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A Positive Mass Theorem for Spaces with Asymptotic SUSY Compactification

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Abstract: We prove a positive mass theorem for spaces which asymptotically approach a flat Euclidean space times a Calabi-Yau manifold (or any special honolomy manifold except the quaternionic Kähler). This is motivated by the very recent work of Hertog-Horowitz-Maeda [HHM].

In general relativity, isolated gravitational systems are modelled by asymptotically flat spacetimes. The spatial slices of such spacetime are then asymptotically flat Riemannian manifolds. That is, Riemannian manifolds (M^n, g) such that $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq \mathbb{R}^n - B_R(0)$ for some $R > 0$ so that in the induced Euclidean coordinates the metric satisfies the asymptotic conditions

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \quad \partial_k g_{ij} = O(r^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = O(r^{-\tau-2}). \tag{0.1}$$

Here $\tau > 0$ is the asymptotic order and r is the Euclidean distance to a base point. The total mass (the ADM mass) of the gravitational system can then be defined via a flux integral [ADM, LP]

$$m(g) = \lim_{R\to\infty} \frac{1}{4\omega_n} \int_{S_R} (\partial_i g_{ij} - \partial_j g_{ii}) * dx_j. \tag{0.2}$$

Here ω_n denotes the volume of the $n-1$ sphere and S_R the Euclidean sphere with radius R centered at the base point.

If $\tau > \frac{n-2}{2}$ and $n \geq 2$, then $m(g)$ is independent of the asymptotic coordinates x_i, and thus is an invariant of the metric. The positive mass theorem [SY1, SY2, SY3, Wi1] says that this total mass is nonnegative provided one has nonnegative local energy density.

Theorem 0.1 (Schoen-Yau, Witten). *Suppose (M^n, g) is an asymptotically flat spin manifold of dimension $n \geq 3$ and of order $\tau > \frac{n-2}{2}$. If the scalar curvature $R \geq 0$, then $m(g) \geq 0$ and $m(g) = 0$ if and only if $M = \mathbb{R}^n$.*

Remark. The scalar curvature R is the local energy density.

According to string theory [CHSW], our universe is really ten dimensional, modelled by $M^{3,1} \times X$, where X is a Calabi-Yau 3-fold. This is the so called Calabi-Yau compactification, which motivates the spaces we now consider.

We consider the complete Riemannian manifolds (M^n, g) such that $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$ for some $R > 0$ and X a compact simply connected Calabi-Yau manifold (or with any other special honolomy except $Sp(m) \cdot Sp(1)$) so that the metric on M_∞ satisfies

$$g = \overset{\circ}{g} + h, \quad \overset{\circ}{g} = g_{\mathbb{R}^k} + g_X, \quad h = O(r^{-\tau}), \quad \overset{\circ}{\nabla} h = O(r^{-\tau-1}), \quad \overset{\circ}{\nabla}\overset{\circ}{\nabla} h = O(r^{-\tau-2}). \tag{0.3}$$

Here $\overset{\circ}{\nabla}$ is the Levi-Civita connection of $\overset{\circ}{g}$, $\tau > 0$ is the asymptotical order. We will call M a space with asymptotic SUSY compactification.

The mass for such a space is then defined by

$$m(g) = \lim_{R\to\infty} \frac{1}{4\omega_k vol(X)} \int_{S_R \times X} (\overset{\circ}{\nabla}_{e_a^0} g_{ja} - \overset{\circ}{\nabla}_{e_j^0} g_{aa}) * dx_j dvol(X). \tag{0.4}$$

Here $\{e_a^0\} = \{\frac{\partial}{\partial x_i}, f_\alpha\}$ is an orthornormal basis of $\overset{\circ}{g}$, the $*$ operator is the one on the Euclidean factor, the index i, j run over the Euclidean factor and the index α runs over X while the index a runs over the full index of the manifold. In fact, this reduces to

$$m(g) = \lim_{R\to\infty} \frac{1}{4\omega_k vol(X)} \int_{S_R \times X} (\partial_i g_{ij} - \partial_j g_{aa}) * dx_j dvol(X).$$

Remark. If $\tau > \frac{k-2}{2}$ and $k \geq 2$, then $m(g)$ is independent of the asymptotic coordinates.

