η-invariants and determinant lines

Xianzhe Dai[a)]
Department of Mathematics, University of Southern California, Los Angeles, California 90089

Daniel S. Freed[b)]
Department of Mathematics, University of Texas at Austin, Austin, Texas 78712

(Received 30 April 1994; accepted for publication 17 May 1994)

The η-invariant of an odd dimensional manifold with boundary is investigated. The natural boundary condition for this problem requires a trivialization of the kernel of the Dirac operator on the boundary. The dependence of the η-invariant on this trivialization is best encoded by the statement that the exponential of the η-invariant lives in the determinant line of the boundary. Our main results are a variational formula and a gluing law for this invariant. These results are applied to reprove the formula for the holonomy of the natural connection on the determinant line bundle of a family of Dirac operators, also known as the "global anomaly formula." The ideas developed here fit naturally with recent work in topological quantum field theory, in which gluing (which is a characteristic formal property of the path integral and the classical action) is used to compute global invariants on closed manifolds from local invariants on manifolds with boundary.

The η-invariant was introduced by Atiyah, Patodi, and Singer (APS)[1] in a series of papers treating index theory on *even*-dimensional manifolds with boundary. It first appears there as a boundary correction in the usual local index formula. Suppose X is a closed *odd*-dimensional spin manifold (which in their index theorem is the boundary of an even-dimensional spin manifold). The Dirac operator D_X is self-adjoint and has discrete real spectrum. (For simplicity we only consider the basic Dirac operator, though as usual in geometric index theory all of our results hold for twisted Dirac operators, i.e., for operators of "Dirac type.") Define

$$\eta_X(s)=\sum_{\lambda\neq 0}\frac{\text{sign }\lambda}{|\lambda|^s},\quad \text{Re}(s)\gg 0,$$

where the sum ranges over the nonzero spectrum of D_X. Then $\eta_X(s)$ is analytic in s and has a meromorphic continuation to $s\in\mathbb{C}$. It is regular at $s=0$, and its value there is the η-invariant. More precisely, what appears in the Atiyah–Patodi–Singer index formula is the ξ-invariant

$$\xi_X=\frac{\eta_X(0)+\dim\ker D_X}{2}.$$

Under a smooth variation of parameters (for example, the metric on X) the ξ-invariant jumps by integers, whereas ξ (mod 1) is smooth. In this paper we are interested in the latter, so consider the *exponentiated* ξ-invariant

$$\tau_X=e^{2\pi i\xi_X}$$

[a)]E-mail address: xdai@math.usc.edu
[b)]E-mail address: dafr@math.utexas.edu

0022-2488/94/35(10)/5155/40/$6.00
J. Math. Phys. 35 (10), October 1994

instead. In fact, our interest is in manifolds with boundary and we use "global" self-adjoint elliptic boundary conditions for the Dirac operator, which are the odd-dimensional analog of the Atiyah–Patodi–Singer boundary conditions.[1] To formulate these boundary conditions we need to choose a "trivialization" of the graded kernel of the Dirac operator on ∂X. (Other authors describe this choice as a Lagrangian subspace of the kernel, or as an involution on the kernel. All of these descriptions are equivalent.) The exponentiated ξ-invariant depends on this trivialization (Theorem 1.4) in a simple way.

Our first observation is that this dependence means that the exponentiated ξ-invariant naturally lives in the *inverse* determinant line of the Dirac operator on the boundary (Proposition 2.15). [An unfortunate choice of sign in the whole index theory—perhaps dating back to Fredholm—explains why it is the *inverse* determinant line which occurs here. An operator $D:H^+ \to H^-$ is an element of $H^- \otimes (H^+)^*$, so the codomain appears with a + sign and the domain with a − sign. It would be better, then, to define the index of D as dim coker $D-$dim ker D. To make the index theorem for manifolds with boundary come out, the ξ-invariant would also be defined with the opposite sign from the usual one, as would the $\hat{A}$-genus. On the other hand, the determinant line (2.7) is defined with the "proper" sign. Regardless of what is proper, this discrepancy explains some of the funny signs which crop up in index theory.] In fact, it has unit norm in the Quillen metric. For a family of Dirac operators this invariant is then a section of the inverse determinant line bundle over the parameter space. In Theorem 1.9 we generalize the usual formula for the variation of the ξ-invariant to a formula for the *covariant derivative* of this section. Here we use the natural connection on the (inverse) determinant line bundle defined by Bismut and Freed.[2] The proof of Theorem 1.9 occupies Sec. III. Our other main result is a *gluing formula* for the exponentiated ξ-invariant, which we state in Theorem 2.20 and prove in Sec. IV. To get the signs right in that theorem we view the determinant line as a *graded* vector space, as explained in Sec. II. In Sec. V we give a new proof of the holonomy formula for the natural connection on the determinant line bundle.[3,4] This formula was originally conjectured by Witten[5] in connection with global anomalies. It expresses the holonomy, or global anomaly, as the adiabatic limit of an exponentiated ξ-invariant. In Sec. VI we explain how our results lead to a conjecture about the geometrical index of families of Dirac operators on odd-dimensional manifolds with boundary. [We understand that ongoing work of Melrose and Piazza is expected to prove this conjecture. (Note added in proof: See the recent preprint "An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary" by R. B. Melrose and P. Piazza.)]

Our results build on previous work treating η-invariants on manifolds with boundary. Many different kinds of boundary conditions appear in these works. Cheeger (Ref. 6, Sec. 6) introduced the η-invariant (for the signature operator) on manifolds with conical singularities, and he noted that this corresponds to global boundary conditions on a manifold with boundary when one attaches a cone to the boundary. Further, his "ideal boundary conditions" correspond to the trivialization of the graded kernel on the boundary. In later work[6] he proves a variational formula for the η-invariant on a manifold with conical singularities. Gilkey and Smith[7] discuss the η-invariant for *local* boundary conditions, which were used in the original proof of the Atiyah–Singer index theorem to show that the index is a bordism invariant.[8] Singer[9] proved a formula relating the difference of η-invariants for two specific local boundary conditions with the determinant of the Laplacian on the boundary. Mazzeo and Melrose[10] assume that the boundary Dirac operator is invertible and then define an η-invariant using Melrose's "b-calculus." With this assumption they prove a gluing law. Dai[11] proved a formula relating this "b-eta invariant" to the η-invariant defined with local boundary conditions. Another approach is to attach a half-cylinder to the boundary and use L^2 spinor fields. This was considered in special cases by the della Pietras[12,13] and more generally by Klimek and Wojciechowski[14] and Müller.[15] Müller proves that this η-invariant is equal to the η-invariant for the global boundary conditions with a certain trivialization of the kernel picked out by the kernel of the Dirac operator on L^2 spinor fields. It is also easy to see that it agrees with the b-eta invariant if the metric near the boundary is asymptotically cylindrical. The self-adjoint global boundary conditions, and certain generalizations, were first studied by Douglas and Wojciechowski.[16] Müller[15] gives a systematic treatment of the analytic aspects of these self-adjoint boundary conditions. Lesch and Wojciechowski[17] determine the dependence of the exponentiated ξ-invariant on the boundary trivialization (Theorem 1.4). Müller[15] derives this result as well; his argument rests on a variational formula. Bunke[18] proves a gluing formula for the (unexponentiated) η-invariant in case a closed manifold is split into two pieces. Recent preprints of Wojciechowski[19,20] also prove gluing formulas for the η-invariant modulo one.

Our contribution here begins with our geometric formulation of the exponentiated ξ-invariant as taking values in the inverse determinant line. For example, this leads to a geometric variational formula (1.10) that is crucial in all of our subsequent work. In particular, the variational formula relates the exponentiated ξ-invariant to the natural connection on the determinant line bundle. The gluing law we prove (Theorem 2.20) is more general than that obtained by cutting a closed manifold into two pieces. This is necessary, for example, in Sec. V, where we glue together cylinders. Thus we must consider gluing along manifolds where the index of the Dirac operator may be nonzero. The most natural formulation of the result is in terms of *graded* determinant lines. This notion is discussed in Knudsen and Mumford[21] who credit the idea to Grothendieck. It also appears in later work of Deligne[22] as clearly the best way to avoid a *cauchemar de signes*! Our proof of the gluing law in Sec. IV is simpler than previous proofs. We begin with the same patching of spinor fields as in Bunke.[18] Then we note a symmetry that allows us to conclude easily with the variation formula. It is tempting to speculate that this approach to gluing may be useful in other linear problems and in nonlinear problems as well.

Our proof of the holonomy theorem—also known as the *global anomaly formula*—is considerably simpler than previous proofs, partly due to our simple proof of the gluing law. We rely heavily on geometric ideas. Thus we avoid any consideration of large time behavior of heat kernels, and we also avoid using nonpseudodifferential operators.[3] Cheeger's argument in Ref. 4, Sec. 9, which proves the adiabatic limit formula for the signature operator in the invertible case, is very closely related to our proof here. He works on a manifold with conical singularities and applies his variational formula and his "singular continuity method;" the latter is analogous to our use of gluing. The idea of considering parallel transport also appears in papers of the della Pietras,[12,13] but they failed to consider gluing. Our proof proceeds as follows: We use gluing to show that the adiabatic limit of exponentiated ξ-invariants on cylinders defines the parallel transport of a connection on the determinant line bundle. Then we apply our geometric variational formula to prove that it agrees with the natural connection. In a sense we use the gluing law to break up the holonomy, a global problem on the circle, into a composition of parallel transports—local problems on small intervals.

The idea of computing global invariants on closed manifolds from local invariants on manifolds with boundary using gluing laws is informed by recent work in *quantum field theory*, particularly *topological* quantum field theory. The gluing is a characteristic property of the path integral, and it follows formally from a similar property of the classical action. These gluing laws are fundamental for computing quantum Chern–Simons invariants, Donaldson polynomials, and other topological and geometric invariants. Older invariants in topology and geometry also obey gluing laws[23,24] and our work here fits the η-invariant into this story. The theory of the *classical* Chern–Simons invariant[25] is very similar, and of course the original papers of Atiyah, Patodi, and Singer[1] discuss the relationship of η-invariants (and so exponentiated ξ-invariants) to Chern–Simons invariants for closed manifolds. We also remark that certain ratios of exponentiated ξ-invariants are topological invariants that live in K^{-1}-theory with $\mathbb{R}/\mathbb{Z}$ coefficients.[1] Our work gives a factorization of these topological invariants as well. It is tempting to say that the exponentiated ξ-invariant is *local* and so can serve as an action for a field theory, just as the Chern–Simons invariant can. (For example, see the recent preprint Ref. 26.) One crucial difference is that the Chern–Simons invariant is multiplicative in coverings, whereas the exponentiated ξ-invariant is *not*. In any case, the gluing law does exhibit some local properties of the η-invariant.

The suggestion that the η-invariant of a (three-) manifold with boundary lives in the determinant line of the boundary was made in a manuscript of Graeme Segal.[27]

I. THE EXPONENTIATED ξ-INVARIANT

Suppose X is a compact odd-dimensional spin manifold with nonempty boundary. (We understand a spin manifold to have a definite metric, orientation, and spin structure. Our work extends to spin^c manifolds and to Dirac operators twisted by a vector bundle with connection, but for simplicity we omit these refinements.) Assume that X has a metric with an explicit product structure near ∂X. Thus in a neighborhood of the boundary there is a given isometry with $(-1,0]\times\partial X$. Let H_X denote the Hilbert space of L^2 spinor fields on X and $D_X:H_X\to H_X$ the formally self-adjoint Dirac operator. We use similar notation for the induced Dirac operator on the boundary.

