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A note on positive energy theorem for spaces with asymptotic SUSY compactification

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We extend the higher dimensional positive mass theorem in [Dai, X., Commun. Math. Phys. **244**, 335–345 (2004)] to the Lorentzian setting. This includes the original higher dimensional positive energy theorem whose spinor proof is given in [Witten, E., Commun. Math. Phys. **80**, 381–402 (1981)] and [Parker, T., and Taubes, C., Commun. Math. Phys. **84**, 223–238 (1982)] for dimension 4 and in [Zhang, X., J. Math. Phys. **40**, 3540–3552 (1999)] for dimension 5. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1862095]

I. INTRODUCTION AND STATEMENT OF THE RESULT

In this note, we formulate and prove the Lorentzian version of the positive mass theorem in Ref. 4. There we prove a positive mass theorem for spaces of any dimension which asymptotically approach the product of a flat Euclidean space with a compact manifold which admits a nonzero parallel spinor (such as a Calabi–Yau manifold or any special honolomy manifold except the quaternionic Kähler manifold). This is motivated by string theory, especially the recent work in Ref. 7. The application of the positive mass theorem of Ref. 4 to the study of stability of Ricci flat manifolds is discussed in Ref. 5.

In general relativity, a space–time is modelled by a Lorentzian 4-manifold (N,g) together with an energy-momentum tensor T satisfying Einstein equation

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}. \tag{1.1}$$

The positive energy theorem[11,14] says that an isolated gravitational system with non-negative local matter density must have non-negative total energy, measured at spatial infinity. More precisely, one considers a complete oriented spacelike hypersurface M of N satisfying the following two conditions.

(a) *M is asymptotically flat*, that is, there is a compact set K in M such that $M-K$ is the disjoint union of a finite number of subsets $M_1, \ldots, M_k$ and each end M_l is diffeomorphic to $\mathbb{R}^3 - B_R(0)$. Moreover, under this diffeomorphism, the metric of M_l is of the form

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \quad \partial_k g_{ij} = O(r^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = O(r^{-\tau-2}). \tag{1.2}$$

Furthermore, the second fundamental form h_{ij} of M in N satisfies

$$h_{ij} = O(r^{-\tau-1}), \quad \partial_k h_{ij} = O(r^{-\tau-2}). \tag{1.3}$$

Here $\tau > 0$ is the asymptotic order and r is the Euclidean distance to a base point.

(b) *M has non-negative local mass density*: for each point $p \in M$ and for each timelike vector e_0 at p, $T(e_0, e_0) \geqslant 0$ and $T(e_0, \cdot)$ is a nonspacelike covector. This implies the dominant energy condition

$$T^{00} \geqslant |T^{\alpha\beta}|, \quad T^{00} \geqslant (-T_{0i} T^{0i})^{1/2}. \tag{1.4}$$

The total energy (the ADM mass) and the total (linear) momentum of M can then be defined as follows[1,10] [for simplicity we suppress the dependence here on l (the end M_l)]:

$$E = \lim_{R\to\infty} \frac{1}{4\omega_n}\int_{S_R} (\partial_i g_{ij} - \partial_j g_{ii}) * \mathrm{d}x_j,$$

$$P_k = \lim_{R\to\infty} \frac{1}{4\omega_n}\int_{S_R} 2(h_{jk} - \partial_{jk} h_{ii}) * \mathrm{d}x_j. \tag{1.5}$$

Here $x_1, \dots, x_n$ are the Euclidean coordinates on the end; $*$ denotes the Hodge star operator; ω_n denotes the volume of the $n-1$ sphere and S_R the Euclidean sphere with radius R centered at the base point. When the asymptotic order $\tau > (n-2)/2$, these quantities are finite and independent of the asymptotic coordinates. [Here $n=3$.]

Theorem 1.1: (Refs. 12–14): *With the assumptions as above and assuming that M is spin, one has*

$$E - |P| \geqslant 0$$

on each end M_l. Moreover, if $E=0$ for some end M_l, then M has only one end and N is flat along M.