Our main result is

Theorem 0.2. *Let (M, g) be a complete spin manifold as above and the asymptotic order $\tau > \frac{k-2}{2}$ and $k \geq 3$. If M has nonnegative scalar curvature, then $m(g) \geq 0$ and $m(g) = 0$ if and only if $M = \mathbb{R}^k \times X$.*

Remark. The result extends without change to the case with more than one end.

Remark. Just like in the usual case, the restriction $k \geq 3$ has to do with getting the correct spin structure at the ends. See Sect. 5 for additional comments regarding the spin structures of the ends.

Our motivation comes from a very recent work of Hertog-Horowitz-Maeda [HHM] on the Calabi-Yau compactifications. Using the existence result of Stolz [S1, S2] on metrics of positive scalar curvature, they constructed classical configurations which have regions of (arbitrarily large) negative energy density as seen from the four dimensional perspective. This should be contrasted with the positivity (nonnegativity) of the total mass, as guaranteed by Theorem 0.2. According to [HHM], physical consequences of the negative energy density include possible violation of Cosmic Censorship and new thermal instability.

The Lorentzian version of Theorem 0.2 will be discussed in a separate paper.

1. Manifolds with Special Holonomy

For a complete Riemannian manifold (M^n, g), the holonomy group $Hol(g)$ (with respect to a base point) is the subgroup of $O(n)$ generated by parallel translations along all loops at the base point. For simply connected irreducible nonsymmetric spaces, Berger has given a complete classification of possible holonomy groups, namely, $SO(n)$ which is the generic situation, $U(m)$ (if $n = 2m$) which is Kähler, $SU(m)$ for Calabi-Yau, $Sp(m) \cdot Sp(1)$ (if $n = 4m$) which is called quaternionic Kähler, $Sp(m)$ which is called hyper-Kähler, $Spin(7)$ (if $n = 8$), and G_2 (if $n = 7$). Except for the generic and Kähler cases, the rest are called special holonomy.

If a Riemannian manifold (M, g) is spin, then one can consider spinors ϕ on M which are sections of the spinor bundle S. The Levi-Civita connection ∇ of g lifts to a connection of the spinor bundle, which will still be denoted by the same notation. In fact, any metric connections lift in the same way. The Dirac operator

$$D\phi = e_i \cdot \nabla_{e_i}\phi,$$

where e_i is a local orthonormal basis of M and $e_i\cdot$ is the Clifford multiplication. A spinor ϕ is parallel if $\nabla\phi = 0$.

Implicitly, all these depend on the underlying spin structure, which is in one-to-one correspondence with elements of $H^1(M,\ \mathbb{Z}_2)$ [LM]. Thus, for simply connected manifolds, one has a unique spin structure. It seems that the issue of spin structure in this context is a subtle one, deserving further study. (See also Sect. 5.)

All manifolds with special holonomy, with the exception of the quaternionic Kähler ones, carry nonzero parallel spinor. In fact, one has the following theorem of McKenzie Wang [Wa].

Theorem 1.1. *Let (M, g) be a complete, simply connected, irreducible Riemannian spin manifold and N be the dimension of parallel spinors. Then $N > 0$ if and only if the holonomy group is one of $SU(m)$, $Sp(m)$, $Spin(7)$, G_2.*

Remark. Wang [Wa] actually characterizes each special holonomy by the number of parallel spinors.

Remark. Manifolds with parallel spinors are called supersymmetric (SUSY) in physics literature.

2. Proof of Theorem 0.2

Our proof is an extension of Witten's spinor proof [Wi1]. Here we follow the idea of Anderson and Dahl [AnD] and use the following alternative formula for the Lichnerowicz formula.