Our first job is to specify self-adjoint elliptic boundary conditions. Our discussion here is somewhat formal. We leave the detailed analysis to the Appendix. Let $J:H_{\partial X}\to H_{\partial X}$ be Clifford multiplication by the *outward* unit normal vector field to the boundary. Then J is skew-adjoint, $J^2=-1$, and $D_{\partial X}J=-JD_{\partial X}$. The $\mp i$-eigenspaces of J induce the usual splitting $H_{\partial X}=H^+_{\partial X}\oplus H^-_{\partial X}$. Now integration by parts yields the formula

$$(D_X\psi,\varphi)_X-(\psi,D_X\varphi)_X=(J\psi,\varphi)_{\partial X},\quad \psi,\varphi\in H_X.$$

Thus if our boundary condition is described by $\psi|_{\partial X}\in W\subset H_{\partial X}$, then the corresponding Dirac operator is self-adjoint if $JW=W^\perp$, at least formally. We also need *elliptic* boundary conditions, so we choose W "close" to the subspace that describes the Atiyah–Patodi–Singer nonlocal boundary conditions.[1]

Our precise choice is this. The non-negative self-adjoint operator $D^2_{\partial X}$ induces decompositions

$$H^\pm_{\partial X}=K^\pm_{\partial X}\oplus\bigoplus_{\lambda>0}E^\pm_{\partial X}(\lambda),$$

where $K^+_{\partial X}\oplus K^-_{\partial X}$ is the kernel of $D_{\partial X}$ and $E^+_{\partial X}(\lambda)\oplus E^-_{\partial X}(\lambda)$ is the eigenspace with eigenvalue λ. The sum is over the spectrum spec$(D^2_{\partial X})$. Note that

$$D_{\partial X}:E^+_{\partial X}(\lambda)\to E^-_{\partial X}(\lambda)$$

is an isomorphism, though it is not unitary—it is $\sqrt{\lambda}$ times a unitary map. Also, by the cobordism invariance of the index[8] we have index $D_{\partial X}=0$ and so $\dim K^+_{\partial X}=\dim K^-_{\partial X}$. Now for any positive $a\notin \mathrm{spec}(D^2_{\partial X})$ let

$$K^\pm_{\partial X}(a)=K^\pm_{\partial X}\oplus\bigoplus_{0<\lambda<a}E^\pm_{\partial X}(\lambda),\quad H^\pm_{\partial X}(a)=\bigoplus_{\lambda>a}E^\pm_{\partial X}(\lambda).\tag{1.1}$$

By ellipticity $K^\pm_{\partial X}(a)$ is finite dimensional. A choice of boundary condition $W_{\langle a,T\rangle}$ is determined by the number a and by a choice of *isometry*

$$T:K^\pm_{\partial X}(a)\to K^-_{\partial X}(a).$$

Let $D_{\partial X}/\sqrt{D^2_{\partial X}}$ denote the operator which restricts to $D_{\partial X}/\sqrt{\lambda}$ on $E^+_{\partial X}(\lambda)$; it is defined on $H^+_{\partial X}\ominus K^+_{\partial X}$. We denote its restriction to $H^+_{\partial X}(a)$ by $D_{\partial X}(a)/\sqrt{D_{\partial X}(a)^2}$. A spinor field $\phi^+\in H^+_{\partial X}$ decomposes according to $H^+_{\partial X}=K^+_{\partial X}(a)\oplus H^+_{\partial X}(a)$. Then

$$W_{\langle a,T\rangle}=\left\{\langle\phi^+,\phi^-\rangle\in H_{\partial X}:\phi^-+\left(T\oplus\frac{D_{\partial X}(a)}{\sqrt{D_{\partial X}(a)^2}}\right)\phi^+=0\right\}. \tag{1.2}$$

This is a generalization of the boundary condition studied by previous authors[15-18] who choose a less than the first eigenvalue of $D^2_{\partial X}$. (Other authors describe the isometry T by its graph, which is a *Lagrangian* subspace of the kernel.) We need this generalization to treat families.

Now for any choice $\langle a,T\rangle$ of boundary conditions the Dirac operator $D_X(a,T)$ is self-adjoint elliptic and has a well-defined η-invariant $\eta_X(a,T)$. (See the Appendix.) We use the more refined ξ-invariant

$$\xi_X(a,T)=\frac{\eta_X(a,T)+\dim\ker D_X(a,T)}{2}$$

and set

$$\tau_X(a,T)=e^{2\pi i\xi_X(a,T)}.$$

Our first result is a generalization of Refs. 17, 18 (Corollary 9.3), and 15 (Theorem 2.21). It computes the dependence of $\tau_X(a,T)$ on $\langle a,T\rangle$. To state it note that if $0<a<b$ with $a,b\notin\operatorname{spec}(D^2_{\partial X})$, and $T:K^+_{\partial X}(a)\to K^-_{\partial X}(a)$ is an isometry, then $T\oplus D_{\partial X}(a,b)/\sqrt{D_{\partial X}(a,b)^2}$: $K^+_{\partial X}(b)\to K^-_{\partial X}(b)$ is also a unitary isomorphism. Here $D_{\partial X}(a,b)$ denotes the restriction of $D_{\partial X}$ to

$$H^{\pm}_{\partial X}(a,b)=\bigoplus_{a<\lambda<b}E^{\pm}_{\partial X}(\lambda). \tag{1.3}$$

Theorem 1.4: Suppose $0<a<b$ with $a,b\notin\operatorname{spec}(D^2_{\partial X})$ and $T,T_1,T_2:K^+_{\partial X}(a)\to K^-_{\partial X}(a)$ are isometries. Then

$$\tau_X(a,T_2)=\det(T_1^{-1}T_2)\tau_X(a,T_1), \tag{1.5}$$

$$\tau_X\left(b,T\oplus\frac{D_{\partial X}(a,b)}{\sqrt{D_{\partial X}(a,b)^2}}\right)=\tau_X(a,T). \tag{1.6}$$

Equation (1.6) is trivial since $W_{\langle b,T\oplus D_{\partial X}(a,b)/\sqrt{D_{\partial X}(a,b)^2}\rangle}=W_{\langle a,T\rangle}$. We defer the proof of (1.5) to Sec. IV (Corollary 4.22).

We can interpret (1.5) and (1.6) as instructions for constructing a Hermitian line $L_{\partial X}$ and an element $\tau_X\in L_{\partial X}$. Namely, let $\mathscr{C}_{\partial X}=\{\langle a,T\rangle\}$ be the set of possible boundary conditions and then define the complex line

$$L_{\partial X}=\{\tau:\mathscr{C}_{\partial X}\to\mathbb{C}:\tau \text{ satisfies (1.5) and (1.6)}\}. \tag{1.7}$$

Since $|\det(T_1^{-1}T_2)|=1$ in (1.5), we see that the expression

$$(\tau_1,\tau_2)=\tau_1(a,T)\overline{\tau_2(a,T)}$$

is independent of $\langle a,T\rangle$ and so defines a Hermitian metric on $L_{\partial X}$. By construction $\tau_X\in L_{\partial X}$ is an element of unit norm.

We use a patching construction to extend to families (cf. Ref. 28). Let $\pi:X\to Z$ be a fiber bundle whose typical fiber is a compact odd-dimensional manifold with boundary, and let $\partial\pi:\partial X\to Z$ be the fiber bundle of the boundaries. A Riemannian structure on $X\to Z$ is a metric on the relative tangent bundle $T(X/Z)$ together with a field of horizontal planes on X, which we specify as the kernel of a projection $P:TX\to T(X/Z)$. Suppose also that $T(X/Z)$ is endowed with an orientation and spin structure. For simplicity we term π a "spin map." For our purposes we also assume that the metrics are products near the boundaries. Now for each $a>0$ define

$$U_a=\{z\in Z : a\notin \operatorname{spec}(D^2_{\partial X_z})\}.$$

On this open set $K^{\pm}_{\partial X_z}(a)$ are smooth vector bundles of equal rank. Choose a cover

$$U_a=\bigcup_i U_{a,i} \tag{1.8}$$

so that these bundles are isomorphic over each $U_{a,i}$. Then choose a smooth family of isomorphisms $T_z(a,i):K^+_{\partial X_z}(a)\to K^-_{\partial X_z}(a)$ and compute $\tau_{X_z}(a,T_z(a,i))$, which is a smooth function of z. The collection of these functions for various choices of a, i, and $T_z(a,i)$ satisfy (1.5) and (1.6). Definition (1.7) extends to this situation—now everything depends smoothly on z—to define a Hermitian line bundle $L_{\partial X/Z}\to Z$. The functions $\tau_{X_z}(a,T_z(a,i))$ patch together to form a smooth section $\tau_{X/Z}$ of $L_{\partial X/Z}$.

In Sec. II we identify $L_{\partial X/Z}$ as the inverse determinant line bundle of the family of Dirac operators on $\partial X\to Z$ with its Quillen metric. This line bundle carries a natural unitary connection ∇, constructed in Ref. 2. (In Sec. V we define a connection ∇' directly on $L_{\partial X/Z}$ using the invariant τ_X. We prove that it agrees with ∇ under the isomorphism with the inverse determinant line bundle.) The following theorem computes the covariant derivative of $\tau_{X/Z}$ with respect to this connection; it generalizes the standard formula on closed manifolds (e.g., Ref. 3, Theorem 2.10).

Theorem 1.9: Let $\pi:X\to Z$ be a spin map whose typical fiber is an odd-dimensional manifold with boundary. Let $\Omega^{X/Z}$ denote the curvature of the relative tangent bundle and $\hat{A}(\Omega^{X/Z})$ its $\hat{A}$-polynomial. Then the covariant derivative of the exponentiated ξ-invariant is

$$\nabla\tau_{X/Z}=2\pi i\left[\int_{X/Z}\hat{A}(\Omega^{X/Z})\right]_{(1)}\cdot\tau_{X/Z}. \tag{1.10}$$

In (1.10) we use the standard sign convention (e.g., Ref. 29) for integration over the fiber. For example, if α is a form on Z and β an n-form on an oriented manifold X^n, then

$$\int_{(Z\times X)/Z}\alpha\wedge\beta=\left(\int_X\beta\right)\alpha.$$

We defer the proof to Sec. III.

II. GRADED DETERMINANT LINES

Our first goal in this section is to identify the Hermitian line $L_{\partial X}$ (1.7) with the *inverse* determinant line $\operatorname{Det}^{-1}_{\partial X}$ of the Dirac operator $D_{\partial X}$. (The inverse L^{-1} of a one-dimensional vector space L is its dual L^*.) The Hermitian structure on $\operatorname{Det}_{\partial X}$ is due to Quillen.[30] We then state various properties of τ_X and $L_{\partial X}$, the most important of which is the gluing law (Theorem 2.20). Here we encounter inverse determinant lines for operators of nonzero index. Then the gluing law involves some signs that are best understood in terms of the *grading* on the determinant line given by the index.[21] Hence we begin this section with an exposition of graded vector spaces.

A *graded vector space* $V=V^+\oplus V^-$ is simply a direct sum of two vector spaces, which in this paper we always take to be complex. We call V^+ (resp. V^-) the even (resp. odd) part of V, and write $|v|=0$ (resp. $|v|=1$) for $v\in V^+$ (resp. $v\in V^-$). For graded vector spaces V,W we write $V\hat{\otimes}W$ for the graded vector space whose underlying vector spaces is $V\otimes W$ and with $|v\otimes w|\equiv|v|+|w|$ (mod 2) for homogeneous elements $v\in V$, $w\in W$. We use the $\hat{\otimes}$ notation to keep track of signs in the isomorphism

$$V\hat{\otimes} W\to W\hat{\otimes} V, \quad v\otimes w\mapsto(-1)^{|v||w|}w\otimes v, \quad v\in V, \quad w\in W. \tag{2.1}$$

Here, as in subsequent expressions, we use homogeneous elements and extend by linearity. The dual space $V^*=(V^+)^*\oplus(V^-)^*$ of a graded vector space is also graded, and we use the natural pairing

$$V^*\hat{\otimes} V\to\mathbb{C}, \quad \check{v}\otimes v\mapsto\check{v}(v), \quad v\in V, \quad \check{v}\in V^*. \tag{2.2}$$

The order of the factors in (2.2) is important! With this choice there is no sign in (2.2), nor is there any in the isomorphisms

$$V^*\hat{\otimes} W^*\to(W\hat{\otimes} V)^*, \quad \check{v}\otimes\check{w}\mapsto(\ell: w\otimes v\mapsto\check{v}(v)\check{w}(w)), \tag{2.3}$$

and

$$W\hat{\otimes} V^*\to\mathrm{Hom}(V,W), \quad w\otimes\check{v}\mapsto(T: v\mapsto\check{v}(v)w). \tag{2.4}$$

Notice that the natural isomorphism

$$V\to V^{**}, \quad v\to(\ell:\check{v}\mapsto(-1)^{|v||\check{v}|}\check{v}(v)), \tag{2.5}$$

picks up a sign in the graded context. The sequence of homomorphisms

$$\mathrm{Tr}_s:\mathrm{End}(V)\xrightarrow{(2.4)}V\hat{\otimes} V^*\xrightarrow{(2.1)}V^*\hat{\otimes} V\xrightarrow{(2.2)}\mathbb{C} \tag{2.6}$$

is the *supertrace*: For $T=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{End}(V^+\oplus V^-)$ we have $\mathrm{Tr}_s\, T=\mathrm{Tr}\, A-\mathrm{Tr}\, D$.