Now, according to string theory,[2] our universe is really 10-dimensional, modelled on $\mathbb{R}^{3,1} \times X$ where X is a Calabi–Yau threefold. This is the so-called Calabi–Yau compactification[6] which motivates the spaces we now consider.

Thus we consider a Lorentzian manifold N [with signature $(-, +, \cdots, +)$] of $\dim N = n+1$, with a energy-momentum tensor satisfying the Einstein equation. Then let M be a complete oriented spacelike hypersurface in N. Further, assume that the Riemannian manifold (M^n, g) with g induced from the Lorentzian metric decomposes $M = M_0 \cup M_1 \cup \cdots \cup M_s$, where M_0 is compact as before but now each of the ends $M_l \simeq (\mathbb{R}^k - B_R(0)) \times X_l$ for some radius $R > 0$ and X_l a compact simply connected spin manifold which admits a nonzero parallel spinor. Moreover the metric on each M_l satisfies

$$g = \mathring{g} + u, \quad \mathring{g} = g_{\mathbb{R}^k} + g_X, \quad u = O(r^{-\tau}), \quad \mathring{\nabla} u = O(r^{-\tau-1}), \quad \mathring{\nabla}\mathring{\nabla} u = O(r^{-\tau-2}), \tag{1.6}$$

and the second fundamental form h of M in N satisfies

$$h = O(r^{-\tau-1}), \quad \mathring{\nabla} h = O(r^{-\tau-2}). \tag{1.7}$$

Here $\mathring{\nabla}$ is the Levi–Civita connection of $\mathring{g}$ (extended to act on all tensor fields), r the Euclidean distance in the Euclidean factor, and $\tau > 0$ is the asymptotical order.

The total energy and total momentum for each end M_l can then be defined by (again we suppress the dependence on l here)

$$E = \lim_{R\to\infty} \frac{1}{4\omega_k vol(X)}\int_{S_R\times X} (\partial_i g_{ij} - \partial_j g_{aa}) * \mathrm{d}x_j \, \mathrm{d}vol(X),$$

$$P_k = \lim_{R\to\infty} \frac{1}{4\omega_k vol(X)}\int_{S_R\times X} 2(h_{jk} - \partial_{jk} h_{ii}) * \mathrm{d}x_j \, \mathrm{d}vol(X). \tag{1.8}$$

Here the $*$ operator is the one on the Euclidean factor, the index i, j run over the Euclidean factor and g_{aa} is the trace of the metric g on the manifold M.

Then we have the following.

Theorem 1.2: *Assuming that M is spin and $\tau > (k-2)/2$, $k \geqslant 2$, one has*

$$E - |P| \geqslant 0$$

on each end M_l. Moreover, if $E=0$ for some end M_l, then M has only one end and N is flat along M.

In particular, this result includes the original positive energy theorem whose spinor proof is given in Refs. 14 and 10 for dimension 4 and in Ref. 15 for dimension 5. The dimension specific nature in these work is due to the use of special isomorphisms of low dimensional spin groups. Here we construct the desired metrics directly using the Clifford algebra .

Remark: If M is globally a product $\mathbb{R}^k \times X$ topologically, then the compact factor X need not be simply connected. The simply connected condition is imposed to guarantee that the spin structure on the ends coincides with the one obtained by restricting the spin structure from the inside.

II. THE HYPERSURFACE DIRAC OPERATOR

We will adapt Witten's spinor method[14] to our situation. For that, we follow the presentation and notations of Ref. 10. The crucial ingredient here is the hypersurface Dirac operator on M, acting on the (restriction of the) spinor bundle of N. Let S be the spinor bundle of N and still denote by the same notation its restriction (or rather, pullback) to M. Denote by ∇ the connection on S induced by the Lorentzian metric on N. The Lorentzian metric on N also induces a Riemannian metric on M, whose Levi–Civita connection gives rise to another connection, $\overline{\nabla}$ on S. The two, of course, differ by a term involving the second fundamental form.