Lemma 2.1. *Given a spinor ϕ on a Riemannian spin manifold, define a 1-form α via*

$$\alpha(X) = \langle(\nabla_X + X \cdot D)\phi,\ \phi\rangle.$$

Then

$$div\,\alpha = \frac{R}{4}|\phi|^2 + |\nabla\phi|^2 - |D\phi|^2.$$

Proof. Choose an orthonormal basis e_a such that $\nabla e_a = 0$ at the given point. Then (Einstein summation enforced)

$$\begin{aligned} div\,\alpha &= (\nabla_{e_a}\alpha)(e_a) = e_a(\alpha(e_a)) \\ &= \langle(\nabla_{e_a} + e_a\cdot D)\phi,\ \nabla_{e_a}\phi\rangle + \langle\nabla_{e_a}(\nabla_{e_a} + e_a\cdot D)\phi,\ \phi\rangle \\ &= |\nabla\phi|^2 - |D\phi|^2 + \langle(\delta_{ab} + e_a\cdot e_b\cdot)\nabla_{e_a}\nabla_{e_b}\phi,\ \phi\rangle. \end{aligned}$$

The last term is just

$$\left\langle \frac{1}{2}[e_a\cdot,\ e_b\cdot]\nabla_{e_a}\nabla_{e_b}\phi,\ \phi\right\rangle = \left\langle \frac{1}{4}[e_a\cdot,\ e_b\cdot]R(e_a,e_b)\phi,\ \phi\right\rangle = \frac{R}{4}|\phi|^2$$

by the usual calculation as in the Lichnerowicz formula [LM]. □

Therefore, for any compact domain $\Omega \subset M$,

$$\begin{aligned} &\int_\Omega \left[\frac{R}{4}|\phi|^2 + |\nabla\phi|^2 - |D\phi|^2\right] dvol(g) \\ &= \int_{\partial\Omega} \sum \langle(\nabla_{e_a} + e_a\cdot D)\phi,\ \phi\rangle\, \mathrm{int}(e_a)\, dvol(g), \\ &= \int_{\partial\Omega} \sum \langle(\nabla_\nu + \nu\cdot D)\phi,\ \phi\rangle\, dvol(g|_{\partial\Omega}), \end{aligned} \tag{2.5}$$

where e_a is an orthonormal basis of g and ν is the unit outer normal of $\partial\Omega$. Also, here $\mathrm{int}(e_a)$ is the interior multiplication by e_a.

In particular, for a harmonic spinor ϕ, i.e., $D\phi = 0$, the left-hand side of (2.5) will be nonnegative provided $R \geq 0$. On the other hand, if the harmonic spinor ϕ can be chosen so that it is asymptotic to a parallel spinor at infinity and we choose the domain Ω so that $\partial\Omega = S_R \times X$, then we will show that the right-hand side of (2.5) converges to the mass (up to a positive normalizing constant). Thus, for the first part of our theorem, we are left with two tasks. First, we need to show the existence of harmonic spinors which are asymptotic to a parallel spinor. Second, we need to show that the limit of the boundary term converges to the mass. The existence of the harmonic spinor is dealt with in Sect. 4 (Lemma 4.1) after the necessary analysis in the next section and the computation of the limit of the boundary term is also left to Sect. 4 (Lemma 4.2).

We now continue with the proof of the rigidity. If $m(g) = 0$, then it follows that ϕ is a (nonzero) parallel spinor on M. This implies that M is Ricci flat, as

$$e_a \cdot R(e_a, X)\phi = -\frac{1}{2}Ric(X)\,\phi.$$

Thus, we are in a position to use the splitting theorem of Cheeger-Gromoll [CG]. To find lines in M, we start with sequences of pairs of points p_i, q_i in $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$. When R is sufficiently large, one can choose p_i, q_i so that their distance is comparable to their Euclidean distance. It follows that one can construct a line in M this way. Similarly, we can construct k lines in M that are almost perpendicular to each other. It follows that $M = \mathbb{R}^k \times X$. □

3. Fibered Boundary Calculus

We will use the fibered boundary calculus of Melrose-Mazzeo [MM] (and further developed by Boris Vaillant in his thesis [V] and in [HHMa]) to solve for the harmonic spinor with the correct asymptotic behavior.

The change of variable $r = \frac{1}{x}$ makes metric into what is called fibered boundary metric, which is defined in the more general setting as follows.

Consider a complete noncompact Riemannian manifold (M, g). Assume that M has a compactification $\bar{M}$ such that $\partial\bar{M}$ comes with a fibration structure $F \to \partial\bar{M} \xrightarrow{\pi} B$. Moreover, in a neighborhood of the boundary $\partial\bar{M}$, the metric g has the form

$$g = \frac{dx^2}{x^4} + \frac{\pi^*(g_B)}{x^2} + g_F, \tag{3.6}$$

where x is a defining function of the boundary, i.e., $x = 0$ on $\partial\bar{M}$ and $dx \neq 0$ on the boundary. Also, g_B is a metric on the base B, g_F is a family of fiberwise metrics.