The *determinant line* Det V of an *ungraded* vector space V is the one-dimensional vector space of totally antisymmetric tensors $\omega=v_1\wedge\cdots\wedge v_n$. We view Det V as a *graded* vector space whose degree is dim V (mod 2). If $V=V^+\oplus V^-$ is graded, then define

$$\mathrm{Det}\, V=(\mathrm{Det}\, V^-)\hat{\otimes}(\mathrm{Det}\, V^+)^{-1}. \tag{2.7}$$

This is again a graded line, the grading given by

$$|\mathrm{Det}\, V|\equiv\dim V\equiv\dim V^+-\dim V^-(\mathrm{mod}\ 2).$$

Using (2.4) we see that if $\dim V^+=\dim V^-$, then the top exterior power of a homomorphism $T: V^+\to V^-$ determines an element

$$\mathrm{Det}\, T\in\mathrm{Det}\, V. \tag{2.8}$$

If $V^+=V^-$, then T has a numerical determinant $\det T\in\mathbb{C}$, and this is related to (2.8) via the supertrace (2.6):

$$\mathrm{Tr}_s(\mathrm{Det}\, T)=(-1)^{\dim V^+}\det T. \tag{2.9}$$

Let $-V$ denote V with the opposite grading: $(-V)^{\pm}=V^{\mp}$. Note the sign in the isomorphism

$$\begin{gathered}\mathrm{Det}(-V)\to\mathrm{Det}(V)^{-1},\\ \omega^+\otimes\check{\omega}^-\mapsto(\ell:\omega^-\otimes\check{\omega}^+\mapsto(-1)^{\dim V^+}\check{\omega}^+(\omega^+)\check{\omega}^-(\omega^-)),\end{gathered} \tag{2.10}$$

where $\omega^{\pm}\in\mathrm{Det}(V^{\pm})$ and $\check{\omega}^{\pm}\in\mathrm{Det}(V^{\pm})^{-1}$. Similarly, if W is another graded vector space, then there is a sign in the isomorphism

$$\begin{gathered}\operatorname{Det}(V\oplus W)\to\operatorname{Det} V\hat{\otimes}\operatorname{Det} W,\\ \omega^-\otimes\eta^-\otimes\check{\eta}^+\otimes\check{\omega}^+\mapsto(-1)^{\dim V^+\dim W}\omega^-\otimes\check{\omega}^+\otimes\eta^-\otimes\check{\eta}^+,\end{gathered}\tag{2.11}$$

where $\omega^{\pm}\in\operatorname{Det}(V^{\pm})$ and $\eta^{\pm}\in\operatorname{Det}(W^{\pm})$.

As a matter of notation, if $\omega\in L$ is a nonzero element of a graded line L, then we denote by $\omega^{-1}\in L^{-1}$ the unique element so that $\omega^{-1}(\omega)=1$ under the pairing (2.2).

Suppose V,W are graded vector spaces with $\dim V^+=\dim V^-$ and $\dim W^+=\dim W^-$. Note in particular that $\dim W$ and $\dim V$ are even. Then for $T:V^+\to V^-$ and $S:W^+\to W^-$ we have

$$\operatorname{Det}(T^{-1})=(-1)^{\dim V^+}(\operatorname{Det} T)^{-1},$$

$$\operatorname{Det}(T\oplus S)=\operatorname{Det} T\otimes\operatorname{Det} S.$$

The equalities here stand for the isomorphisms (2.10) and (2.11).

Next, we review the construction of the determinant line of a Dirac operator (see Ref. 28 for details), but now as a *graded* line. Let Y be a closed even-dimensional spin manifold. The spinor fields $H_Y=H_Y^+\oplus H_Y^-$ on Y are graded, and the Dirac operator $D_Y:H_Y^+\to H_Y^-$ anticommutes with the grading. We use the notations $K_Y(a)$, $H_Y(a)$, and $H_Y(a,b)$ from (1.1) and (1.3), where $a<b$ are positive numbers not in $\operatorname{spec}(D_Y^2)$. Now $D_Y(a,b)=D_Y:H_Y^+(a,b)\to H_Y^-(a,b)$ is an isomorphism, so

$$\operatorname{Det} D_Y(a,b)\in\operatorname{Det} H_Y(a,b)$$

is a nonzero element. Define an isomorphism

$$\begin{gathered}\theta_Y(a,b):\operatorname{Det} K_Y(a)\to\operatorname{Det} K_Y(a)\hat{\otimes}\operatorname{Det} H_Y(a,b)\cong\operatorname{Det} K_Y(b),\\ \omega(a)\mapsto\omega(a)\otimes\operatorname{Det} D_Y(a,b).\end{gathered}\tag{2.12}$$

Then an element of the determinant line is defined to be a set of compatible elements $\omega(a)\in\operatorname{Det} K_Y(a)$:

$$\operatorname{Det}_Y=\{\omega=\{\omega(a)\in\operatorname{Det} K_Y(a)\}_{a\notin\operatorname{spec}(D_Y^2)}:\omega(b)=\theta_Y(a,b)\omega(a)\}.$$

Note that

$$|\operatorname{Det}_Y|\equiv\operatorname{index} D_Y\pmod 2.$$

Now the lines $\operatorname{Det} K_Y(a)$ and $\operatorname{Det} H_Y(a,b)$ inherit Hermitian metrics from the L^2 metric on H_Y, and we compute

$$|\theta(a,b)\omega(a)|^2_{K_Y(b)}=\left(\prod_{a<\lambda<b}\lambda\right)|\omega(a)|^2_{K_Y(a)},\quad\omega(a)\in\operatorname{Det} K_Y(a).$$

Hence the expression

$$|\omega|^2_{\operatorname{Det}_Y}=\left(\prod_{\lambda>a}\lambda\right)|\omega(a)|^2_{\operatorname{Det} K_Y(a)}$$

is independent of a, where the product is defined using a ζ-function. This defines the *Quillen metric* on Det_Y.

A careful computation shows that (2.10) and (2.11) are compatible with the "patching" isomorphism $\theta(a,b)$ in (2.12), so they determine isometries

$$\mathrm{Det}_{-Y} \cong \mathrm{Det}_Y^{-1}, \tag{2.13}$$

$$\mathrm{Det}_{Y_1 \sqcup Y_2} \cong \mathrm{Det}_{Y_1} \hat{\otimes} \mathrm{Det}_{Y_2}. \tag{2.14}$$

Here Y, Y_1, Y_2 are closed spin manifolds, '$-Y$' denotes the spin manifold Y with the opposite orientation, and '$Y_1 \sqcup Y_2$' denotes the disjoint union of Y_1 and Y_2. [Let $\mathrm{Spin}(Y) \to Y$ denote the principal Spin_n bundle which defines the spin structure of Y; it is a double cover of the bundle of oriented orthonormal frames. Then the spin structure on $-Y$ is defined by the complement of $\mathrm{Spin}(Y)$ in the Pin_n bundle of frames $\mathrm{Spin}(Y) \times_{\mathrm{Spin}_n} \mathrm{Pin}_n \to Y$.]

The patching isomorphism used to patch the inverse determinant line (which appears in (2.13), for example) is

$$(\theta_Y(a,b)^*)^{-1} : (\mathrm{Det}\, K_Y(a))^{-1} \to (\mathrm{Det}\, H_Y(a,b))^{-1} \hat{\otimes} (\mathrm{Det}\, K_Y(a))^{-1} \cong (\mathrm{Det}\, K_Y(b))^{-1},$$

$$\eta(a) \mapsto (\mathrm{Det}\, D_Y(a,b))^{-1} \otimes \eta(a).$$

With this understood we can identify the Hermitian line determined by the exponentiated ξ-invariant.

Proposition 2.15: Let X be a compact odd-dimensional spin manifold and $L_{\partial X}$ the Hermitian line defined in (1.7). Then

$$L_{\partial X} \to \mathrm{Det}_{\partial X}^{-1},$$
$$\{\tau(a,T) \in \mathbb{C}\} \mapsto \left\{ \eta(a) = \tau(a,T) \left(\prod_{\lambda > a} \lambda \right)^{1/2} (\mathrm{Det}\, T)^{-1} \in (\mathrm{Det}\, K_{\partial X}(a))^{-1} \right\} \tag{2.16}$$

is an isometry.

The proof is straightforward. First, (1.5) and (1.6) imply that $\{\eta(a)\}$ defines an element of $\mathrm{Det}_{\partial X}^{-1}$. Then (1.7) and (2.21) imply that the isomorphism (2.16) is an isometry. Here, following Ray and Singer,[31] we use a ζ-function to define the infinite product in this isometry.

From now on we identify $L_{\partial X}$ as the inverse determinant line. So for *any* closed even-dimensional spin manifold Y the Hermitian line L_Y is defined.

Now we state some properties of the lines L_Y and the exponentiated ξ-invariant τ_X. (It might be illuminating to compare with the analogous assertions about the Chern–Simons invariant in Ref. 25, Theorem 2.19.) For simplicity we state these for a single manifold X rather than for families. However, they work as stated for families, and the proofs are designed to work with the patching construction of Sec. I. [Recall that this is our motivation to allow arbitrary a in (1.2).]

First, (2.13) and (2.14) imply that there are isometries

$$L_{-Y} \cong L_Y^{-1}, \tag{2.17}$$

$$L_{Y_1 \sqcup Y_2} \cong L_{Y_1} \hat{\otimes} L_{Y_2}. \tag{2.18}$$

[Note that (2.17) is *not* the inverse of (2.13); the sign in (2.5) enters. Also, one must keep in mind (2.3) when comparing (2.14) and (2.18).] For the exponentiated ξ-invariant we have

$$\tau_{-X} = \tau_X^{-1}, \quad \tau_{X_1 \sqcup X_2} = \tau_{X_1} \otimes \tau_{X_2},$$

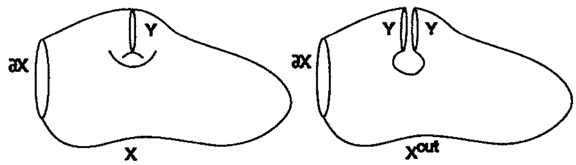

FIG. 1. Cutting a manifold X along Y.

where we use the isomorphisms (2.17) and (2.18) to compare the left- and right-hand sides of these equalities.

If Y,Y' are spin manifolds, then we define a *spin isometry* $\tilde{f}$ to be an ordinary isometry $f:Y'\to Y$ together with a lift $\tilde{f}:\mathrm{Spin}(Y')\to\mathrm{Spin}(Y)$ to the spin bundle of frames. A spin isometry induces an isometry

$$L_{Y'}\xrightarrow{\tilde{f}_*}L_Y$$

of inverse determinant lines. If $\tilde{F}:\mathrm{Spin}(X')\to\mathrm{Spin}(X)$ is a spin isometry, then

$$(\partial\tilde{F})_*(\tau_{X'})=\tau_X.$$

Any spin manifold Y has a naturally defined spin isometry $\tilde{\iota}:\mathrm{Spin}(Y)\to\mathrm{Spin}(Y)$ that is multiplication by $-1\in\mathrm{Spin}_n$; it covers the identity diffeomorphism on Y. The induced map on the inverse determinant line is

$$\tilde{\iota}_*=(-1)^{\mathrm{index}\,D_Y}. \tag{2.19}$$

The most important property of the exponentiated ξ-invariant is the *gluing law*.

Theorem 2.20: Let X be a compact odd-dimensional spin manifold, $Y\hookrightarrow X$ a closed oriented hypersurface, and X^{cut} the manifold obtained by cutting X along Y. (See Fig. 1.) We assume that the metric on X^{cut} is a product near $\partial X^{\mathrm{cut}}=\partial X\sqcup Y\sqcup -Y$. Then

$$\tau_X=\mathrm{Tr}_s(\tau_{X^{\mathrm{cut}}}), \tag{2.21}$$

where Tr_s is the contraction

$$L_{\partial X^{\mathrm{cut}}}\xrightarrow{(2.18)}L_{\partial X}\hat{\otimes}L_Y\hat{\otimes}L_{-Y}\xrightarrow{(2.17)}L_{\partial X}\hat{\otimes}L_Y\hat{\otimes}L_Y^{-1}\xrightarrow{\mathrm{Tr}_s}L_{\partial X} \tag{2.22}$$

using the supertrace (2.6).

