j_3^{p,q} = \text{Ident}$, for $q > (n+3)/2$.

Again, since $j_3^{p,(n+3)/2}$ is induced by

$$\text{Ident} : E_2^{p,(n+1)/2}(M) \to E_2^{p,(n+3)/2}(C_\pi M, M)\,,$$

one sees that $j_3^{p,(n+3)/2}$ is the natural inclusion. It follows that

$$\tilde{d}_2^{p,q} = \begin{cases} d_2^{p,q-1} & q > (n+3)/2 \\ 0 & q \leq (n+3)/2 \end{cases},$$

and therefore,

$$E_4^{p,q}(C_\pi M, M) = E_4^{p,q}(M), \qquad \text{if } q > (n+5)/2,$$

$$E_4^{p,(n+5)/2}(C_\pi M, M) = E_4^{p,(n+3)/2}(M) \oplus \operatorname{Im}(d_3^{p,(n+3)/2})^*,$$

$$E_4^{p,(n+3)/2}(C_\pi M, M) = E_4^{p,(n+1)/2}(M) \oplus \operatorname{Im}(d_3^{p,(n+1)/2})^* \oplus \operatorname{Im} d_2^{p,(n+1)/2})^*\,.$$

By proceeding inductively, the desired result follows. ∎

We are now ready to prove Theorem 1.3.

Proof. From the general theory of the spectral sequence, the filtration on $\Omega^*_{(2)}(C_\pi M)$ induces a filtration on $H^*_{(2)}(C_\pi M)$,

$$H^*_{(2)}(C_\pi M) \supset F^0 H^*_{(2)}(C_\pi M) \supset F^1 H^*_{(2)}(C_\pi M) \supset \cdots \supset 0,$$

such that

$$E_\infty^{p,q}(C_\pi M) \cong F^p H^{p+q}_{(2)}(C_\pi M)/F^{p+1} H^{p+q}_{(2)}(C_\pi M).$$

Therefore,

$$\begin{aligned} H^k(C_\pi M) &\cong \oplus_{p+q=k} E_\infty^{p,q}(C_\pi M) \\ &\cong \oplus_{p+q=k, q\leq (n-1)/2}[E_\infty^{p,q}(M) \oplus \operatorname{Im}(d^{k-(n+3)/2,(n+1)/2}_{(n+3)/2-q}) \oplus \operatorname{Im}(d^{k-(n+5)/2,(n+3)/2}_{(n+5)/2-q}) \oplus \cdots]. \end{aligned}$$

Similarly

$H^k(C_\pi(M),M) \cong \oplus_{p+q=k,q\geq(n+3)/2}[E_\infty^{p,q-1}(M)\oplus \mathrm{Im}(d^{p,q-1}_{q-(n-1)/2})^* \oplus \mathrm{Im}(d^{p,q-1}_{q-(n-1)/2+1})^* \oplus \cdots]$.

We claim that in terms of this identification, the map,

$$H^k(C_\pi(M),M) \to H^k(C_\pi(M))\,,$$

is given by applying appropriate d_r's to the appropriate factors. To verify that this is the case, we need to trace back the isomorphisms. It suffices to look at, for example, $\mathrm{Im}(d_2^{p,(n+1)/2})^*$.

Let $[\theta] \in \mathrm{Im}(d_2^{p,(n+1)/2})^* \subset E_2^{p,(n+1)/2}(M)$. In this case, θ can be represented by a $(p+(n+1)/2)$-form, such that

$$\theta \in F^p,\ d\theta \in F^{p+2}.$$

Therefore, v-deg $d\theta \leq p+(n+1)/2-(p+2) \leq (n-1)/2$. By Lemma 3.1, $d\theta$ can be extended to an L^2-form on $C_\pi(M)$. To make $[0,\theta]$ an element of $E_\infty^{p,(n+1)/2}(C_\pi(M),M)$, we modify its representative slightly:

$$(d\theta,\theta) \in [0,\theta] \text{ and } [d\theta,\theta] \in E_\infty^{p,(n+1)/2}(C_\pi(M),M)\,.$$

This implies that

$$(d\theta,\theta) \in F^pH_{(2)}^{p+(n+1)/2}(C_\pi M,M),$$

which is mapped to

$$d\theta \in F^pH_{(2)}^{p+(n+1)/2}(C_\pi M),$$

which, via the identification with the spectral sequence terms, is

$$d\theta \,|\, M = d_M\theta.$$

This is exactly $d_2[\theta]$. The rest of the terms can be treated in exactly the same fashion. ■

6 L^2-signatures of Generalized Thom Spaces

Assume that that $\dim T = m+1$ is divisible by 4.

