An Introduction to L^2 Cohomology

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This paper consists of two parts. In the first part, we give an introduction to L^2 cohomology. This is partly based on [8]. We focus on the analytic aspect of the L^2 cohomology theory. For the topological story, we refer to [1, 22, 31] and of course the original papers [16, 17]. For the history and comprehensive literature, see [29]. The second part is based on our joint work with Jeff Cheeger [11] which gives the contribution to the L^2 signature from non-isolated conical singularity.

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1 L^2 Cohomology—What and Why

1.1 What is L^2 cohomology?

The de Rham theorem provides one of the most useful connections between the topological and differential structure of a manifold. The differential structure enters the de Rham complex, which is the cochain complex of smooth exterior differential forms on a manifold M, with the exterior derivative as differential:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots$$

The de Rham Theorem says that the de Rham cohomology, the cohomology of the de Rham complex, $H^k_{\mathrm{dR}}(M) \stackrel{\mathrm{def}}{=} \ker d_k / \mathrm{Im}\, d_{k-1}$, is isomorphic to the singular cohomology:

$$H^k_{\mathrm{dR}}(M) \cong H^k(M; \mathbf{R}).$$

The situation can be further rigidified by introducing geometry into the picture. Let g be a Riemannian metric on M. Then g induces an L^2-metric on $\Omega^k(M)$. As usual, let δ denote the formal adjoint of d. In terms of a choice of local orientation for M, we have $\delta = \pm * d *$, where $*$ is the Hodge star operator. Define the Hodge Laplacian to be

$$\Delta = d\delta + \delta d.$$

A differential form ω is harmonic if $\Delta\omega = 0$.

The great theorem of Hodge then states that, for a closed Riemannian manifold M, every de Rham cohomology class is represented by a unique harmonic form. This theorem provides a direct bridge between topology and analysis of manifolds through geometry, and has had many remarkable applications.

Naturally then, one would like to extend the theory to noncompact manifolds and manifolds with singularity. The de Rham cohomology is still defined (one would restrict to the smooth open submanifold of a manifold with singularity). However, it does not capture the information at infinity or at the singularity.

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One way of remedying this is to restrict to a subcomplex of the usual de Rham complex, namely that of the square integrable differential forms—this leads us to the L^2 cohomology.

More precisely, let (Y, g) denote an open (possibly incomplete) Riemannian manifold, $\Omega^i = \Omega^i(Y)$ the space of C^∞ i-forms on Y and $L^2 = L^2(Y)$ the L^2 completion of Ω^i with respect to the L^2-metric. Define d to be the exterior differential with the domain

$$\operatorname{dom} d = \{\alpha \in \Omega^i(Y) \cap L^2(Y);\ d\alpha \in L^2(Y)\}.$$

Put

$$\Omega^i_{(2)}(Y) = \Omega^i(Y) \cap L^2(Y).$$

Then, one has cochain complex

$$0 \to \Omega^0_{(2)}(Y) \overset{d}{\to} \Omega^1_{(2)}(Y) \overset{d}{\to} \Omega^2_{(2)}(Y) \overset{d}{\to} \Omega^3_{(2)}(Y) \to \cdots .$$

The L^2-cohomology of Y is defined by to be the cohomology of this cochain complex:

$$H^i_{(2)}(Y) = \ker d_i / \operatorname{Im} d_{i-1}\,.$$

Thus defined, the L^2 cohomology is in general no longer a topological invariant. However, the L^2 cohomology depends only on the quasi-isometry class of the metric.

Examples

- The real line: For the real line $\mathbb{R}$ with the standard metric, one has

  $$H^i_{(2)}(\mathbb{R}) = \begin{cases} 0, \quad i = 0 \\ \infty \text{ dimensional!}, \quad i = 1. \end{cases}$$

  For the first part, this is because constant functions can never be L^2, unless they are zero. For the second part, a 1-form $\phi(x)\,dx$, with $\phi(x)$ having compact support, is obviously closed and L^2, but can never be the exterior derivative of an L^2 function, unless the total integral of ϕ is zero.