Notice that index D_Y is not necessarily zero, which is why we introduce graded determinant lines. We prove Theorem 2.20 in Sec. III.

To illustrate the gluing law consider an arbitrary closed even-dimensional spin manifold Y and form the cylinder $C=[-1,1]\times Y$ with the product metric and spin structure. Then $\tau_C\in L_Y\hat{\otimes}L_{-Y}\cong\mathrm{End}(L_Y)$. If we cut C along $\{0\}\times Y$, we obtain a manifold "spin isometric" to $C\sqcup C$. Then (2.21) asserts that $\tau_C=\tau_C\circ\tau_C$, where '∘' denotes composition in $\mathrm{End}(L_Y)$. We conclude

$$\tau_C=\mathrm{id}\in\mathrm{End}(L_Y). \tag{2.23}$$

This equation is derived assuming the gluing law (2.21). In Sec. IV we compute it directly (Proposition 4.7) as part of our proof of (2.21).

Recall that the circle S^1 admits two inequivalent spin structures, and we denote the corresponding spin manifolds 'S^1_{bounding}' and '$S^1_{\text{nonbounding}}$.' The former is the boundary of the disk (with its unique spin structure), while for the latter the bundle $\text{Spin}(S^1_{\text{nonbounding}}) \to \text{SO}(S^1)$ is the trivial double cover of the bundle of oriented orthonormal frames $\text{SO}(S^1)$. Now consider $S^1_{\text{nonbounding}} \times Y$ with the product metric and product spin structure. If we cut along $\{pt\} \times Y$, we obtain C, and the gluing law (2.21) asserts

$$\tau_{S^1_{\text{nonbounding}} \times Y} = \text{Tr}_s(\tau_C) = \text{Tr}_s(\text{id}) = (-1)^{\text{index } D_Y}. \tag{2.24}$$

On the other hand, if we apply the spin isometry ι to one boundary component of C and then glue, we obtain $S^1_{\text{bounding}} \times Y$. It follows from (2.19) that

$$\tau_{S^1_{\text{bounding}} \times Y} = (-1)^{\text{index } D_Y} \, \text{Tr}_s(\tau_C) = 1. \tag{2.25}$$

Equations (2.24) and (2.25) agree with known results and provide a simple check of the signs in the gluing law.

III. THE VARIATION FORMULA

The purpose of this section is to present the proof of Theorem 1.9.

Let $\pi: X \to Z$ be a spin map whose typical fiber is a compact odd-dimensional manifold with boundary. Since the assertion to be proved is local, it suffices to work over an open set $U_{a,i}$, defined in (1.8). Over $U_{a,i}$ we have smooth isomorphic Hermitian bundles $K^{\pm}_{\partial X/Z}(a)$ and we choose a smooth family of isometries

$$T = T_z(a,i) : K^+_{\partial X_z}(a) \to K^-_{\partial X_z}(a). \tag{3.1}$$

By Proposition 2.15 over the open set $U_{a,i}$, the smooth section $\tau_{X/Z}$ of $L_{\partial X/Z} \to Z$ can be identified with

$$\tau_{X/Z} = e^{2\pi i \xi_X(a,T)} u^{-1},$$

where

$$u = (\text{Det } T) / \sqrt{\det D^2_{\partial X/Z}(a)} \in \text{Det}_{\partial X/Z} \tag{3.2}$$

is a smooth section of unit Quillen norm. Clearly, then, Theorem 1.9 is equivalent to the following.

Theorem 3.3: Modulo the integers $\xi_X(a,T(a,i))$ defines a smooth function on $U_{a,i}$ and

$$d\xi_X(a,T) = \left[\int_{X/Z} \hat{A}(\Omega^{X/Z}) \right]_{(1)} + \frac{1}{2\pi i} u^{-1} \nabla u.$$

As we mentioned earlier the connection ∇ here is the natural unitary connection on the determinant line bundle introduced in Ref. 2 by Bismut–Freed. It is a natural generalization of the induced connection in the finite-dimensional case to the infinite-dimensional setting and uses the heat equation regularization. For our purpose we recall its construction. (See Ref. 28 for a treatment in terms of ζ-functions.)

Let $\pi: Y = \partial X \to Z$ be a spin map and $D^+ = D^+_{Y/Z}$ be the family of fiber Dirac operators. (Everything works even if Y is not a boundary.) Now D^+ can be considered as a smooth section of $\text{Hom}(H^+, H^-)$, where $H^{\pm}$ are infinite-dimensional Hermitian bundles over Z (see Ref. 2 for details). Assume for the moment that $H^{\pm}$ are finite-dimensional Hermitian bundles over Z. In this case the determinant line bundle can be identified with $(\mathrm{Det}\,H^-)\otimes(\mathrm{Det}\,H^+)^{-1}$, and so is naturally endowed with a Hermitian metric. Clearly $\mathrm{Det}\,D^+$ is a smooth section. Now if $H^\pm$ are also endowed with unitary connections $\tilde{\nabla}$, then they induce a unitary connection ∇ on the determinant line bundle. In fact when D^+ is invertible,

$$\nabla \mathrm{Det}\,D^+ = \mathrm{Tr}[(D^+)^{-1}\tilde{\nabla}D^+]\cdot \mathrm{Det}\,D^+. \tag{3.4}$$

Further, if $H^\pm = K^\pm \oplus H_1^\pm$ is an orthogonal decomposition invariant under D^+, then

$$\nabla = \nabla^K + \nabla^{H_1}. \tag{3.5}$$

These two properties fully suggest how to define it in the infinite-dimensional setting.

Thus over U_a let

$$H^\pm = K^\pm(a)\oplus H^\pm(a)$$

be the orthogonal decomposition defined in Sec. I. The infinite-dimensional Hermitian bundles $H^\pm$ are equipped with the unitary connection $\tilde{\nabla}$ defined in Ref. 3, Def. 1.3. (Note that the notation there for that connection is '$\tilde{\nabla}^u$.') Over U_a we have smooth finite-dimensional subbundles $K^\pm(a)$ of $H^\pm$. Hence they inherit a unitary connection, which in turn induces a unitary connection ∇^a on $(\mathrm{Det}\,K^-(a))\hat{\otimes}(\mathrm{Det}\,K^+(a))^{-1}$. By the additivity (3.5) this is the $K^\pm(a)$-part of the connection.

To define the $H^\pm(a)$-part of the connection one makes sense of (3.4) in the infinite-dimensional setting by the heat equation regularization. Note that the restriction $D^+(a)$ of D^+ to $H^+(a)$ is indeed invertible. When there is no confusion we also use '$D^2(a)$' [instead of '$D^-(a)D^+(a)$'] to denote the restriction of D^2 to $H^+(a)$. The formal expression $\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)]$ will be defined by

$$\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)] = \mathrm{f.p.}\{\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}]\}, \tag{3.6}$$

where f.p. is a suitably defined finite part of the right-hand side of (3.6) as $t\to 0$.

To define this finite part, note that

$$\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}] = \int_t^\infty \mathrm{Tr}[(D^-(a))\tilde{\nabla}D^+(a)e^{-sD^2(a)}]ds.$$

It follows that as $t\to 0$

$$\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}] \sim \sum_{j=-n/2}^{-1} a_j t^j + a_0 + a_{0,1}\log t + \sum_{j=1}^{\infty} a_j t^j.$$

Then the finite part is defined as

$$\mathrm{f.p.}\{\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}]\} = a_0 + \Gamma'(1)a_{0,1},$$

or symbolically,

$$\mathrm{f.p.}\{\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}]\} = \mathrm{LIM}_{t\to 0}\,\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}] + \Gamma'(1)\mathrm{LIM}_{t\to 0}\frac{1}{\log t}\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}], \tag{3.7}$$

Finally the Bismut–Freed connection is defined as

$$\nabla = \nabla^a + \text{f.p.}\{\mathrm{Tr}[(D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}]\}.$$

Coming back to Theorem 3.3, when $D_{\partial X/Z}$ is invertible we can choose a less than the smallest nonzero eigenvalue of $D_{\partial X/Z}$. In this case $u = \mathrm{Det}\, D^+_{\partial X/Z}/\|\mathrm{Det}\, D^+_{\partial X/Z}\|$ and thus $u^{-1}\nabla u = \mathrm{Im}\,\omega$, where ω is the connection form for the Bismut–Freed connection:

$$\nabla(\mathrm{Det}\, D^+_{\partial X/Z}) = \omega \cdot \mathrm{Det}\, D^+_{\partial X/Z}.$$

The imaginary part of ω has the following explicit formula:

$$\mathrm{Im}\,\omega = \frac{1}{2}\int_0^\infty \mathrm{Tr}_s(D_{\partial X/Z}\tilde{\nabla}D_{\partial X/Z}e^{-tD^2_{\partial X/Z}})dt. \tag{3.8}$$

That the integral in (3.8) is well defined comes from the following cancellation result:[3,4]

$$\mathrm{Tr}_s(D_{\partial X/Z}\tilde{\nabla}D_{\partial X/Z}e^{-tD^2_{\partial X/Z}}) = O(1) \quad \text{as } t\to 0. \tag{3.9}$$

This result holds without the assumption on the invertibility of $D_{\partial X/Z}$ and is also crucial in our proof of Theorem 3.3.

Thus in the invertible case our formula states

$$d\xi_X = \left[\int_{X/Z}\hat{A}(\Omega^{X/Z})\right]_{(1)} + \frac{1}{4\pi i}\int_0^\infty \mathrm{Tr}_s(\tilde{\nabla}D_{\partial X/Z}\cdot D_{\partial X/Z}e^{-tD^2_{\partial X/Z}})dt = \left[\int_{X/Z}\hat{A}(\Omega^{X/Z}) - \tilde{\eta}\right]_{(1)},$$

where $\tilde{\eta}$ is the differential form generalization of η introduced in Ref. 32. We point out that Cheeger[4] has also proven a formula similar to the above in the context of manifolds with conical singularities.

The proof of Theorem 3.3 is divided into several lemmas and two propositions.

The first lemma deals with a special case. Namely, we assume that the metrics along the fibers are of the form

$$g_z = du^2 + g_{\partial X_z},$$

near the boundary, where $g_{\partial X_z}$ is independent of z, i.e., the metrics near the boundary are all the same (and of product type). Fix a choice of boundary condition $\langle a,T\rangle$.

Lemma 3.10: Under these conditions $\xi(a,T)$ (mod 1) is a smooth function on U_a and

$$d\xi(a,T) = -\frac{1}{\sqrt{\pi}}\operatorname*{LIM}_{t\to 0} t^{1/2}\,\mathrm{Tr}(\tilde{\nabla}D(a,T)e^{-tD^2(a,T)}),$$

where LIM means taking the constant term in the asymptotic expansion.

Proof: This is a slight generalization of Ref. 15, Prop. 2.15. His proof can be easily generalized to this situation and is given in Proposition A17.

In general the boundary geometry and the boundary conditions vary. The idea here is to conjugate to a family with fixed boundary conditions.

Thus write the metric g_z near the boundary as

$$g_z = du^2 + g_{\partial X_z}.$$

and let $\Pi_a(z)$ denote the orthogonal projection onto the space spanned by eigensections of $D_{\partial M}(z)$ with eigenvalues $\lambda > \sqrt{a}$. Then $\Pi_a(z)$ is a smooth family of (pseudodifferential) projections on $L^2(\partial X/Z,S)$ (for $z \in U_a$), and let $\Pi_T(z)$ denote the corresponding orthogonal projection onto the graph of $T_z(a,i)$, defined in (3.1). Then

$$\Pi_{(a,T)}(z) = \Pi_a(z) + \Pi_T(z)$$

is a smooth family of pseudodifferential projections that describes the family of the boundary conditions. From the general perturbation theory, for any fixed $z_0 \in Z$ there is a smooth family of unitary operator $U(z)$ on $L^2(\partial X_z,S)$ (see Ref. 33, Lemma 2.9, for example) such that

$$U(z)\Pi_{(a,T)}(z_0)U^{-1}(z) = \Pi_{(a,T)}(z), \quad U(z_0) = \mathrm{Id}.$$

In fact, as we will see later,

$$U(z) = \begin{pmatrix} B^{-1}(z)B(z_0) & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.11}$$

where $B(z) = T(z) \oplus D_{\partial X_z}(a)/\sqrt{D^2_{\partial X_z}(a)} : H^+ \to H^-$.