Thus, in the setting of spaces with asymptotic SUSY compactification, one has a trivial fibration $S^{k-1} \times X$ and $x = \frac{1}{r}$.

We will use the notation M, $\bar{M}$, and ∂M, $\partial\bar{M}$ interchangeably. For a manifold with boundary, the Lie algebra of b-vector fields consists of vector fields tangent to the boundary

$$\mathcal{V}_b(M) = \{V \mid V \text{ is tangent to the boundary } \partial M\}.$$

The Lie algebra of vector fields associated with the fibered boundary metric is

$$\mathcal{V}_{fb} = \{V \in \mathcal{V}_b(M) \mid V \text{ is tangent to the fibers } F \text{ at } \partial M,\ Vx = O(x^2)\}. \tag{3.7}$$

If y is a local coordinate of B and z is a local coordinate of F, then $\mathcal{V}_{fb}$ is spanned by $x^2\partial_x,\ x\partial_y,\ \partial_z$. The fibered boundary vector fields $\mathcal{V}_{fb}$ generate the ring of fibered boundary differential operators. The Dirac operator D associated to the fibered boundary metric is such a fibered boundary differential operator of first order.

Define the L^2 and Sobolev spaces as follows:

$$L^2(M, S) = L^2(M, S; dvol(g)) = L^2(M, S, \frac{dxdydz}{x^{2+l}}),$$

if $\dim B = l$,

$$L^{p,2}(M, S) = \{\, \phi \in L^2(M, S) \mid \nabla_{V_1} \cdots \nabla_{V_j}\phi \in L^2(M, S),\ \forall j \leq p,\ V_i \in \mathcal{V}_b \,\}.$$

For $\gamma \in \mathbb{R}$, the space of conormal sections of order γ is defined to be

$$\mathcal{A}^\gamma(M, S) = \{\, \phi \in C^\infty(M, S) \mid \nabla_{V_1} \cdots \nabla_{V_j}\phi| \leq Cx^\gamma,\ \forall j,\ V_i \in \mathcal{V}_b \,\},$$

while the space of polyhomogeneous sections is

$$\mathcal{A}^*_{phg}(M, S) = \{\, \phi \in \mathcal{A}^*(M, S) \mid \phi \sim \sum_{\mathrm{Re}\gamma_j \to \infty} \sum_{k=0}^{N_j} \psi_{jk} x^{\gamma_j} (\log x)^k,\ \psi_{jk} \in C^\infty(\partial M, S) \,\}.$$

Here the expansion is the usual asymptotic expansion, uniform with all the derivatives. We usually specify all possible pairs (γ_j, N_j) that can appear in the expansion and the collection of (γ_j, N_j) is called the index set.

Assume that ker D_F has constant dimension so it forms a vector bundle on the base B. Let Π_0 be the orthogonal projection onto ker D_F and $\Pi_\perp = I - \Pi_0$. The following is a summary of the results developed in [MM, V, HHMa].

Theorem 3.1. *Suppose that a is not an indicial root of* $\Pi_0 x^{-1} D \Pi_0$. *Then*

$$D : x^a L^{1,2}(M, S) \to x^{a+1}\Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$$

is Fredholm. If $D\phi = 0$ *for* $\phi \in x^a L^2(M, S)$*, then* ϕ *is polyhomogeneous with exponents in its expansion determined by the indicial roots of* $\Pi_0 x^{-1} D \Pi_0$ *and truncated at* a*. If* $D\xi = \psi$ *for* $\psi \in \mathcal{A}^a(M, S)$ *and* $\xi \in x^{c-1}\Pi_0 L^{1,2}(M, S) \oplus x^c \Pi_\perp L^{1,2}(M, S)$ *and* $c < a$*, then* $\xi \in \Pi_0 \mathcal{A}^I_{phg}(M, S) + \mathcal{A}^a(M, S)$.