- Finite cone: Let $C(N) = C_{[0,1]}(N) = (0,\ 1) \times N$, N a closed manifold of dimension n, with the conical metric $g = dr^2 + r^2 g_N$. Then a result of Cheeger [8] gives:

  $$H^i_{(2)}(C(N)) = \begin{cases} H^i(N) & \text{if } i < (n+1)/2, \\ 0 & \text{if } i \geq (n+1)/2. \end{cases}$$

  Intuitively this can be explained by the fact that some of the differential forms that define classes for the cylinder $N \times (0,1)$ cannot be L^2 on the cone if their degrees are too big. More specifically, let ω be an i-form on N and extend it trivially to $C(N)$, i.e., constant along the radial directions. Then

  $$\int_{C(N)} |\omega|_g^2 d\,\mathrm{vol}_g = \int_0^1 \int_N |\omega|_{g_N} r^{n-2i} dx\ dr.$$

  Thus, the integral is infinite if $i \geq (n+1)/2$.

As we mentioned, the L^2 cohomology is in general no longer a topological invariant. Now clearly, there is a natural map

$$H^i_{(2)}(Y) \longrightarrow H^i(Y, \mathbb{R})$$

via the usual de Rham cohomology. However, this map is generally neither injective nor surjective. On the other hand, in the case when (Y, g) is a compact Riemannian manifold with corner (for a precise definition see the article by Gilles Carron in this volume), the map above is an isomorphism because the L^2 condition is automatically satisfied for any smooth forms.

Also, another natural map is from the compact supported cohomolgy to the L^2 cohomology:

$$H_c^i(Y) \longrightarrow H_{(2)}^i(Y).$$

As above, this map is also neither injective nor surjective in general.

Instead, the L^2 cohomology of singular spaces is intimately related to the intersection cohomology of Goresky-MacPherson ([16, 17], see also Greg Friedman's article in this volume for the intersection cohomology). This connection was pointed out by Dennis Sullivan who observed that Cheeger's local computation of L^2 cohomology for isolated conical singularity agrees with that of Goresky-MacPherson for the middle intersection homology. In [8], Cheeger established the isomorphism of the two cohomology theories for admissible pseudomanifolds. One of the fundamental questions has been the topological interpretation of the L^2 cohomology in terms of the intersection cohomology of Goresky-MacPherson.

1.2 Reduced L^2 Cohomology and L^2 Harmonic Forms

In analysis, one usually works with complete spaces. That means, in our case, the full L^2 space instead of just smooth forms which are L^2. Now the coboundary operator d has well defined strong closure $\bar{d}$ in L^2: $\alpha \in \operatorname{dom} \bar{d}$ and $\bar{d}\alpha = \eta$ if there is a sequence $\alpha_j \in \operatorname{dom} d$ such that $\alpha_j \to \alpha$ and $d\alpha_j \to \eta$ in L^2. Similarly, δ has the strong closure $\bar{\delta}$.

One can also define the L^2-cohomology using the strong closure $\bar{d}$. Thus, define

$$H_{(2),\#}^i(Y) = \ker \bar{d}_i / \operatorname{Im} \bar{d}_{i-1}\,.$$

Then, the natural map,

$$\iota_{(2)}:\ H_{(2)}^i(Y) \longrightarrow H_{(2),\#}^i(Y)\,,$$

turns out to be always an isomorphism [8].

This is good, but does not produce any new information ....yet! The crucial observation is that, in general, the image of $\bar{d}$ need not be closed. This leads to the notion of reduced L^2-cohomology, which is defined by quotienting out by the closure instead:

$$\bar{H}_{(2)}^i(Y) = \ker \bar{d}_i / \overline{\operatorname{Im} \bar{d}_{i-1}}\,.$$

The reduced L^2-cohomology is generally not a cohomology theory but it is intimately related to the Hodge theory as we will see.

Now we define the space of L^2-harmonic i-forms $\mathcal{H}_{(2)}^i(Y)$ to be the space

$$\mathcal{H}_{(2)}^i(Y) = \{\theta \in \Omega^i \cap L^2; d\theta = \delta\theta = 0\}.$$

We remark that some authors define the L^2-harmonic forms differently (cf. [31]). The definitions coincide when the manifold is complete. The advantage of our definition is that, when Y is oriented, the Hodge star operator induces

$$*\ : \mathcal{H}_{(2)}^i(Y) \to \mathcal{H}_{(2)}^{n-i}(Y),$$

which is naturally the Poincaré duality isomorphism.

Now the big question is,"do we still have a Hodge theorem?".