Now extend $U(z)$ to a smooth family of unitary operators on $L^2(X/Z,S)$ such that $U(z)$ is constant along normal directions to a neighborhood of $\partial X/Z$ and identity in the interior and interpolate in between. This can be done, at least in a neighborhood of z_0. For example, let $\chi(u)$ be a smooth function on $[0,1]$ such that $\chi(u)=0$ for $u \geq \frac{3}{4}$ and $\chi(u)=1$ for $u \leq \frac{1}{2}$. Then $U(\chi(u)z + (1-\chi(u))z_0)$ does the job. (Here we interpret z as local coordinates around z_0.) For simplicity we still denote this extension by $U(z)$.

Lemma 3.12: Modulo the integers $\xi(a,T(z))$ defines a smooth function on U_a and

$$d\xi(a,T(z)) = -\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t \to 0} t^{1/2}\, \mathrm{Tr}(\tilde{\nabla} D(a,T) e^{-tD^2(a,T)}) - \frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t \to 0} t^{1/2}\, \mathrm{Tr}[D(a,T), \tilde{\nabla} U e^{-tD^2(a,T)}]. \tag{3.13}$$

Proof: Since $D(a,T(z))$ and $U(z)^{-1}D(a,T(z))U(z)$ have the same eigenvalues, we have

$$\xi(a,T) = \xi(U(z)^{-1}D(a,T(z))U(z)).$$

However, now $U(z)^{-1}D(a,T(z))U(z)$ is a smooth family of operators satisfying conditions (Ha), (Hb), and (Hc), which are defined in the Appendix preceding Lemma A14 and Lemma A15. Therefore, we apply Lemma A17 of the Appendix to obtain

$$\begin{aligned} d\xi(U(z)^{-1}D(a,T(z))U(z)) &= -\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t \to 0} t^{1/2}\, \mathrm{Tr}(\tilde{\nabla}[U(z)^{-1}D(a,T(z))U(z)] e^{-t(U(z)^{-1}D(a,T(z))U(z))^2}) \\ &= -\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t \to 0} t^{1/2}\, \mathrm{Tr}(\tilde{\nabla} D(a,T) e^{-tD^2(a,T)}) - \frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t \to 0} t^{1/2}\, \mathrm{Tr}[D(a,T), \tilde{\nabla} U e^{-tD^2(a,T)}]. \end{aligned}$$

Remark: In the second term of (3.13), $[D(a,T), \tilde{\nabla} U e^{-tD^2(a,T)}]$ should be interpreted as an operator acting on the Sobolev space $H^1(X,S)$. As we see from the proof, this term comes from $[D(a,T), \tilde{\nabla} U] e^{-tD^2(a,T)}$, which is clearly trace class on $L^2(X,S)$. Of course, both traces are equal.

We now look at the first term in (3.13).

Proposition 3.14: We have

$$-\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} t^{1/2} \operatorname{Tr}(\tilde{\nabla} D(a,T)e^{-tD^2(a,T)}) = \left[\int_{X/Z} \hat{A}(\Omega^{X/Z})\right]_{(1)}.$$

Proof: By the explicit construction of the heat kernel $e^{-tD^2(a,T)}$ [see (A8)], the asymptotic expansion separates into an interior part and a boundary part, and by the corresponding result for closed manifold we have

$$-\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} t^{1/2} \operatorname{Tr}(\tilde{\nabla} D(a,T)e^{-tD^2(a,T)}) = \left[\int_{X/Z} \hat{A}(\Omega^{X/Z})\right]_{(1)} + \text{boundary term}.$$

As to computing the boundary term we can replace the manifold X/Z by the half cylinder $\mathbb{R}_+ \times \partial X/Z$, with the family of the metrics given by

$$g_z = du^2 + g_{\partial X_z}.$$

To compute the heat kernel $e^{-tD^2(a,T)}$ on the half cylinder, we let $\{\varphi_\lambda\}$ be an orthonormal basis of eigensections of $D_{\partial X/Z}$ such that $J\varphi_\lambda = \varphi_{-\lambda}$. Then

$$e^{-tD^2(a,T)} = E_{>a}(t) + E_{<a}(t), \tag{3.15}$$

where

$$E_{>a}(t) = \sum_{\lambda > \sqrt{a}} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} (e^{-(u-v)^2/4t} - e^{-(u+v)^2/4t}) \varphi_\lambda \otimes \varphi_\lambda^* + \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} (e^{-(u-v)^2/4t} + e^{-(u+v)^2/4t}) J\varphi_\lambda$$
$$\otimes J\varphi_\lambda^* - \lambda e^{\lambda(u+v)} \operatorname{erfc}\left(\frac{u+v}{2\sqrt{t}} + \lambda\sqrt{t}\right) J\varphi_\lambda \otimes J\varphi_\lambda^*,$$

with

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, d\xi,$$

and $E_{<a}(t)$ is the heat kernel of the following system on the half-line $u \geqslant 0$:

$$\begin{cases} (\partial_t - \partial_u^2 + A^2) E_{<a}(t,u,v) = 0, \\ E_{<a}|_{t=0} = \mathrm{Id}, \\ \Pi_T E_{<a}|_{u=0} = 0, \\ J\Pi_T J(\partial_u + A) E_{<a}|_{u=0} = 0. \end{cases}$$

Here $A = D_{\partial X/Z}|_{K(a)}$. Note that A is a smooth family of finite-dimensional (symmetric) endomorphisms and the boundary condition here is local.

Therefore,

$$\operatorname{tr}(\tilde{\nabla} D(a,T)E_{>a}(t))(u)=\sum_{\lambda>\sqrt{a}} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}(1-e^{-u^2/t})\langle J\tilde{\nabla} D\varphi_\lambda,\varphi_\lambda\rangle+\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}(1+e^{-u^2/t})$$

$$\times\langle \tilde{\nabla} D\varphi_\lambda,J\varphi_\lambda\rangle-\lambda e^{2\lambda u}\operatorname{erfc}\left(\frac{u}{\sqrt{t}}+\lambda\sqrt{t}\right)\langle \tilde{\nabla} D\varphi_\lambda,J\varphi_\lambda\rangle$$

$$=\sum_{\lambda>\sqrt{a}} \frac{d}{du}\left[\frac{1}{2}e^{2\lambda u}\operatorname{erfc}\left(\frac{u}{\sqrt{t}}+\lambda\sqrt{t}\right)\right]\langle J\tilde{\nabla} D\varphi_\lambda,\varphi_\lambda\rangle.$$

Here, and also in what follows, we have suppressed the subscript $\partial X/Z$ of D. Integrating with respect to u from 0 to ∞ yields

$$\operatorname{Tr}(\tilde{\nabla} D(a,T)E_{>a}(t))=\sum_{\lambda>\sqrt{a}} \frac{1}{2}\operatorname{erfc}(\lambda\sqrt{t})\langle J\tilde{\nabla} D\varphi_\lambda,\varphi_\lambda\rangle$$

$$=\frac{i}{2\sqrt{\pi}}\int_{\sqrt{t}}^{\infty}\operatorname{Tr}_s(D(a)\tilde{\nabla} D(a)e^{-s^2D^2(a)})ds.$$

Here the last equation follows from the fact that

$$\langle J\tilde{\nabla} D\varphi_{-\lambda},\varphi_{-\lambda}\rangle=-\langle J\tilde{\nabla} D\varphi_\lambda,\varphi_\lambda\rangle$$

which is a consequence of the following equations:

$$J\tilde{\nabla} D=-\tilde{\nabla} DJ,\quad J\varphi_\lambda=\varphi_{-\lambda}. \tag{3.16}$$

Now,

$$\operatorname{Tr}_s(D(a)\tilde{\nabla} D(a)e^{-s^2D^2(a)})=O(1)\quad \text{as } t\to 0,$$

as it follows from (3.9). Consequently,

$$\operatorname*{LIM}_{t\to 0} t^{1/2}\operatorname{Tr}(\tilde{\nabla} D(a,T)E_{>a}(t))=0.$$

On the other hand,

$$\operatorname{Tr}(\tilde{\nabla} D(a,T)E_{<a}(t))=\operatorname{Tr}(J\tilde{\nabla} DE_{<a}(t))=i\operatorname{Tr}_s(\tilde{\nabla} DE_{<a}(t)).$$

By (3.16) $\tilde{\nabla} D$ is an odd operator. However, the heat kernel $E_{<a}(t)$ is not even because of the boundary condition. The crucial observation here is that the leading asymptotic as $t\to 0$ is indeed even, for local boundary conditions do not contribute to the leading asymptotic. Since the underlying manifold here is one dimensional, the leading asymptotic is $t^{-1/2}$, which implies

$$\operatorname{Tr}_s(\tilde{\nabla} DE_{<a}(t))=O(1)\quad \text{as } t\to 0.$$

Therefore,

$$\operatorname*{LIM}_{t\to 0} t^{1/2}\operatorname{Tr}(\tilde{\nabla} D(a,T)E_{<a}(t))=0.$$

Thus the boundary term is zero. This finishes our proof.

We now turn to the computation of the commutator term in (3.13). In general the trace of the commutator of a bounded linear operator with a trace class operator is zero. On a closed manifold, this can be extended to

$$\mathrm{Tr}[D,\,K]=0$$

for D a differential operator and K a smoothing operator (say). This is no longer true on a manifold with boundary. However, we have the following.

Lemma 3.17: For D the Dirac operator and K a smoothing operator with smooth kernel $K(x,x')$ on a compact spin manifold M with boundary we have

$$\mathrm{Tr}[D,\,K]=-\int_{\partial M}\mathrm{tr}(JK(y,y))d\ \mathrm{vol}(y). \tag{3.18}$$

Remark: This is quite similar to the characteristic feature of the b-trace introduced by Melrose[34] in the context of manifolds with asymptotically cylindrical ends.

Proof: Clearly DK is a smoothing operator with kernel given by $D_xK(x,x')$. Thus

$$\mathrm{Tr}(DK)=\int_M \mathrm{tr}(D_xK(x,x')|_{x'=x})d\ \mathrm{vol}(x).$$

On the other hand,

$$(KD)f(x)=\int_M K(x,x')(Df\,)(x')d\ \mathrm{vol}(x')=\int_M D_{x'}K(x,x')f(x')d\ \mathrm{vol}(x')$$
$$+\int_{\partial M}JK(x,y')f(y')d\ \mathrm{vol}(y').$$

Therefore the kernel of KD is given by $D_{x'}K(x,x')+JK(x,x')\,\delta_{\partial M}$, and hence

$$\mathrm{Tr}[D,K]=\mathrm{Tr}(DK)-\mathrm{Tr}(\overline{(DK)^*})-\int_M \mathrm{tr}\ JK(x,x)\,\delta_{\partial M}\ d\ \mathrm{vol}(x)=-\int_{\partial M}\mathrm{tr}(JK(y,y))d\ \mathrm{vol}(y).$$

It should be pointed out that for the above equation the Lidskii's theorem does not apply immediately to $JK(x,x')\,\delta_{\partial M}$. However, this can be overcome by approximating the delta function via smooth functions and estimating the trace norm of the approximation via the Hilbert–Schmidt norms.

With this lemma at our disposal we now turn to the commutator term. Recall the definition of u from (3.2).

Proposition 3.19: We have

$$\underset{t\to 0}{\mathrm{LIM}}\ t^{1/2}\ \mathrm{Tr}[D(a,T),\tilde{\nabla}Ue^{-tD^2(a,T)}]=\frac{i}{2\sqrt{\pi}}\,u^{-1}\nabla u.$$

Proof: Clearly $\tilde{\nabla}Ue^{-tD^2(a,T)}$ is a smoothing operator. Therefore, according to (3.18) the trace of the commutators part can be computed by taking pointwise trace of the Schwartz kernel of $\tilde{\nabla}Ue^{-tD^2(a,T)}$ and integrated over the boundary. Thus U can be taken to be the original family of unitary operators on the boundary, extended radially constant to the whole cylinder. For our computation we need the precise construction of U.