By definition, the L^2-signature, $\mathrm{sign}_{(2)}(T)$, of the generalized Thom space, (T,g), is the signature of the pairing,

$$H_{(2)}^{(m+1)/2}(T)\otimes H_{(2)}^{(m+1)/2}(T) \to \mathbf{R}\,,$$

induced by wedge product and integration.

The L^2-signature of $C_\pi M$ as a (singular) manifold with boundary, $\partial(C_\pi M)=M$, is, by definition, the signature of the (possibly degenerate) pairing

$$H_{(2)}^{(m+1)/2}(C_\pi M,M)\otimes H_{(2)}^{(m+1)/2}(C_\pi M,M) \to \mathbf{R}\,, \tag{6.17}$$

defined via the map

$$H^k(C_\pi(M), M) \to H^k(C_\pi(M))\,, \tag{6.18}$$

and the (nondegenerate) pairing

$$H^{m+1-k}_{(2)}(C_\pi M, M) \otimes H^k_{(2)}(C_\pi M) \to \mathbf{R}\,, \tag{6.19}$$

given by

$$([\omega,\theta],\ [\alpha]) \to \int_{C_\pi M} \omega\wedge\alpha - \int_M \theta\wedge\alpha. \tag{6.20}$$

As in [9], one has

$$\mathrm{sign}_{(2)}(T) = \mathrm{sign}_{(2)}(C_\pi M).$$

Thus, to prove Theorem 1.1, we just have to compute the L^2-signature of $C_\pi M$. The computation is in the spirit of Example 2 of the introduction.

Proof of Theorem 1.1:

The outline of the proof is as follows. First, we consider the intersection matrix with respect to the block decompositions in terms of $E_\infty^{r,s}(C_\pi M, M)$ and $E_\infty^{p,q}(C_\pi M)$, and show that the matrix is lower anti-diagonal. Then we consider each of these anti-diagonal blocks and show that they are also anti-diagonal with respect to the block decompositions given by the Thom isomorphism. Finally, we identify the pairings along the anti-diagonals and show that they give rise to the τ invariant.

Consider the pairing between $H^k_{(2)}(C_\pi M)$ and $H^{m+1-k}_{(2)}(C_\pi M, M)$. Let $\ell = \dim B$, $n = \dim Z$. (In what follows below, we make a change of index.)

First of all, we show that in terms of the identifications,

$$H^k(C_\pi(M)) \cong$$
$$\oplus_{p+q=k, q\le (n-1)/2}[E_\infty^{p,q}(M) \oplus \mathrm{Im}\, d_{(n+3)/2-q}^{k-(n+3)/2,(n+1)/2} \oplus \mathrm{Im}\, d_{(n+5)/2-q}^{k-(n+5)/2,(n+3)/2} \oplus \cdots,]$$

and

$$H^{m+1-k}(C_\pi(M), M) \cong$$
$$\oplus_{p'+q'=k, q'\le (n-1)/2}[E_\infty^{l-p',n-q'}(M) \oplus \mathrm{Im}\,(d_{(n+3)/2-q'}^{l-p',n-q'})^* \oplus \mathrm{Im}\,(d_{(n+5)/2-q'}^{l-p',n-q'})^* \oplus \cdots]$$

the pairing has the form of the block matrix,

$$\begin{pmatrix} 0 & \cdots & 0 & * \\ 0 & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \end{pmatrix},$$

with the blocks on the anti-diagonal coming from the pairing between $E_\infty^{p,q}(M)$ and $E_\infty^{l-p,n-q}(M)$, $E^{p,q}_{(n+3)/2-q}(M) \supset \mathrm{Im}\, d_{(n+3)/2-q}^{k-(n+3)/2,(n+1)/2}$ and $E^{l-p,n-q}_{(n+3)/2-q}(M) \supset \mathrm{Im}\, d_{q-(n-1)/2}^{l-p,n-q})^*$, etc..

To see this, we observe that in terms of the identification,

$$H^k(C_\pi M) \cong \oplus_{p+q=k} E_\infty^{p,q}(C_\pi M)\,,$$

and

$$H^{m+1-k}(C_\pi M, M) \cong \oplus_{p'+q'=k} E_\infty^{l-p',n+1-q'}(C_\pi M, M)\,,$$

the pairing is of the form

$$\begin{pmatrix} 0 & \cdots & 0 & * \\ 0 & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \end{pmatrix},$$

with the entries on the anti-diagonal coming from the pairing between $E_\infty^{l-p,n+1-q}(C_\pi M, M)$ and $E_\infty^{p,q}(C_\pi M)$. This is a consequence of the following formal properties of the spectral sequence:

$$E_\infty^{p,k-p} = F^p H^k / F^{p+1} H^k\,,$$

$$F^p(C_\pi M, M) \cdot F^{p'}(C_\pi M) \subset F^{p+p'}((C_\pi M, M))\,,$$

and

$$F^{p+p'} = 0\,, \text{ if } p + p' > l\,.$$

We now consider the pairings between $E_\infty^{l-p,n+1-q}(C_\pi M, M)$ and $E_\infty^{p,q}(C_\pi M)$. For this we need to once again look at the isomorphisms in Propositions 5.1 and 5.2. As we have shown before, an element $[\theta] \in E_r^{l-p,n-q}(M)$ lifts to $[d\theta, \theta] \in E_\infty^{l-p,n+1-q}(C_\pi(M), M)$, where $r = (n+3)/2 - q + k$ with k a nonnegative integer. On the other hand, for $q \le (n-1)/2$, an element, $[\omega] \in E_\infty^{p,q}(M)$, $\operatorname{Im} d_{(n+3)/2-q}^{k-(n+3)/2,(n+1)/2}$, $\cdots$, lifts to $[\omega] \in E_\infty^{p,q}(C_\pi M)$. Hence, by tracing through the isomorphisms, we find that the pairing is given by

$$\langle [\theta], [\omega] \rangle = \int_{C_\pi M} d\theta \wedge \omega - \int_M \theta \wedge \omega\,.$$

Now

$$\text{h-}\deg d\theta \wedge \omega \ge \text{h-}\deg d\theta + \text{h-}\deg \omega \ge l + 2\,,$$

which implies

$$\omega \wedge d\theta = 0\,.$$

Thus,

$$\langle [\theta], [\omega] \rangle = - \int_M \theta \wedge \omega.$$

If both $[\theta]$ and $[\omega]$ are elements of E_r, i.e if both are at the same level, this yields

$$\langle [\theta], [\omega] \rangle = -\langle [\theta] \cdot [\omega], \xi_r \rangle.$$

We now show that if $[\theta]$ and $[\omega]$ are in different levels of the spectral sequence, then the pairing,

$$\int_M \theta \wedge \omega \,,$$

is of the desired triangular form. We do it for

$$[\theta] \in \mathrm{Im}\, d^{k-(n+3)/2,(n+1)/2}_{(n+3)/2-q}, \;\; [\omega] \in \mathrm{Im}\, (d^{l-p,n-q}_{(n+5)/2-q})^* \,.$$

The remaining cases are similar.

By assumption $\theta = d\alpha$, for some $\alpha \in F^{k-(n+3)/2}$. Therefore, by the L^2-Stokes' theorem,

$$\int_M \theta \wedge \omega = (-1)^k \int_M \alpha \wedge d\omega.$$

Since

$$h\text{-}\deg \alpha \wedge d\omega \geq k - (n+3)/2 + l - p + (n+5)/2 - q = l + 1 \,,$$

it follows that $\alpha \wedge d\omega = 0$ as claimed.

To compute the signature of $C_\pi M$, we note that the pairing (6.17) factors through the pairing (6.19), via the map in (6.18). It is known that the radical for this pairing is

$$\mathrm{Im}(H^{k-1}(M) \to H^k(C_\pi M, M)) \,,$$

which in terms of the Leray spectral sequence, is given by

$$\oplus_{p'+q'=k, q' \leq (n-1)/2} E^{l-p',n-q'}_\infty(M) \,.$$

It follows that we can choose our decomposition so that the pairing will look like

$$\begin{pmatrix} 0 & \cdots 0 \\ 0 & \begin{pmatrix} 0 & \cdots & 0 & * \\ 0 & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \end{pmatrix} \end{pmatrix} ,$$

with the entries on the anti-diagonal given by the nondegenerate pairing

$$\begin{array}{ccccc} \mathrm{Im}\,(d^{p,q}_r)^* & \otimes & \mathrm{Im}\,(d^{p,q}_r)^* & \to & \mathbf{R} \\ \cap & & \cap & & \\ E^{p,q}_r & \otimes & E^{p,q}_r & \to & \mathbf{R} \\ \varphi & & \psi & \to & -\langle \varphi \cdot d_r\psi, \xi_r \rangle \end{array} \tag{6.21}$$

for $[\frac{n+r}{2}] \leq q \leq [\frac{n+r}{2}] + r - 2$ and $p + q = \frac{m-1}{2}$. Now, one can deform the matrix, without changing its signature, to one whose entries are all zero except on the anti-diagonal. Consequently, $\mathrm{sign}_{(2)}(T)$ is given by the signature of the pairing (6.21) restricted to the direct sum of the $E^{p,q}_r$, with $[\frac{n+r}{2}] \leq q \leq [\frac{n+r}{2}] + r - 2$ and $p + q = \frac{m-1}{2}$.