1.3 Kodaira decomposition, L^2 Stokes and Hodge Theorems

To answer the question, let's look at the natural map, the Hodge map

$$\mathcal{H}^i_{(2)}(Y) \longrightarrow H^i_{(2)}(Y)\,.$$

Then the question becomes when this map is an isomorphism. Following Cheeger [8], when the Hodge map is an isomorphism, we will say that the Strong Hodge Theorem holds.

The most basic result in this direction is the Kodaira decomposition [23] (see also [14]),

$$L^2 = \mathcal{H}^i_{(2)} \oplus \overline{d\Lambda^{i-1}_0} \oplus \overline{\delta\Lambda^{i+1}_0},$$

an orthogonal decomposition which leaves invariant the subspaces of smooth forms. Here subscript "0" denotes having compact support. This result is essentially the elliptic regularity.

It follows from the Kodaira decomposition that

$$\ker \bar{d}_i = \mathcal{H}^i_{(2)} \oplus \overline{d\Lambda^{i-1}_0}.$$

Therefore the question is reduced to what the space $\operatorname{Im} \bar{d}_{i-1}$ is in the decomposition. We divide the discussion into two parts: surjectivity and injectivity.

Surjectivity: If $\operatorname{Im} \bar{d}$ is closed, then $\operatorname{Im} \bar{d} \supset \overline{d\Lambda^{i-1}_0}$. Hence, the Hodge map is surjective in this case.

In particular, this holds if the L^2-cohomology is finite dimensional.

Injectivity: The issue of injectivity of the Hodge map has to do with the L^2 Stokes theorem. We say that Stokes' theorem holds for Y in the L^2 sense, if

$$\langle \bar{d}\alpha,\ \beta\rangle = \langle \alpha,\ \bar{\delta}\beta\rangle$$

for all $\alpha \in \operatorname{dom} \bar{d}$, $\beta \in \operatorname{dom} \bar{\delta}$; or equivalently, one has

$$\langle d\alpha,\ \beta\rangle = \langle \alpha,\ \delta\beta\rangle$$

$\alpha \in \operatorname{dom} d$, $\beta \in \operatorname{dom} \delta$.

If the L^2 Stokes theorem holds, then one has

$$\mathcal{H}^i_{(2)}(Y) \perp \operatorname{Im} \bar{d}_{i-1},$$

and consequently, the Hodge map is injective in this case. Moreover,

$$H^i_{(2)}(Y) = \mathcal{H}^i_{(2)}(Y) \oplus \overline{\operatorname{Im} \bar{d}_{i-1}}/\operatorname{Im} \bar{d}_{i-1}.$$

Here, by the closed graph theorem, the last summand is either 0 or infinite dimensional. Note also, since it follows that

$$\mathcal{H}^i_{(2)}(Y) \perp \overline{\operatorname{Im} \bar{d}_{i-1}}\,,$$

one has,

$$\bar{H}^i_{(2)}(Y) \cong \mathcal{H}^i_{(2)}(Y).$$

That is, when the L^2 Stokes theorem holds, the reduced L^2 cohomology is simply the space of L^2 harmonic forms.

Summarizing the discussion above, if the L^2-cohomology of Y is finite dimensional and Stokes' Theorem holds on Y in the L^2-sense, then the Hodge theorem holds in this case, and L^2-cohomology of Y is isomorphic to the space of L^2-harmonic forms. Therefore, when Y is orientable, Poincáre duality holds as well. Consequently, the L^2 signature of Y is well-defined in this case.

Regarding the L^2 Stokes theorem, there are several now classical results. By Gaffney [15], L^2 Stokes theorem holds for complete Riemannian manifolds. On the other hand, for manifolds with conical singularity $M = M_0 \cup C(N)$, the general result of Cheeger [9] says that the L^2 Stokes theorem holds provided that L^2 Stokes holds for N and in addition the middle dimensional (L^2) cohomology group of N vanishes if $\dim N$ is even. In particular, if N is a closed manifold of odd dimension, or $H^{\dim N/2}(N) = 0$ if $\dim N$ is even, the L^2 Stokes theorem holds for M.

Remark: There are various extensions of L^2 cohomology, for example, to cohomology with coefficients or the Dolbeault cohomology for complex manifolds.

2 L^2 Signature of Non-isolated Conical Singularity

2.1 Non-isolated Conical Singularity

We now consider manifolds with non-isolated conical singularity whose strata are smooth manifolds themselves. In other words, singularities are of the following type:
a). Singular stratum consists of disjoint unions of smooth submanifolds.
b). Singularity structure along the normal directions is conical.