There are two choices of metrics on S, which is another subtlety here. Since part of the treatment in Ref. 10 is special to dimension 4, we will give a construction directly using the Clifford algebra Ref. 8.

Let $\mathrm{SO}(n,1)$ denote the identity component of the groups of orientation preserving isometries of the Minkowski space $\mathbb{R}^{n,1}$. A choice of a unit timelike covector e^0 gives rise to injective homomorphisms α, $\hat{\alpha}$, and a commutative diagram

$$\begin{array}{cccc} \alpha: & \mathrm{SO}(n) & \rightarrow & \mathrm{SO}(n,1) \\ & \uparrow & & \uparrow \\ \hat{\alpha}: & \mathrm{Spin}(n) & \rightarrow & \mathrm{Spin}(n,1). \end{array} \tag{2.1}$$

We now fix a choice of unit timelike normal covector e^0 of M in N. Let $F(N)$ denote the $\mathrm{SO}(n,1)$ frame bundle of N and $F(M)$ the $\mathrm{SO}(n)$ frame bundle of M. Then $i^*F(N)=F(M) \times_{\alpha} \mathrm{SO}(n,1)$, where $i:M \hookrightarrow N$ is the inclusion. If N is spin, then we have a principal $\mathrm{Spin}(n,1)$ bundle $P_{\mathrm{Spin}(n,1)}$ on N, whose restriction on M is then $i^*P_{\mathrm{Spin}(n,1)}=P_{\mathrm{Spin}(n)} \times_{\hat{\alpha}} \mathrm{Spin}(n,1)$, where $P_{\mathrm{Spin}(n)}$ is the principal $\mathrm{Spin}(n)$ bundle of M. Thus, even if N is not spin, $i^*P_{\mathrm{Spin}(n,1)}$ is still well-defined as long as M is spin.

Similarly, when N is spin, the spinor bundle S on N is the associated bundle $P_{\mathrm{Spin}(n,1)} \times_{\rho_{n,1}} \Delta$, where $\Delta=\mathbb{C}^{2^{[(n+1)/2]}}$ is the complex vector space of spinors and

$$\rho_{n,1}:\mathrm{Spin}(n,1) \rightarrow \mathrm{GL}(\Delta) \tag{2.2}$$

is the spin representation. Its restriction to M is given by $i^*P_{\mathrm{Spin}(n,1)} \times_{\rho_{n,1}} \Delta = P_{\mathrm{Spin}(n)} \times_{\rho_n} \Delta$ with

$$\rho_n:\mathrm{Spin}(n) \overset{\hat{\alpha}}{\hookrightarrow} \mathrm{Spin}(n,1) \xrightarrow{\rho_{n,1}} \mathrm{GL}(\Delta). \tag{2.3}$$

Again, the restriction is still well defined as long as M is spin.

Let e^0, e^i be an orthonormal basis of the Minkowski space $\mathbb{R}^{n,1}$ such that $|e^0|^2=-1$ (in this section the indices i and j range from 1 to n).

Lemma 2.1: There is a positive definite Hermitian inner product $\langle , \rangle$ on Δ which is Spin(n)-invariant. Moreover, $(s,s')=\langle e^0 \cdot s, s' \rangle$ defines a Hermitian inner product which is also Spin(n)-invariant but not positive definite. In fact

$$(v \cdot s, s') = (s, v \cdot s')$$

for all $v \in \mathbb{R}^{n,1}$.