To get the final result, we observe the following symmetry of the pairing on $E_r^{\frac{m-1}{2}} = \oplus_{p+q=\frac{m-1}{2}} E_r^{p,q}$. Recall the inner product, $(\ ,\ )_r$, on E_r, defined at the end of Section 4. Also, there is a natural basis, $\xi_r \in E_r^{l,n}$, (i.e. a volume element) constructed from the orientation. Define the finite dimensional star operator, $\star_r$, by

$$\langle \varphi \cdot \psi, \xi_r \rangle = (\varphi, \star_r \psi)_r .$$

In fact, $\star_2$ coincides with the usual $*$ operator when E_2 is identified with the harmonic differential forms on the base with values in the harmonic differential forms along the fibres. Then τ_r, the signature of the pairing on $E_r^{\frac{m-1}{2}}$, is exactly the signature of the self adjoint operator, $\star_r d_r$, on $E_r^{\frac{m-1}{2}}$. Now,

$$\star_r d_r : \ E_r^{p,q} \to E_r^{l-p-r,n-q+r-1}.$$

Moreover $q = n - q + r - 1$ iff $q = \frac{n+r-1}{2}$. It follows that we have the decomposition,

$$E_r^{\frac{m-1}{2}} = A_0 \oplus A_1 \oplus A_2 ,$$

where if r is even,

$$A_0 = E_r^{(l-r)/2,(n+r-1)/2} ,$$

and otherwise, $A_0 = 0$.

Also,

$$A_1 = \sum_{p+q=\frac{m-1}{2}, q<\frac{n+r-1}{2}} E_r^{p,q} ,$$

and

$$A_2 = \sum_{p+q=\frac{m-1}{2}, q>\frac{n+r-1}{2}} E_r^{p,q} .$$

With respect to this decomposition, the operator $\star_r d_r$ restricts to $\star_r d_r$ on A_0, and has the form

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} .$$

Hence, the spectrum of this operator is symmetric about 0. This shows that only the term, $E_r^{(l-r)/2,(n+r-1)/2}$, contributes to the signature, τ_r, and in fact, only when r is even. This completes the proof of Theorem 0.1. ■

7 L^2-cohomology of hyperkähler manifolds

In recent years, there has been considerable interest in L^2-harmonic forms on noncompact moduli spaces arising in gauge theory. Typically, these spaces come equipped with hyperkähler structures. In [17], using Gromov's Kähler hyperbolicity trick as adapted by Jost-Zuo

[20], (see also [8]) Hitchen showed that for complete hyper-kähler manifolds, with one of the Kähler forms having linear growth, the L^2-harmonic forms are all concentrated in the middle dimension. Moreover, these L^2-harmonic forms are either all self-dual or all anti-self-dual. Thus, on these hyper-kähler manifolds, the L^2-cohomology is completely determined by the L^2-signature.

Theorem 7.1 (Hitchin)**.** *Let M be a complete hyper-kähler manifold of real dimension $4k$ such that one of the Káhler forms $\omega_i = d\beta$ where β has linear growth. Then any L^2 harmonic forms is primitive and of the type (k,k) with respect to all complex structures. Therefore they are anti-self-dual when k is odd and self-dual when k is even.*

The complete hyper-kähler manifolds that are ALE have been classified by Kronheimer [21]. The general classification remains open. So far, all known examples come with a fibered (or more generally, stratified) geometric structure at infinity. In such cases, our results can be applied. More precisely, consider a complete hyperkähler manifold, M, of real dimension $4k$, such that

$$M = M_0 \cup M_1, \cup \cdots \cup M_r \,, \tag{7.22}$$

where M_0 is a compact manifold with boundary and each of $M_i,\ 1 \le i \le r$, is a geometrically fibered ends ([16]). By this we mean that there is a fibration

$$Z_i \to Y_i \xrightarrow{\pi} B_i$$

such that $M_i = [1,\ \infty) \times Y_i$, and themetric is quasi-isometric to the fibered cusp metric,

$$g_{M_i} = dr^2 + \pi^* g_{B_i} + e^{-2r} g_{Z_i} \,, \tag{7.23}$$

respectively, the fibered boundary metric

$$g_{M_i} = dr^2 + r^2 \pi^* g_{B_i} + g_{Z_i} \,. \tag{7.24}$$

Such metrics appear in the ALF and ALG gravitational instantons, for example, Taub-NUT space.

Theorem 7.2. *With the same hypothesis as above, we have*

$$\operatorname{sign}_{(2)}(M) = \operatorname{sign}(M_0) + \sum_{i=1}^{r} \tau(M_i),$$

Proof. We note that our result on the conical singularity extends easily to cusp singularities. This is because it depends only on two ingredients: Lemma 3.1 on the radially constant forms and the Poincaré Lemma, Lemma 3.2. Both of these are true for cusp singularities; see [31, 32]. Also, the fibered conical end is conformally equivalent to a fibered cusp end. Since the middle dimensional L^2-cohomology is conformally invariant, the theorem follows. ∎

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