For the precise definition of the indicial root, and in particular, the indicial root of $\Pi_0 x^{-1} D \Pi_0$, we refer the reader to [MM, HHMa]. For our purpose, we only note that it is a discrete set.

Remark. Strictly speaking, only $\mathring{g}$ is a fibered boundary metric in the pure sense but it is easy to see that the result generalize to the metric g. In any case, the metric perturbation produces only a lower order term (cf. Sect. 4).

Lemma 3.2. *If* $R \geq 0$ *and* $a > \frac{k-2}{2}$ *is not an indicial root, then*

$$D : x^a L^{1,2}(M, S) \to x^{a+1}\Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$$

is an isomorphism.

Proof. We first see that it is injective. If $D\phi = 0$ for $\phi \in x^a L^2(M, S)$, then by Theorem 3.1, $\phi \in \mathcal{A}^a_{phg}(M, S)$. Now, from (2.5),

$$\int_\Omega \left[|\nabla\phi|^2 + \frac{R}{4}|\phi|^2 \right] dvol = \int_{\partial\Omega} \langle \nabla_\nu \phi, \phi \rangle \, dvol(\partial\Omega).$$

By taking Ω so that $\partial\Omega = S_r \times X$ and $r \to \infty$ we see that the right hand side goes to zero since $\phi \in \mathcal{A}^a_{phg}(M, S)$ and $a > \frac{k-2}{2}$. It follows then by the assumption $R \geq 0$ that ϕ is parallel and hence zero.

Now, if ω is in the cokernel of D, then, by the Fredholm property, $\omega \in x^{a+1}\Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$ and ω is a weak solution of Dirac equation:

$$\langle \omega, D\xi \rangle = 0, \ \forall \xi \in x^a L^{1,2}(M, S).$$

It follows by the regularity part of Theorem 3.1, $\omega \in \mathcal{A}^a_{phg}(M, S)$. Therefore the same argument as above shows $\omega = 0$. □

4. Computation of the Mass

Recall that $g = \overset{\circ}{g} + h$ with $\overset{\circ}{g} = g_{\mathbb{R}^k} + g_X$ and $h = O(r^{-\tau})$, $\overset{\circ}{\nabla} h = O(r^{-\tau-1})$, $\overset{\circ}{\nabla}\overset{\circ}{\nabla} h = O(r^{-\tau-2})$. Let e_a^0 be the orthonormal basis of $\overset{\circ}{g}$ which consists of $\frac{\partial}{\partial x_i}$ followed by an orthonormal basis f_α of g_X. Orthonormalizing e_a^0 with respect to g gives rise an orthonormal basis e_a of g. Moreover,

$$e_a = e_a^0 - \frac{1}{2} h_{ab} e_b^0 + O(r^{-2\tau}). \tag{4.8}$$

This gives rise to a gauge transformation

$$A: \; SO(\overset{\circ}{g}) \ni e_a^0 \to e_a \in SO(g)$$

which identifies the corresponding spin groups and spinor bundles.

To compare ∇ and $\overset{\circ}{\nabla}$, in particular their lifts to the spinor bundles, one introduces a new connection $\nabla^0 = A \circ \overset{\circ}{\nabla} \circ A^{-1}$. This connection is compatible with the metric g but has a torsion

$$T(X,Y) = \nabla_X^0 Y - \nabla_Y^0 X - [X,Y] = -(\overset{\circ}{\nabla}_X A) A^{-1} Y + (\overset{\circ}{\nabla}_Y A) A^{-1} X. \tag{4.9}$$

The difference of ∇ and ∇^0 is then expressible in terms of the torsion

$$2\langle \nabla_X^0 Y - \nabla_X Y, Z\rangle = \langle T(X,Y), Z\rangle - \langle T(X,Z), Y\rangle - \langle T(Y,Z), X\rangle, \tag{4.10}$$

where we use the metric g for the inner product $\langle\ ,\ \rangle$.