Proof: Detailed study via Γ matrices (Ref. 3, pp. 10 and 11) shows that there is a positive definite Hermitian inner product $\langle,\rangle$ on Δ with respect to which e^i is skew-Hermitian while e^0 is Hermitian. It follows then that $\langle,\rangle$ is Spin(n)-invariant. We now show that $(s,s')=\langle e^0\cdot s,s'\rangle$ defines a Spin(n)-invariant Hermitian inner product. Since e^0 is Hermitian with respect to $\langle,\rangle$, (,) is clearly Hermitian. To show that (,) is Spin(n)-invariant, we take a unit vector v in the Minkowski space, $v=a_0e^0+a_ie^i$, a_0, $a_i\in\mathbb{R}$, and $-a_0^2+\Sigma_{i=1}^n a_i^2=1$. Then

$$\begin{aligned}(vs,vs') = \langle e^0vs,vs'\rangle &= a_0^2\langle e^0e^0s,e^0s'\rangle + a_ia_0\langle e^0e^is,e^0s'\rangle + a_0a_i\langle e^0e^0s,e^is'\rangle + a_ia_j\langle e^0e^is,e^js'\rangle \\ &= a_0^2\langle s,e^0s'\rangle - a_ia_j\langle e^je^0e^is,s'\rangle = a_0^2\langle e^0s,s'\rangle + a_ia_j\langle e^0e^je^is,s'\rangle \\ &= a_0^2\langle e^0s,s'\rangle - a_i^2\langle e^0s,s'\rangle = -(s,s').\end{aligned}$$

Consequently, (,) is Spin(n)-invariant. The above computation also implies that $v\cdot$ acts as Hermitian operator on Δ with respect to (,). ■

Thus the spinor bundle S restricted to M inherits an Hermitian metric (,) and a positive definite metric $\langle,\rangle$. They are related by the equation

$$(s,s') = \langle e^0\cdot s,s'\rangle. \tag{2.4}$$

Now the hypersurface Dirac operator is defined by the composition

$$\mathcal{D}:\Gamma(M,S) \xrightarrow{\nabla} \Gamma(M,T^*M\otimes S) \xrightarrow{c} \Gamma(M,S), \tag{2.5}$$

where c denotes the Clifford multiplication. In terms of a local orthonormal basis $e_1,e_2,\ldots,e_n$ of TM,

$$\mathcal{D}\psi = e^i\cdot\nabla_{e_i}\psi,$$

where e^i denotes the dual basis.

The two most important properties of hypersurface Dirac operator are the self-adjointness with respect to the metric $\langle,\rangle$ and the Bochner–Lichnerowicz–Weitzenbock formula.[14,10]

Lemma 2.2: Define a $(n-1)$*-form on M by* $\omega=\langle\phi,e^i\cdot\psi\rangle\mathrm{int}(e_i)\mathrm{d}vol$, *where* d*vol is the volume form of the Riemannian metric g and* int(e_i) *is the interior multiplication by* e_i*. We have*

$$[\langle\phi,\mathcal{D}\psi\rangle - \langle\mathcal{D}\phi,\psi\rangle]\mathrm{d}vol = \mathrm{d}\omega.$$

Thus $\mathcal{D}$ *is formally self-adjoint with respect to the* L^2 *metric defined by* $\langle,\rangle$ *and* d*vol.*

Proof: Since ω is independent of the choice of the orthonormal basis, we do our computation locally using a preferred basis. For any given point $p\in M$, choose a local orthonormal frame e_i of TM near p such that $\overline{\nabla}e_i=0$ at p. Extend e_0, e_i to a neighborhood of p in N by parallel translating along e_0 direction. Then, at p, $\nabla_{e_i}e^j=-h_{ij}e^0$ and $\nabla_{e_i}e^0=-h_{ij}e^j$. Therefore (again at p),