Recall that U is constructed to conjugate the family of boundary conditions, which are described by [see (1.2)]

$$W_{\langle a,T\rangle}=\left\{\langle\phi^+,\phi^-\rangle\in H_{\partial X/Z}:\phi^-+\left(T\oplus\frac{D_{\partial X/Z}(a)}{\sqrt{D^2_{\partial X/Z}(a)}}\right)\phi^+=0\right\}.$$

In other words, they are described by the graph of the pseudodifferential operator:

$$B(z)=T(z)\oplus\frac{D_{\partial X_z}(a)}{\sqrt{D^2_{\partial X_z}(a)}}:H^+_{\partial X_z}\to H^-_{\partial X_z}.$$

Then it is not hard to verify that formula (3.11) defines such a unitary conjugation. One easily finds

$$\tilde{\nabla}U(z_0)=\begin{pmatrix}-B^{-1}(z_0)\tilde{\nabla}B(z_0) & 0\\ 0 & 0\end{pmatrix}$$

and

$$B^{-1}\tilde{\nabla}B=T^{-1}\tilde{\nabla}T\oplus((D^+(a))^{-1}\tilde{\nabla}D^+(a)-\tfrac{1}{2}(D^2(a))^{-1}\tilde{\nabla}(D^2(a))).$$

Using these and (3.15) we obtain

$$-\int_{\partial X/Z}\operatorname{tr}J\tilde{\nabla}Ue^{-tD^2(a,T)}=\operatorname{tr}(JT^{-1}\tilde{\nabla}TE_{<a}(t)|_{u=0})+\int_{\partial X/Z}\operatorname{tr}\left(J\left[(D^+(a))^{-1}\tilde{\nabla}D^+(a)\right.\right.$$
$$\left.\left.-\frac{1}{2}(D^2(a))^{-1}\tilde{\nabla}(D^2(a))\right]E_{>a}(t)\right). \qquad (3.20)$$

For the first term we have

$$\underset{t\to 0}{\mathrm{LIM}}\ t^{1/2}\operatorname{tr}(JT^{-1}\tilde{\nabla}TE_{<a}(t)|_{u=0})=\frac{1}{\sqrt{4\pi}}\operatorname{tr}(JT^{-1}\tilde{\nabla}T)=\frac{i}{2\sqrt{\pi}}\frac{\nabla^a\operatorname{Det}T}{\operatorname{Det}T}, \qquad (3.21)$$

where again we have made use of the observation that the leading asymptotic of $\operatorname{tr}(JT^{-1}\tilde{\nabla}TE_{<a}(t))$ is independent of the boundary condition.

The second term is a little bit more complicated. We first note that

$$E_{>a}(t)|_{\partial X/Z}=\sum_{\lambda>\sqrt{a}}\left(\frac{e^{-\lambda^2t}}{\sqrt{\pi t}}-\lambda\operatorname{erfc}(\lambda\sqrt{t})\right)J\varphi_\lambda\otimes J\varphi^*_\lambda,$$

and

$$\lambda\operatorname{erfc}(\lambda\sqrt{t})=\frac{2\lambda^2}{\sqrt{\pi}}\int_t^\infty\frac{1}{2\sqrt{s}}e^{-s\lambda^2}\,ds=\frac{1}{\sqrt{\pi t}}e^{-t\lambda^2}-\frac{1}{2\sqrt{\pi}}\int_t^\infty s^{-3/2}e^{-s\lambda^2}\,ds.$$

Hence

$$E_{>a}(t)|_{\partial X/Z}=\sum_{\lambda>\sqrt{a}}\frac{1}{2\sqrt{\pi}}\int_t^\infty s^{-3/2}e^{-s\lambda^2}\,ds\ J\varphi_\lambda\otimes J\varphi^*_\lambda,$$

and

$$\int_{\partial X/Z} \operatorname{tr}\left(J\left[(D^+(a))^{-1}\tilde{\nabla} D^+(a) - \frac{1}{2}((D^2(a))^{-1}\tilde{\nabla}(D^2(a))\right] E_{>a}(t) \right)$$
$$= \frac{i}{4\sqrt{\pi}} \int_t^\infty s^{-3/2} \operatorname{Tr}((D^+(a))^{-1}\tilde{\nabla} D^+(a) e^{-sD^2(a)}) ds$$
$$- \frac{i}{8\sqrt{\pi}} \int_t^\infty s^{-3/2} \operatorname{Tr}((D^2(a))^{-1}\tilde{\nabla}(D^2(a)) e^{-sD^2(a)}) ds.$$

One finds

$$\operatorname*{LIM}_{t\to 0} t^{1/2} \int_{\partial X/Z} \operatorname{tr}\left(J\left[(D^+(a))^{-1}\tilde{\nabla} D^+(a) - \frac{1}{2}((D^2(a))^{-1}\tilde{\nabla}(D^2(a))\right] E_{>a}(t) \right)$$
$$= \frac{i}{2\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} \operatorname{Tr}((D^+(a))^{-1}\tilde{\nabla} D^+(a) e^{-tD^2(a)}) + \frac{i}{\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} \frac{1}{\log t} \operatorname{Tr}((D^+(a))^{-1}$$
$$\times \tilde{\nabla} D^+(a) e^{-tD^2(a)}) - \frac{i}{4\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} \operatorname{Tr}((D^2(a))^{-1}\tilde{\nabla}(D^2(a)) e^{-tD^2(a)})$$
$$- \frac{i}{2\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} \frac{1}{\log t} \operatorname{Tr}((D^2(a))^{-1}\tilde{\nabla}(D^2(a)) e^{-tD^2(a)}). \tag{3.22}$$

From (3.9) and the identity

$$\operatorname{Tr}_s[(D(a))^{-1}\tilde{\nabla} D(a) e^{-tD^2(a)}] = \int_t^\infty \operatorname{Tr}_s[(D(a))\tilde{\nabla} D(a) e^{-sD^2(a)}] ds, \tag{3.23}$$

we find

$$\operatorname*{LIM}_{t\to 0} \frac{1}{\log t} \operatorname{Tr}_s[(D(a))^{-1}\tilde{\nabla} D(a) e^{-tD^2(a)}] = 0,$$

or, equivalently,

$$\operatorname*{LIM}_{t\to 0} \frac{1}{\log t} \operatorname{Tr}((D^+(a))^{-1}\tilde{\nabla} D^+(a) e^{-tD^2(a)}) = \frac{1}{2} \operatorname*{LIM}_{t\to 0} \frac{1}{\log t} \operatorname{Tr}((D^2(a))^{-1}\tilde{\nabla} D^2(a) e^{-tD^2(a)}). \tag{3.24}$$

Thus the right-hand side of (3.22) reduces to

$$\frac{i}{2\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} \operatorname{Tr}((D^+(a))^{-1}\tilde{\nabla} D^+(a) e^{-tD^2(a)}) - \frac{i}{4\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} \operatorname{Tr}((D^2(a))^{-1}\tilde{\nabla}(D^2(a)) e^{-tD^2(a)}).$$

On the other hand, we have by (3.7)

$$\frac{\nabla \operatorname{Det} T}{\operatorname{Det} T} = \frac{\nabla^a \operatorname{Det} T}{\operatorname{Det} T} + \operatorname*{LIM}_{t\to 0} \operatorname{Tr}((D^+(a))^{-1}\tilde{\nabla} D^+(a) e^{-tD^2(a)})$$

$$+\Gamma'(1)\underset{t\to 0}{\mathrm{LIM}}\frac{1}{\log t}\,\mathrm{Tr}((D^+(a))^{-1}\tilde{\nabla}D^+(a)e^{-tD^2(a)}) \tag{3.25}$$

and

$$\frac{d(\sqrt{\det D^2(a)})}{\sqrt{\det D^2(a)}}=\frac{1}{2}\underset{t\to 0}{\mathrm{LIM}}\,\mathrm{Tr}((D^2(a))^{-1}\tilde{\nabla}(D^2(a))e^{-tD^2(a)})$$

$$+\frac{1}{2}\Gamma'(1)\underset{t\to 0}{\mathrm{LIM}}\frac{1}{\log t}\,\mathrm{Tr}((D^2(a))^{-1}\tilde{\nabla}D^2(a)e^{-tD^2(a)}). \tag{3.26}$$

We combine (3.20)–(3.26) to complete the proof.

IV. THE GLUING FORMULA

In this section we prove Theorem 2.20. We assume the notation of that theorem and of Sec. I. Fix a positive number $a'\notin\mathrm{spec}(D^2_{\partial X})$. Choose an isometry

$$T':K^+_{\partial X}(a')\to K^-_{\partial X}(a').$$

Then according to (1.7) and (2.16), the pair $\langle a',T'\rangle$ induces a trivialization of $L_{\partial X}$. This trivialization is simply carried along in the computation below. Much more essential is the following. Choose $a\notin\mathrm{spec}(D^2_Y)$ and denote

$$K^\pm=K^\pm_Y(a)=K^\mp_{-Y}(a).$$

Now choose an isometry

$$T:K^+\oplus K^-\to K^+\oplus K^-. \tag{4.1}$$

Note that T has a numerical determinant $\det T\in\mathbb{C}$. Now $K^+_{Y\sqcup -Y}\cong K^+\oplus K^-$ and $K^-_{Y\sqcup -Y}\cong K^-\oplus K^+$ [note the swap in factors from the right-hand side of (4.1)], so there is an induced trivialization

$$(-1)^{\dim K^+\dim K^-}(\mathrm{Det}\,T)^{-1}\in L_{Y\sqcup -Y}. \tag{4.2}$$

Our first task is to compute the image of (4.2) under the sequence of maps (2.22), where we leave off the $L_{\partial X}$ factor for convenience. Recall that (2.22) is the composition

$$\mathrm{Tr}_s\circ(2.17)\circ(2.18). \tag{4.3}$$

Each of the three maps in (4.3) involves a factor, and these factors are computed in (2.9)–(2.11). The total factor [including the factor in (4.2)] is

$$(-1)^{\dim K^+\dim K^-}(-1)^{\dim K^+ +\dim K^-}(-1)^{\dim K^+}(-1)^{\dim K^+(\dim K^+ +\dim K^-)}=(-1)^{\mathrm{index}\,D_Y},$$

from which it follows that the image of (4.2) is

$$(-1)^{\dim K^+\dim K^-}(\mathrm{Det}\,T)^{-1}\overset{(2.22)}{\mapsto}(-1)^{\mathrm{index}\,D_Y}(\det T)^{-1}. \tag{4.4}$$

Thus Eq. (2.21) is equivalent to the following statement.

Proposition 4.5: Let X be a compact odd-dimensional spin manifold, $Y \hookrightarrow X$ be a closed oriented hypersurface, and X^{cut} be the manifold obtained by cutting X along Y. We assume that the metric on X^{cut} is a product near $\partial X^{\text{cut}} = \partial X \sqcup Y \sqcup -Y$. Choose a, a', T, T' as above. Then

$$\tau_{X^{\text{cut}}}(a',T';a,T) = (-1)^{\text{index } D_Y} \det T \cdot \tau_X(a',T'). \tag{4.6}$$

Equation (4.6) is an equality of complex numbers.

As a preliminary to proving Proposition 4.5 we compute directly the exponentiated ξ-invariant of the cylinder. This generalizes Ref. 17, Sec. 3.

Proposition 4.7: Let Y be a closed even-dimensional spin manifold and $C = [-1, 1] \times Y$ be the corresponding cylinder. Choose a, T as above to define boundary conditions for the Dirac operator on C. Then

$$\tau_C(a,T) = \det T. \tag{4.8}$$

This is compatible with (2.23), which we derived in Sec. II as a consequence of the gluing law. (Of course, that derivation was not a proof as the proof of the gluing law depends on Proposition 4.7.) Namely, the element of $\text{End}(L_Y)$ corresponding to (4.8) is $\tau_C(a,T)(\text{Det } T)^{-1}$—the ζ-factor in (2.19) cancels out for $\text{End}(L_Y)$—and as in (4.4) we compute

$$\tau_C(a,T)((-1)^{\dim K^+ \dim K^-}(\text{Det } T)^{-1}) \overset{(2.22)}{\mapsto} (-1)^{\text{index } D_Y} \tau_C(a,T)(\det T)^{-1} = (-1)^{\text{index } D_Y}, \tag{4.9}$$

which agrees with the supertrace of $\text{id} \in \text{End}(L_Y)$.