Since ∇ and ∇^0 are both g-compatible, their induced connections on the spinor bundle differ by

$$\nabla_{e_a} - \nabla_{e_a}^0 = -\frac{1}{4}\sum_{b,c}(\omega_{bc}(e_a) - \overset{\circ}{\omega}_{bc}(e_a)) e_b e_c, \tag{4.11}$$

where e_b, e_c act on the spinors by the Clifford multiplication and the connection 1-forms

$$\omega_{bc}(e_a) = \langle \nabla_{e_a} e_b, e_c\rangle, \quad \overset{\circ}{\omega}_{bc}(e_a) = \langle \overset{\circ}{\nabla}_{e_a} e_b, e_c\rangle.$$

From (4.10) and (4.9) we obtain

$$\nabla_{e_a} - \nabla_{e_a}^0 = \frac{1}{8}\sum_{b\neq c}(\overset{\circ}{\nabla}_{e_b} g_{ac} - \overset{\circ}{\nabla}_{e_c} g_{ab}) e_b e_c + O(r^{-2\tau-1}) \tag{4.12}$$

for the difference of the two connections acting on spinors.

Lemma 4.1. *There exists a harmonic spinor on $(M,\ g)$ which is asymptotic to a parallel spinor at infinity.*

Proof. Our manifold $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$. Since $k \geq 3$ and X is simply connected, the end M_∞ is also simply connected, and therefore has a unique spin structure coming from the product of the restriction of the spin structure on $\mathbb{R}^k$ and the spin structure on X.

Now pick a unit norm parallel spinor ψ_0 of $(\mathbb{R}^k, g_{\mathbb{R}^k})$ and a unit norm parallel spinor ψ_1 of (X, g_X). Then $\phi_0 = A(\psi_0 \otimes \psi_1)$ defines a spinor of M_∞. We extend ϕ_0 smoothly inside. Then $\nabla^0 \phi_0 = 0$ outside the compact set. Thus, it follows from (4.12) that

$$\nabla \phi_0 = O(r^{-\tau-1}). \tag{4.13}$$

We now construct our harmonic spinor by setting $\phi = \phi_0 + \xi$ and solve $D\xi = -D\phi_0 \in O(r^{-\tau-1})$. By using Lemma 3.2, adjusting τ slightly if necessary so that it is not one of the indicial roots, we have a solution $\xi \in O(r^{-\tau})$. □

Lemma 4.2. *For the harmonic spinor ϕ constructed above, we have*

$$\lim_{R\to\infty} \int_{S_R \times X} \sum \langle (\nabla_{e_a} + e_a \cdot D)\phi,\ \phi\rangle \operatorname{int}(e_a)\, dvol(g) = \omega_k vol(X) m(g).$$

Proof. By (2.5),

$$\begin{aligned} &\int_{S_R \times X} \sum \langle (\nabla_{e_a} + e_a \cdot D)\phi,\ \phi\rangle \operatorname{int}(e_a)\, dvol(g) \\ &\quad = \operatorname{Re} \int_{S_R \times X} \sum \langle (\nabla_{e_a} + e_a \cdot D)\phi,\ \phi\rangle \operatorname{int}(e_a)\, dvol(g). \end{aligned}$$

Now,

$$\begin{aligned} \langle (\nabla_{e_a} + e_a \cdot D)\phi,\ \phi\rangle &= \left\langle \frac{1}{2}[e_a\cdot, e_b\cdot]\, \nabla_{e_b}\phi,\ \phi \right\rangle \\ &= \left\langle \frac{1}{2}[e_a\cdot, e_b\cdot]\, \nabla_{e_b}\phi_0,\ \phi_0 \right\rangle + \left\langle \frac{1}{2}[e_a\cdot, e_b\cdot]\, \nabla_{e_b}\phi_0,\ \xi \right\rangle \\ &\quad + \left\langle \frac{1}{2}[e_a\cdot, e_b\cdot]\, \nabla_{e_b}\xi,\ \phi_0 \right\rangle + \left\langle \frac{1}{2}[e_a\cdot, e_b\cdot]\, \nabla_{e_b}\xi,\ \xi \right\rangle. \end{aligned} \tag{4.14}$$

The second term and the last term are $O(r^{-2\tau-1})$ and therefore contribute nothing in the limit. For the third term, one notices that if β is the $n-2$ form,

$$\beta = \langle [e_a\cdot, e_b\cdot]\, \phi,\ \psi\rangle \operatorname{int}(e_a) \operatorname{int}(e_b)\, dvol(g)$$