$$\begin{aligned}\mathrm{d}\omega &= \nabla_{e_i}\langle\phi,e^i\cdot\psi\rangle\mathrm{d}vol \\ &= [((\nabla_{e_i}e^0)\cdot\phi,e^i\cdot\psi) + (e^0\cdot\nabla_{e_i}\phi,e^i\cdot\psi) + (e^0\cdot\phi,(\nabla_{e_i}e^i)\cdot\psi) + (e^0\cdot\phi,e^i\cdot\nabla_{e_i}\psi)]\mathrm{d}vol \\ &= [-h_{ij}(e^j\cdot\phi,e^i\cdot\psi) + (e^i\cdot e^0\cdot\nabla_{e_i}\phi,\psi) - h_{ii}(e^0\cdot\phi,e^0\cdot\psi) + \langle\phi,\mathcal{D}\psi\rangle]\mathrm{d}vol \\ &= [-h_{ij}(e^i\cdot e^j\cdot\phi,\psi) - \langle\mathcal{D}\phi,\psi\rangle - h_{ii}(e^0\cdot\phi,e^0\cdot\psi) + \langle\phi,\mathcal{D}\psi\rangle]\mathrm{d}vol \\ &= [-\langle\mathcal{D}\phi,\psi\rangle + \langle\phi,\mathcal{D}\psi\rangle]\mathrm{d}vol.\end{aligned}$$

■

The second property we need is the Bochner–Lichnerowicz–Weitzenbock formula. For a proof, see Ref. 10.

Lemma 2.3: One has

$$\mathcal{D}^2 = \nabla^* \nabla + \mathcal{R},$$
$$\mathcal{R} = \tfrac{1}{4}(R + 2R_{00} + 2R_{0i}e^0 \cdot e^i \cdot) \in \mathrm{End}(S). \tag{2.6}$$

Here the adjoint ∇^* *is with respect to the metric* $\langle,\rangle$.

III. PROOF OF THE THEOREM

By the Einstein equation,

$$\mathcal{R} = 4\pi(T_{00} + T_{0i}e^0 \cdot e^i \cdot).$$

It follows then from the dominant energy condition (1.4) that

$$\mathcal{R} \geqslant 0. \tag{3.1}$$

Now, for $\phi \in \Gamma(M,S)$ and a compact domain $\Omega \subset M$ with smooth boundary, the Bochner–Lichnerowicz–Weitzenbock formula yields

$$\int_\Omega [|\nabla \phi|^2 + \langle \phi, \mathcal{R}\phi\rangle - |\mathcal{D}\phi|^2] \mathrm{d}vol(g) = \int_{\partial\Omega} \sum \langle (\nabla_{e_a} + e_a \cdot \mathcal{D})\phi, \phi\rangle \mathrm{int}(e_a) \mathrm{d}vol(g) \tag{3.2}$$

$$= \int_{\partial\Omega} \sum \langle (\nabla_\nu + \nu \cdot \mathcal{D})\phi, \phi\rangle \mathrm{d}vol(g|_{\partial\Omega}), \tag{3.3}$$

where e_a is an orthonormal basis of g and ν is the unit outer normal of $\partial\Omega$.

Now without loss of generality, assume that M has only one end. That is, let the manifold $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$, and (X, g_X) a compact Riemannian manifold with nonzero parallel spinors. Moreover, the metric g on M satisfies (1.6). Let e_a^0 be the orthonormal basis of $\mathring{g}$ which consists of $\partial/\partial x_i$ followed by an orthonormal basis f_α of g_X. Orthonormalizing e_a^0 with respect to g gives rise an orthonormal basis e_a of g. Moreover,

$$e_a = e_a^0 - \tfrac{1}{2}u_{ab}e_b^0 + O(r^{-2\tau}). \tag{3.4}$$

This gives rise to a gauge transformation

$$A{:}\mathrm{SO}(\mathring{g}) \ni e_a^0 \to e_a \in \mathrm{SO}(g)$$

which identifies the corresponding spin groups and spinor bundles.

We now pick a unit norm parallel spinor ψ_0 of $(\mathbb{R}^k, g_{\mathbb{R}^k})$ and a unit norm parallel spinor ψ_1 of (X, g_X). Then $\phi_0 = A(\psi_0 \otimes \psi_1)$ defines a spinor of M_∞. We extend ϕ_0 smoothly inside. Then $\nabla^0 \phi_0 = 0$ outside the compact set.