Proof: We first prove (4.8) assuming that $a = \epsilon$ is less than the first positive eigenvalue of D_Y^2. In other words, $K = K^+ \oplus K^-$ is the kernel of D_Y. Then we use the variation formulas of Sec. III to derive the general formula.

A spinor field on C is a sum of fields of the form

$$\psi = f(t)\phi_\lambda^+ + g(t)\phi_\lambda^-, \tag{4.10}$$

where $f,g:[-1, 1] \to \mathbb{C}$ and $\phi_\lambda^\pm \in E_Y^\pm(\lambda)$ are eigenfunctions of D_Y^2. If $\lambda > 0$, we choose $\phi_\lambda^- = D_Y \phi_\lambda^+$, and then

$$D_C \psi = (-if'(t) + i\lambda g(t))\phi_\lambda^+ + (-if(t) + ig'(t))D_Y \phi_\lambda^+.$$

In this case the involution

$$f(t)\phi_\lambda^+ + g(t)D_Y\phi_\lambda^+ \mapsto \sqrt{\lambda} g(t)\phi_\lambda^+ + \frac{f(t)}{\sqrt{\lambda}} D_Y \phi_\lambda^+$$

anticommutes with D_C and preserves the boundary conditions (1.2), which reduce to the equations

$$g(1) + \frac{f(1)}{\sqrt{\lambda}} = 0, \quad f(-1) + \sqrt{\lambda} g(-1) = 0. \tag{4.11}$$

Therefore, the part of the spectrum of D_C coming from spinor fields (4.9) with $\lambda > 0$ is symmetric about the origin, and so does not contribute to the η-invariant. An easy computation shows that $\text{Ker } D_C$ contains no nonzero spinor fields that are sums of fields of the form (4.10) subject to the boundary constraint (4.11). So there is no contribution to the ξ-invariant.

We are left to consider spinor fields

$$\psi=f(t)\phi^{+}+g(t)\phi^{-}, \quad \phi\in K^{+}, \quad \phi^{-}\in K^{-},$$

subject to the boundary condition

$$\begin{pmatrix} f(-1)\phi^{+} \\ g(1)\phi^{-} \end{pmatrix} + T\begin{pmatrix} f(1)\phi^{+} \\ g(-1)\phi^{-} \end{pmatrix} = 0. \tag{4.12}$$

Now

$$D_C\psi=-if'(t)\phi^{+}+ig'(t)\phi^{-},$$

and it is straightforward to see that $D_C\psi=\mu\psi$ subject to (4.12) if and only if

$$\psi=e^{i\mu t}\phi^{+}+e^{-i\mu t}\phi^{-}$$

with

$$T\begin{pmatrix} \phi^{+} \\ \phi^{-} \end{pmatrix} = -e^{-2i\mu}\begin{pmatrix} \phi^{+} \\ \phi^{-} \end{pmatrix}.$$

So each eigenvalue ν of T contributes a set of the form $\mu+\pi\mathbb{Z}$ to the spectrum of D_C, where $0\leq\mu<\pi$ satisfies $-e^{-2i\mu}=\nu$. A standard computation (e.g., Ref. 1) shows that the η-invariant of the set $\mu+\pi\mathbb{Z}$ is $1-2\mu/\pi$ if $\mu\neq 0$. Thus if $\mu\neq 0$, the ξ-invariant is $\frac{1}{2}-\mu/\pi$, and exponentiating we obtain $e^{2\pi i(1/2-\mu/\pi)}=-e^{-2i\mu}=\nu$. This is the correct value of the exponentiated ξ-invariant for $\mu=0$ as well. Combining the contribution from all of the eigenvalues we obtain (4.8).

Now for $a>0$ the boundary condition is a unitary map

$$T:K_Y^{+}(a)\oplus K_Y^{-}(a)\to K_Y^{+}(a)\oplus K_Y^{-}(a). \tag{4.13}$$

If $T=T_0$ has the form $T_0 = T' \oplus D/\sqrt{D^2}$ for $D=D_{\partial C}(\epsilon,a)$ and some isometry $T':K_Y^{+}(\epsilon)\oplus K_Y^{-}(\epsilon)\to K_Y^{+}(\epsilon)\oplus K_Y^{-}(\epsilon)$, then the desired result follows from the previous argument and (1.6). [Recall that (1.6) is a triviality.] Another isometry T (4.13) is connected to T_0 via a path of isometries T_t, and by Theorem 3.3 and (3.2) we have

$$\frac{1}{\tau_C(a,T_t)}\frac{d\tau_C(a,T_t)}{dt}=\frac{1}{\det T_t}\frac{d(\det T_t)}{dt}.$$

It follows that $\tau_C(a,T_t)=\det T_t$ as desired.

Proof of Proposition 4.5: Following Bunke[18] we will first construct an isometry

$$U:H_{X^{\mathrm{cut}}}(a',T';a,T)\to H_X(a',T')\oplus H_C(a,\tilde{T}), \tag{4.14}$$

where the notation means the subspace of spinor fields which satisfy the appropriate boundary condition (1.2). Note the appearance of

$$\tilde{T}=\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} T \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}. \tag{4.15}$$

We then compute

$$Q=U^{-1}(D_X\oplus D_C)U-D_{X^{\mathrm{cut}}}, \tag{4.16}$$

which turns out to be a bundle endomorphism supported on the disjoint union of two cylinders. It follows that

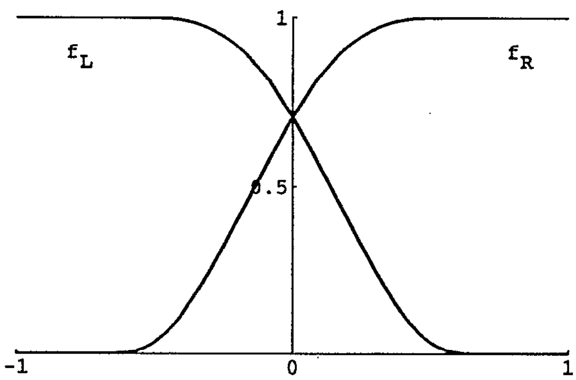

FIG. 2. The cutoff functions f_L and f_R.

$$\frac{d}{du}\,e^{2\pi i\xi(D_{X^{\text{cut}}}+uQ)} \tag{4.17}$$

may be computed locally, and we use a symmetry argument to prove that it vanishes. Equating the values at $u=0$ and $u=1$ we see that

$$\tau_{X^{\text{cut}}}(a',T';a,T)=\tau_X(a',T')\tau_C(a,\tilde{T}), \tag{4.18}$$

which reduces to (4.6) using (4.8).

To begin let $f_L,f_R:[-1,1]\to[0,1]$ be smooth cutoff functions which satisfy (Fig. 2)

$$\begin{gathered} f_L([-1,-\tfrac{1}{2}])=f_R([\tfrac{1}{2},1])=1,\\ f_L([1/2,1])=f_R([-1,-1/2])=0,\\ f_L^2+f_R^2=1,\quad f_L(-x)=f_R(x). \end{gathered} \tag{4.19}$$

The functions f_L,f_R lift to functions on $C=[-1,1]\times Y$.

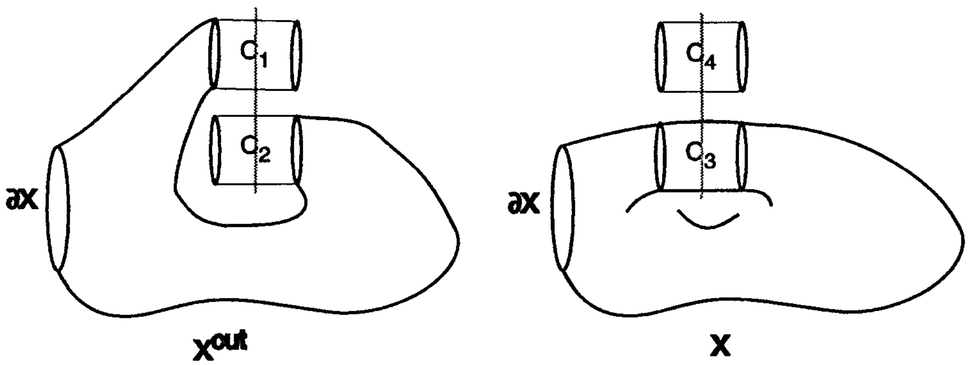

FIG. 3. The map U.

As in Fig. 3 we choose isometric embeddings $C \hookrightarrow X^{\mathrm{cut}}$ near the boundary pieces Y and $-Y$. Denote the image cylinders by C_1 and C_2, respectively. Similarly, we choose an isometric embedding $C \hookrightarrow X$ with image C_3 so that we obtain X^{cut} from X by cutting along $\{0\}\times Y \subset C_3$. If we cut X^{cut} along $\{0\}\times Y \subset C_1$ and $\{0\}\times Y \subset C_2$, then two extra pieces fall out, and they reassemble to form an extra cylinder C_4. Define U as follows. Let ψ be a spinor field on X^{cut}. Let U map its restriction to the complement of $C_1 \sqcup C_2$ unchanged to the complement of C_3 in X. Then let ψ_1, ψ_2 be the restrictions of ψ to C_1, C_2, and define

$$U: \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} f_L & f_R \\ -f_R & f_L \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{4.20}$$

The right-hand side of (4.20) is an element of $H_{C_3} \oplus H_{C_4}$, and it patches to ψ on $X - C_3$ to give a smooth spinor field on $X \sqcup C_4$. Note the change in the boundary values on C_4, as indicated in (4.14) and (4.15). It is easy to check that U is unitary.

Next we compute Q, which is defined in (4.16). Since U is the identity on the complement of $C_1 \sqcup C_2$, the operator Q has support on $C_1 \sqcup C_2$. An easy computation yields

$$Q: \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where the one-form

$$\theta = f_L df_R - f_R df_L$$

acts by Clifford multiplication. Notice that θ is supported in the interior of $C_1 \sqcup C_2$.

Consider the map

$$I: \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} dx \cdot \psi_2(-x) \\ dx \cdot \psi_1(-x) \end{pmatrix},$$

where "$\cdot$" denotes Clifford multiplication. This is the map on spinor fields induced by the orientation preserving diffeomorphism $\langle x_1, x_2 \rangle \mapsto \langle -x_2, x_1 \rangle$ of $C_1 \sqcup C_2$. We only apply I on the domain of Q, so we need only consider $\langle \psi_1, \psi_2 \rangle$ with support in the interior of $C_1 \sqcup C_2$. It is easy to verify

$$I^2 = -1, \quad ID = -DI, \quad IQ = -QI, \tag{4.21}$$

where $D = D_C$ is the Dirac operator on C. For the second equation, note that any orientation-reversing isometry anticommutes with the Dirac operator. For the third, note that

$$\theta(-x) = \theta(x)$$

from Eqs. (4.19).

Let ξ_u denote the ξ-invariant of $D_u = D_{X^{\mathrm{cut}}} + uQ$. As in Lemma 3.10 its variation is computed by the formula

$$\frac{d\xi_u}{du} = \frac{-1}{\sqrt{\pi}} \lim_{t \to 0} t^{1/2} \operatorname{Tr}_{X^{\mathrm{cut}}}(Q e^{-tD_u^2}).$$

Now the right-hand side is the integral over X^{cut} of a locally computed quantity, and since Q has support in $C_1 \sqcup C_2$ the integral may be computed there. However, from (4.21) we have

$$\operatorname{Tr}(Qe^{-tD_u^2}) = -\operatorname{Tr}(I^2 Q e^{-tD_u^2}) = \operatorname{Tr}(IQIe^{-tD_u^2}) = \operatorname{Tr}(IQe^{-tD_u^2} I) = \operatorname{Tr}(I^2 Q e^{-tD_u^2}) = -\operatorname{Tr}(Qe^{-tD_u^2}).$$

This proves that (4.17) vanishes, from which (4.18) and then (4.6) follow.

As a corollary of Proposition 4.5 we derive (1.5), which is a generalization of Ref. 17, Theorem 3.1.