More precisely, a neighborhood of a singular stratum of positive dimension can be described as follows. Let

$$Z^n \to M^m \xrightarrow{\pi} B^l \tag{1}$$

be a fibration of closed oriented smooth manifolds. Denote by $C_\pi M$ the mapping cylinder of the map $\pi : M \to B$. This is obtained from the given fibration by attaching a cone to each of the fibres. Indeed, we have

$$C_{[0,1]}(Z) \to C_\pi M \to B\,.$$

The space $C_\pi M$ also comes with a natural quasi-isometry class of metrics. A metric can be obtained by choosing a submersion metric on M:

$$g_M = \pi^* g_B + g_Z.$$

Then, on the nonsingular part of $C_\pi M$, we take the metric,

$$g_1 = dr^2 + \pi^* g_B + r^2 g_Z. \tag{2}$$

The general class of spaces with non-isolated conical singularities as above can be described as follows. A space X in the class will be of the form

$$X = X_0 \cup X_1 \cup \cdots \cup X_k,$$

where X_0 is a compact smooth manifold with boundary, and each X_i (for $i = 1, \ldots, k$) is the associated mapping cylinder, $C_{\pi_i} M_i$, for some fibration, (M_i, π_i), as above.

More generally, one can consider the iterated construction where we allow manifolds in our initial fibration to have singularities of the type considered above. However we will restrict ourselves to the simplest situation where the initial fibrations are all modeled on smooth manifolds.

Remark: An n-dimensional stratified pseudomanifold X is a topological space together with a filtration by closed subspaces

$$X = X_n = X_{n-1} \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0$$

such that for each point $p \in X_i - X_{i-1}$ there is a distinguished neighborhood U in X which is filtered homeomorphic to $C(L) \times B^i$ for a compact stratified pseudomanifold L of dimension $n - i - 1$. $X_i - X_{i-1}$ is an i-dimensional manifold called the i-dimensional stratum. A conical metric on X is a Riemannian metric on the regular set of X such that on each distinguished neighborhood it is quasi-isometric to a metric of the type (2) with $B = B^i, Z = L$ and g_B the standard metric on B^i, g_Z a conical metric on L. Such conical metrics always exist on a stratified pseudomanifold.

2.2 L^2 Signature of Generalized Thom Spaces

A generalized Thom space T is obtained by coning off the boundary of the space $C_\pi M$.

Namely,

$$T = C_\pi M \cup_M C(M)$$

is a compact stratified pseudomanifold with two singular strata, B and a single point (unless B is a sphere).

Example Let $\xi \xrightarrow{\pi} B$ be a vector bundle of rank k. Then we have the associated sphere bundle:

$$S^{k-1} \to S(\xi) \xrightarrow{\pi} B.$$

The generalized Thom space constructed out of this fibration coincides with the usual Thom space equipped with a natural metric.

Now consider the generalized Thom space constructed from an oriented fibration (1) of closed manifolds, i.e., both the base B and fiber Z are closed oriented manifolds and so is the total space M. Then T will be a compact oriented stratified pseudomanifold with two singular strata. Since we are interested in the L^2 signature, we assume that the dimension of M is odd (so $\dim T$ is even). In addition, we assume the Witt conditions; namely, either the dimension of the fibers is odd or its middle dimensional cohomology vanishes. Under the Witt cnditions, the strong Hodge theorem holds for T. Hence the L^2 signature of T is well defined.

Question: What is the L^2 signature of T?

Let's go back to the case of the usual Thom space.

Example (continued) In this case,

$$\mathrm{sign}_{(2)}(T) = -\mathrm{sign}(D(\xi)),$$

the signature of the disk bundle $D(\xi)$ (as a manifold with boundary).

Let Φ denote the Thom class and χ the Euler class. Then the Thom isomorphism gives the commutative diagram

$$\begin{array}{ccccc}
H^{*+k}(D(\xi), S(\xi)) & \otimes & H^{*+k}(D(\xi), S(\xi)) & \to & \mathbf{R} \\
\uparrow \pi^*(\cdot) \cup \Phi & & \uparrow \pi^*(\cdot) \cup \Phi & & \\
H^*(B) & \otimes & H^*(B) & \to & \mathbf{R} \\
\phi & & \psi & \to & [\phi \cup \psi \cup \chi][B]\,.
\end{array}$$

Thus, $\mathrm{sign}_{(2)}(T)$ is the signature of this bilinear form on $H^*(B)$.