Lemma 3.1: If a spinor ϕ *is asymptotic to* ϕ_0: $\phi = \phi_0 + O(r^{-\tau})$, *then we have*

$$\lim_{R\to\infty} \Re \int_{S_R \times X} \sum \langle (\nabla_{e_a} + e_a \cdot D)\phi, \phi\rangle \mathrm{int}(e_a) \mathrm{d}vol(g) = \omega_k vol(X) \langle \phi_0, E\phi_0 + P_k\, \mathrm{d}x^0 \cdot \mathrm{d}x^k \cdot \phi_0\rangle,$$

where $\Re$ *means taking the real part.*

Proof. Recall that $\overline{\nabla}$ denote the connection on S induced from the Levi–Civita connection on M. We have

$$\nabla_{e_a}\psi = \overline{\nabla}_{e_a}\psi - \tfrac{1}{2}h_{ab}e^0 \cdot e^b \cdot \psi. \tag{3.5}$$

By the Clifford relation,

$$\langle(\nabla_{e_a} + e_a \cdot D)\phi, \phi\rangle = -\tfrac{1}{2}\langle[e^a \cdot , e^b \cdot]\nabla_{e_b}\phi, \phi\rangle.$$

Hence

$$\int_{S_R\times X} \sum \langle(\nabla_{e_a} + e_a \cdot D)\phi, \phi\rangle \mathrm{int}(e_a)\mathrm{d}vol(g)$$
$$= -\frac{1}{2}\int_{S_R\times X} \langle[e^a \cdot , e^b \cdot]\overline{\nabla}_{e_b}\phi, \phi\rangle \mathrm{int}(e_a)\mathrm{d}vol(g)$$
$$+ \frac{1}{4}\int_{S_R\times X} \sum \langle[e^a \cdot , e^b \cdot]h_{bc}e^0 \cdot e^c \cdot \phi, \phi\rangle \mathrm{int}(e_a)\mathrm{d}vol(g).$$

Using (3.4) and the asymptotic conditions (1.7), the second term on the right-hand side can be easily seen to give us

$$\lim_{R\to\infty} \frac{1}{4}\int_{S_R\times X} \langle 2(h_{ac} - \delta_{ac}h_{bb})e^0 \cdot e^c \cdot \phi, \phi\rangle \mathrm{int}(e_a)\mathrm{d}vol(g) = \omega_k vol(X)\langle \phi_0, P_k\, \mathrm{d}x^0 \cdot \mathrm{d}x^k \cdot \phi_0\rangle.$$

The first term is computed in Ref. 4 to limit

$$\omega_k vol(X)\langle \phi_0, E\phi_0\rangle.$$

■

The following lemma is standard, see Refs. 10 and 14.

Lemma 3.2: If

$$\langle \phi_0, E\phi_0 + P_k\, \mathrm{d}x^0 \cdot \mathrm{d}x^k \cdot \phi_0\rangle \geqslant 0$$

for all constant spinors ϕ_0, *then*

$$E - |P| \geqslant 0.$$

As usual, the trick to get the positivity now is to find a harmonic spinor ϕ asymptotic to ϕ_0.

Lemma 3.3: There exists a harmonic spinor ϕ *on* (M,g) *which is asmptotic to the parallel spinor* ϕ_0 *at infinity*,

$$\mathcal{D}\phi = 0, \quad \phi = \phi_0 + O(r^{-\tau}).$$

Proof: The proof is essentially the same as in Ref. 4 [Cf. Refs. 6 and 9]. We use the Fredholm property of $\mathcal{D}$ on a weighted Sobolev space and $\mathcal{R} \geqslant 0$ to show that it is an isomorphism. The harmonic spinor ϕ can then be obtained by setting $\phi = \phi_0 + \xi$ and solving $\xi \in O(r^{-\tau})$ from the equation $\mathcal{D}\xi = -\mathcal{D}\phi_0$. ■

Thus, with the choice of harmonic spinor as above, the left-hand side of (3.4) will be non-negative since $\mathcal{R} \geqslant 0$. Taking the limit as $R \to 0$ and using Lemma 3.1 and Lemma 3.2 then give us the desired result.[4,9,10,14]

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