(Einsterin summation here and below), then

$$\begin{aligned} d\,\beta &= -2\big(\langle [e_a\cdot, e_b\cdot]\, \nabla_{e_b}\phi,\ \psi\rangle \operatorname{int}(e_b)\, dvol(g) \\ &\qquad + \langle [e_a\cdot, e_b\cdot]\, \phi,\ \nabla_{e_b}\psi\rangle \operatorname{int}(e_b)\, dvol(g)\big) \\ &= -4\big(\langle [e_a\cdot, e_b\cdot]\, \nabla_{e_b}\phi,\ \psi\rangle \operatorname{int}(e_b)\, dvol(g) \\ &\qquad - \langle \phi,\ [e_a\cdot, e_b\cdot]\, \nabla_{e_b}\psi\rangle \operatorname{int}(e_b)\, dvol(g)\big) \end{aligned} \tag{4.15}$$

which yields

$$\int_{\partial\Omega} \langle [e_a\cdot, e_b\cdot]\, \nabla_{e_b}\phi,\ \psi\rangle \operatorname{int}(e_b)\, dvol(g) = \int_{\partial\Omega} \langle \phi,\ [e_a\cdot, e_b\cdot]\, \nabla_{e_b}\psi\rangle \operatorname{int}(e_b)\, dvol(g).$$

It follows then that the third term is similarly dealt with as the second. Thus the only contribution is coming from the first term, for which we note that

$$\left\langle \frac{1}{2}[e_a\cdot, e_b\cdot]\,\nabla_{e_b}\phi_0,\ \phi_0 \right\rangle = \left\langle \frac{1}{2}[e_a\cdot, e_b\cdot](\nabla_{e_b} - \nabla^0_{e_b})\phi_0,\ \phi_0 \right\rangle$$
$$= \frac{1}{16}\sum_{c\neq d}(\overset{\circ}{\nabla}_{e_c}\, g_{bd} - \overset{\circ}{\nabla}_{e_d}\, g_{bc})\langle [e_a\cdot,\ e_b\cdot]\, e_c\cdot e_d\cdot\phi_0,\ \phi_0\rangle$$
$$+ O(r^{-2\tau-1})$$

by (4.12). Now

$$\frac{1}{16}\sum_{c\neq d}(\overset{\circ}{\nabla}_{e_c}\, g_{bd} - \overset{\circ}{\nabla}_{e_d}\, g_{bc})\langle [e_a\cdot,\ e_b\cdot]\, e_c\cdot e_d\cdot\phi_0,\ \phi_0\rangle$$
$$= \frac{1}{8}\sum_{c\neq d}(\overset{\circ}{\nabla}_{e_c}\, g_{bd} - \overset{\circ}{\nabla}_{e_d}\, g_{bc})\langle e_a\cdot e_b\cdot e_c\cdot e_d\cdot\phi_0,\ \phi_0\rangle$$
$$+ \frac{1}{8}\sum_{c\neq d}(\overset{\circ}{\nabla}_{e_c}\, g_{ad} - \overset{\circ}{\nabla}_{e_d}\, g_{ac})\langle e_c\cdot,\ e_d\cdot\phi_0,\ \phi_0\rangle$$
$$= \frac{1}{8}\sum_{c\neq d}\overset{\circ}{\nabla}_{e_c}\, g_{bd}\langle e_a\cdot e_b\cdot e_c\cdot e_d\cdot\phi_0,\ \phi_0\rangle + \frac{1}{8}\sum_{c\neq d}\overset{\circ}{\nabla}_{e_d}\, g_{bb}\langle e_a\cdot e_d\cdot\phi_0,\ \phi_0\rangle$$
$$+ \frac{1}{8}\sum_{c\neq d}(\overset{\circ}{\nabla}_{e_c}\, g_{bd} - \overset{\circ}{\nabla}_{e_d}\, g_{bc})\langle e_c\cdot e_d\cdot\phi_0,\ \phi_0\rangle$$
$$= \frac{1}{8}\sum_{c\neq d}\overset{\circ}{\nabla}_{e_c}\, g_{bb}\langle e_a\cdot e_c\cdot\phi_0,\ \phi_0\rangle + \frac{1}{4}\sum_{c\neq d}\overset{\circ}{\nabla}_{e_b}\, g_{bd}\langle e_a\cdot e_d\cdot\phi_0,\ \phi_0\rangle$$
$$+ \frac{1}{8}\sum_{c\neq d}\overset{\circ}{\nabla}_{e_d}\, g_{bb}\langle e_a\cdot e_d\cdot\phi_0,\ \phi_0\rangle + \frac{1}{8}\sum_{c\neq d}(\overset{\circ}{\nabla}_{e_c}\, g_{bd} - \overset{\circ}{\nabla}_{e_d}\, g_{bc})\langle e_c\cdot e_d\cdot\phi_0,\ \phi_0\rangle.$$