We now introduce the topological invariant which gives the L^2-signature for a generalized Thom space. In [13], in studying adiabatic limits of eta invariants, the second author introduced a global topological invariant associated with a fibration. (For adiabatic limits of eta invariants, see also [32, 5, 10, 3].) Let (E_r, d_r) be the E_r-term with differential, d_r, of the Leray spectral sequence of the fibration (1) in the construction of the generalized Thom space T. Define a pairing

$$\begin{array}{lll}
E_r \otimes E_r & \to & \mathbf{R} \\
\phi \otimes \psi & \mapsto & \langle \phi \cdot d_r \psi, \xi_r \rangle,
\end{array}$$

where ξ_r is a basis for E_r^m naturally constructed from the orientation. In case $m = 4k-1$, when restricted to $E_r^{\frac{m-1}{2}}$, this pairing becomes symmetric. We define τ_r to be the signature of this symmetric pairing and put

$$\tau = \sum_{r \geq 2} \tau_r\,.$$

When the fibration is a sphere bundle with the typical fiber a $(k-1)$-dimensional sphere, then the spectral sequence satisfies $E_2 = \cdots = E_k$, $E_{k+1} = E_\infty$ with $d_2 = \cdots = d_{k-1} = 0$, $d_k(\psi) = \psi \cup \chi$. Hence τ coincides with the signature of the bilinear form from the Thom isomorphism theorem. The main result of [11] is the following

Theorem 1 (Cheeger-Dai) *Assume that the fibre Z is either odd dimensional or its middle dimensional cohomology vanishes. Then the L^2-signature of the generalized Thom space T is equal to $-\tau$:*

$$\mathrm{sign}_{(2)}(T) = -\tau.$$

In spirit, our proof of the theorem follows the example of the sphere bundle of a vector bundle. Thus, we first establish an analog of Thom's isomorphism theorem in the context of generalized Thom spaces. In part, this consists of identifying the L^2-cohomology of T in terms of the spectral sequence of the original fibration; see [11] for complete details.

Corollary 2 *For a compact oriented space X with non-isolated conical singularity satisfying the Witt conditions, the L^2-signature is given by*

$$\mathrm{sign}_{(2)}(X) = \mathrm{sign}(X_0) + \sum_{i=1}^{k} \tau(X_i)\,.$$

The study of the L^2-cohomology of the type of spaces with conical singularities discussed here turns out to be related to work on the L^2-cohomology of noncompact hyper-kähler manifolds which is motivated by Sen's conjecture; see e.g [19], [18] . Hyper-kähler manifolds often arise as moduli spaces of (gravitational) instantons and monopoles, and so-called S-duality predicts the dimension of the L^2-cohomology of these moduli spaces (Sen's conjecture). Many of these spaces can be compactified to given a space with non-isolated conical singularities. In such cases, our results can be applied. We would also like to refer the reader to the work of Hausel-Hunsicker-Mazzeo, [18], which studies the L^2-cohomology and L^2-harmonic forms of noncompact spaces with fibered geometric ends and their relation to the intersection cohomology of the compactification. Various applications related to Sen's conjecture are also considered there.

Combining the index theorem of [4] with our topological computation of the L^2-signature of T, we recover the following adiabatic limit formula of [13]; see also [32, 5, 9, 3].

Corollary 3 *Assume that the fibre Z is odd dimensional. Then we have the following adiabatic limit formula for the eta invariant of the signature operator.*

$$\lim_{\epsilon \to 0} \eta(A_{M,\epsilon}) = \int_B \mathcal{L}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \tau.$$

In the general case i.e. with no the dimension restriction on the fibre, the L^2-signature for generalized Thom spaces is discussed in [21]. In particular, Theorem 1 is proved for the general case in [21]. However, one of ingredients there is the adiabatic limit formula of [13], rather than the direct topological approach taken here. One of our original motivations was to give a simple topological proof of the adiabatic limit formula. In [20], the methods and techniques in [11] are used in the more general situation to derive a very interesting topological interpretation for the invariant τ_r. On the other hand, in [7], our result on the generalized Thom space, together with the result in [13], is used to derive the signature formula for manifolds with non-isolated conical singularity.

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