For the last equality, we use $e_c\cdot e_d\cdot = \frac{1}{2}[e_c\cdot,\ e_d\cdot]$ for $c\neq d$, and $[e_c\cdot,\ e_d\cdot]$ skew-hermitian to see that its real part is zero. Finally, one uses $e_a\cdot e_d\cdot = \frac{1}{2}[e_a\cdot,\ e_d\cdot] - \delta_{ad}$ and the skew-hermitian property of the commutators to obtain

$$\mathrm{Re}\left(\langle\frac{1}{2}[e_a\cdot, e_b\cdot]\,\nabla_{e_b}\phi_0,\ \phi_0\rangle\right) = \frac{1}{4}(\overset{\circ}{\nabla}_{e_b}\, g_{ab} - \overset{\circ}{\nabla}_{e_a}\, g_{bb})|\phi_0|^2 + O(r^{-2\tau-1}).$$

This yields

$$\lim_{R\to\infty}\int_{S_R\times X}\sum\langle(\nabla_{e_a} + e_a\cdot D)\phi,\ \phi\rangle\,\mathrm{int}(e_a)\,dvol(g)$$
$$= \lim_{R\to\infty}\int_{S_R\times X}\frac{1}{4}(\overset{\circ}{\nabla}_{e_b}\, g_{ab} - \overset{\circ}{\nabla}_{e_a}\, g_{bb})|\phi_0|^2\mathrm{int}(e_a)\,dvol(g).$$

To see that this reduces to the definition of the mass, we first note that one can replace e_a by e^0_a in the integrand on the right-hand side, producing only an error of $O(r^{-2\tau-1})$, then replace $dvol(g)$ by $dxdvol_X$ with a similar error term. □

5. Negative Energy Solutions in Kaluza-Klein Theory

It was observed by Witten that positive energy theorems do not extend immediately to Kaluza-Klein theory [Wi2]. He observed that there are two zero energy solutions on a space asymptotic to $M_4 \times S^1$ which should lead to perturbatively negative energy solutions. The explicit negative energy solutions were constructed later in [BP, BH]. The following example is from [BH].

The analytically continued Reissner-Nordström metric

$$ds^2 = (1 - \frac{2m}{r} - \frac{q^2}{r^2})d\theta^2 + (1 - \frac{2m}{r} - \frac{q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2,$$

where $r \geq r_+ = m + \sqrt{m^2 + q^2}$, $\theta \in \mathbb{R}/\frac{2\pi r_+^2}{r_+ - m}\mathbb{Z}$ and $d\Omega^2$ is the standard metric on the 2-sphere. This is a scalar flat metric on $\mathbb{R}^2 \times S^2$ and asymptotic to $\mathbb{R}^3 \times S^1$ at infinity. The mass can be computed via (0.4), which is

$$m(g) = \frac{1}{2} m \frac{r_+ - m}{2\pi r_+^2}. \tag{5.16}$$

For fixed asymptotic geometry, i.e., fixed circle size $\frac{2\pi r_+^2}{r_+ - m} = l$, this can be made arbitrarily negative if one takes $m < 0$ sufficiently large, while $q \neq 0$ is chosen appropriately (which will necessarily be large as well).

The reason here is that the end $\mathbb{R}^3 \times S^1$, and in particular, S^1 has the wrong spin structure! Recall that S^1 has two spin structures which correspond to the trivial double cover of S^1 and the nontrivial double cover of S^1. Here, since S^1 bounds the disk inside, it has the spin structure corresponding to the nontrivial double cover. It therefore has no parallel spinor.

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