Corollary 4.22: Let X be a compact odd-dimensional spin manifold with boundary. Choose a positive number $a \notin \operatorname{spec}(D^2_{\partial X})$ and isometries $T_1, T_2 : K^+_{\partial X}(a) \to K^-_{\partial X}(a)$. Then

$$\tau_X(a,T_2) = \det(T_1^{-1}T_2)\,\tau_X(a,T_1). \tag{4.23}$$

Proof: Let $C=[-1,\ 1]\times\partial X \hookrightarrow X$ be an isometric embedding mapping $\{1\}\times\partial X$ onto ∂X, and let Y be the image of $\{0\}\times\partial X$. Cutting along Y we obtain X^{cut} which is (spin) isometric to $X\sqcup C$. Consider the boundary conditions defined by T_2 on ∂X. On $Y\sqcup -Y$ we use the boundary conditions

$$T=\begin{pmatrix} 0 & T_1^{-1} \\ T_2 & 0 \end{pmatrix}.$$

Note that

$$\det T = (-1)^{\dim K^+_{\partial X}(a)} \det(T_1^{-1}T_2).$$

The induced boundary conditions on C are

$$\tilde{T}=\begin{pmatrix} 0 & T_1^{-1} \\ T_1 & 0 \end{pmatrix},$$

and

$$\det \tilde{T} = (-1)^{\dim K^+_{\partial X}(a)}.$$

Now (4.6) and (4.8) imply the desired result (4.23).

V. ADIABATIC LIMITS AND HOLONOMY

In this section we reprove the main result in Ref. 3 that computes the holonomy of the natural connection ∇ on the (inverse) determinant line bundle as the *adiabatic* limit of exponentiated ξ-invariants (on a closed manifold). Our proof here uses the curvature formula proved in Refs. 2 and 3, the variation formula (1.10), and the gluing law (2.21). (In fact, it suffices to consider the case where the base Z is a circle, and then the curvature obviously vanishes. So the curvature formula is not really needed.) We define a new connection ∇' by specifying its *parallel transport* as the adiabatic limit of exponentiated ξ-invariants, now defined on manifolds with boundary. We then show that $\nabla'=\nabla$.

Let $\pi: Y\to Z$ be a spin map whose typical fiber is a closed even-dimensional manifold, and let $L\to Z$ denote the inverse determinant line bundle. According to Ref. 2 it comes equipped with a (Quillen) metric and a natural unitary connection ∇. The curvature of ∇ is (see Ref. 3, Theorem 1.21)

$$\Omega^L = -2\pi i\left[\int_{Y/Z} \hat{A}(\Omega^{Y/Z})\right]_{(2)}. \tag{5.1}$$

[Since we use the inverse determinant line bundle the sign in (5.1) differs from that in Ref. 3.]

We now define ∇'. Let $\mathscr{P}Z$ denote the space of smooth parametrized paths $\gamma:[0,\ 1]\to Z$ with $\gamma|_{[0,\,0.1]}$ and $\gamma|_{[0.9,\,1]}$ constant. For $\gamma\in\mathscr{P}Z$ let $Y_\gamma=\gamma^*Y$ denote the pullback of $\pi: Y\to Z$ via γ; then

$\pi_\gamma : Y_\gamma \to [0, 1]$ is a spin map. Let $g_{[0,1]}$ denote an arbitrary metric on [0, 1] and $g_{Y/Z}$ the metric on the relative tangent bundle $T(Y/Z)$. Define a family of metrics on Y_γ by the formula

$$g_\epsilon = \frac{g_{[0,1]}}{\epsilon^2} \oplus g_{Y/Z}, \quad \epsilon \neq 0. \tag{5.2}$$

The metric g_ϵ on Y_γ is determined by requiring that π_γ be a Riemannian submersion. Physicists term 'lim'$_{\epsilon\to 0}$ the *adiabatic limit.* The spin structure on $T(Y_\gamma/Z)$ induces one on TY_γ since

$$TY_\gamma \cong \pi_\gamma^* T([0, 1]) \oplus T(Y/Z) \tag{5.3}$$

and $T[0,1]$ is trivial. Now the exponentiated ξ-invariant is a map

$$\tau_{Y_\gamma}(\epsilon) : L_{\gamma(0)} \to L_{\gamma(1)}.$$

Here we use the isomorphisms (2.17) and (2.18).

Lemma 5.4: The adiabatic limit $\tau_\gamma = \lim_{\epsilon\to 0} \tau_{Y_\gamma}(\epsilon)$ exists and is independent of the choice of metric $g_{[0,1]}$.

Proof: As a preliminary we state without proof a simple result about the Riemannian geometry of adiabatic limits. Let $\nabla^{Y_\gamma}(\epsilon)$ denote the Levi–Civita connection on Y_γ with the metric (5.2) and $\Omega^{Y_\gamma}(\epsilon)$ its curvature. Then $\lim_{\epsilon\to 0} \nabla^{Y_\gamma}(\epsilon)$ exists and is torsionfree. Furthermore, the curvature of this limiting connection is the limit of the curvatures of $\nabla^{Y_\gamma}(\epsilon)$ and has the form

$$\lim_{\epsilon\to 0} \Omega^{Y_\gamma}(\epsilon) = \begin{pmatrix} 0 & 0 \\ * & \Omega^{Y_\gamma/[0,1]} \end{pmatrix} \tag{5.5}$$

relative to the decomposition (5.3). We will apply this result in families, where it also holds.

Consider the spin map $p : Y_\gamma \times (\mathbb{R} - \{0\}) \to \mathbb{R} - \{0\}$, where the metric on the fiber at ϵ is (5.2). According to Theorem 1.9 we have

$$\frac{d}{d\epsilon} \tau_{Y_\gamma}(\epsilon) = 2\pi i \left[\int_p \hat{A}(\Omega^p) \right]_{(1)}.$$

Now (5.5) immediately implies that the component of the integrand in the [0, 1] direction approaches zero as $\epsilon \to 0$. In other words, if t is the coordinate in the [0, 1] direction, then any term in the integrand involving dt approaches zero as $\epsilon \to 0$. Hence $\lim_{\epsilon\to 0} \frac{d}{d\epsilon} \tau_{Y_\gamma}(\epsilon) = 0$ and so $\lim_{\epsilon\to 0} \tau_{Y_\gamma}(\epsilon)$ exists.

A similar argument proves that τ_γ is independent of $g_{[0,1]}$. Let $\mathscr{M}$ denote the space of metrics on [0, 1] and consider the spin map

$$Y_\gamma \times (\mathbb{R} - \{0\}) \times \mathscr{M} \to (\mathbb{R} - \{0\}) \times \mathscr{M},$$

where the metric on the fiber over $\langle \epsilon, g_{[0,1]} \rangle$ is (5.2). As in the previous argument we see that the differential of $\tau_{Y_\gamma}(\epsilon, g_{[0,1]})$ with respect to $g_{[0,1]}$ vanishes as $\epsilon \to 0$. The desired conclusion follows immediately.

An immediate corollary is that τ_γ is invariant under reparametrization of paths. Also, if $\gamma_1, \gamma_2 \in \mathscr{P}Z$ with $\gamma_1(1) = \gamma_2(0)$, and $\gamma_2 \circ \gamma_1$ denotes the composed path, then $\tau_{\gamma_2\circ\gamma_1} = \tau_{\gamma_2} \circ \tau_{\gamma_1}$. This follows from the gluing law (Theorem 2.20). Now a general theorem (Ref. 25, Appendix B) applies to construct a connection ∇' on L whose parallel transport is τ.

Now we compute the holonomy of ∇'. Let $\bar{\gamma} : S^1 \to Z$ be a loop in Z and $Y_{\bar{\gamma}} \to S^1$ the corresponding fibered manifold. Realize $\bar{\gamma}$ as the gluing of a path $\gamma : [0, 1] \to Z$; then $Y_{\bar{\gamma}}$ is obtained by identifying the ends of $Y_\gamma \to [0, 1]$. This identification induces the spin structure on $Y_{\bar\gamma}$ obtained by lifting the *nonbounding* spin structure on S^1. The gluing law Theorem 2.20 implies [compare (2.24)]

$$\lim_{\epsilon\to 0} \tau_{Y_{\bar\gamma}}(\epsilon) = \mathrm{Tr}_Y(\lim_{\epsilon\to 0} \tau_{Y_\gamma}(\epsilon)) = \mathrm{Tr}_Y(\text{parallel transport along } \gamma)$$
$$= (-1)^{\mathrm{index}\, D_Y} \cdot (\text{holonomy around } \bar\gamma). \tag{5.6}$$

If $L = L_{\gamma(0)}$ then the parallel transport is an element of $L \hat\otimes L^*$. The sign comes since the composition $L \hat\otimes L^* \to L^* \hat\otimes L \to \mathbb{C}$ is $(-1)^{|L|} = (-1)^{\mathrm{index}\, D_Y}$ times the usual contraction. Let $Y'_{\bar\gamma}$ denote $Y_{\bar\gamma}$ with spin structure induced by lifting the *bounding* spin structure on S^1. If we substitute $Y'_{\bar\gamma}$ for $Y_{\bar\gamma}$ in (5.6), then the resulting equation has no factor $(-1)^{\mathrm{index}\, D_Y}$. This follows as in (2.25). (Compare Ref. 28, Theorem 1.31.)

Our main result in this section is the following.

Proposition 5.7: $\nabla' = \nabla$.

To prove Proposition 5.7 we compare the covariant derivative of their parallel transports using the following general lemma.

Lemma 5.8: Let $L \to Z$ be an arbitrary line bundle with connection ∇ and curvature Ω^L. Denote the parallel transport of ∇ along a path γ by ρ_γ. Then

$$\nabla \rho = -\left(\int_{p_2} \mathrm{ev}^* \Omega^L \right) \cdot \rho, \tag{5.9}$$

where ev and p_2 are the maps

$$\begin{array}{ccc} [0,1] \times \mathcal{P}Z & \xrightarrow{\mathrm{ev}} & Z \\ p_2 \downarrow & & \\ \mathcal{P}Z & & \end{array}$$

To interpret (5.9) view ρ as a section of the line bundle $(\mathrm{ev}_0^*(L))^* \otimes (\mathrm{ev}_1^*(L)) \to \mathcal{P}Z$ with its connection induced from ∇. Here $\mathrm{ev}_t(\gamma) = \gamma(t)$. The proof is elementary.

Corollary 5.10: If ∇, ∇' are connections on $L \to Z$ with parallel transports ρ, τ, and if $\nabla\rho/\rho = \nabla\tau/\tau$, then $\nabla' = \nabla$.

For if $\nabla' = \nabla + \alpha$ for a one-form α on Z, then

$$\frac{\nabla \tau}{\tau} - \frac{\nabla \rho}{\rho} = -\left(d \int_{p_2} \mathrm{ev}^* \alpha \right),$$

and if $\alpha \neq 0$, then the right-hand side is nonzero.

We now verify the hypotheses of Corollary 5.10 for the natural connection ∇ and the new connection ∇' on the inverse determinant line bundle. We use the diagram

$$\begin{array}{ccc} \mathrm{ev}^* Y & \to & Y \\ \pi' \downarrow & & \downarrow \pi \\ [0,1] \times \mathcal{P}Z & \xrightarrow{\mathrm{ev}} & Z \\ p_2 \downarrow & & \\ \mathcal{P}Z & & \end{array}$$

We compute $\nabla\tau$ using the variation formula (1.10). Namely, τ_γ is the adiabatic limit of τ_{Y_γ}, and Y_γ is the fiber $(p_2\circ\pi')^{-1}(\gamma)$. So by the variation formula

$$\nabla\tau=2\pi i\left[\int_{p_2\circ\pi'}\text{a-lim}\,\hat{A}(\Omega^{p_2\circ\pi'})\right]_{(1)}\cdot\tau=2\pi i\int_{p_2}\left[\int_{\pi'}\hat{A}(\Omega^{\pi'})\right]_{(2)}\cdot\tau$$

$$=\int_{p_2}\text{ev}^*\left[2\pi i\int_{\pi}\hat{A}(\Omega^{\pi})\right]_{(2)}\cdot\tau,$$

where we use (5.5) to pass from the first equation to the second. (Of course, "a-lim" is the adiabatic limit.) But by (5.9) and the curvature formula (5.1) this latter expression is the covariant derivative of the parallel transport of ∇. This concludes the proof of Proposition 5.7.

Therefore, (5.6) also computes the holonomy of the canonical connection ∇ on the inverse determinant line bundle as the adiabatic limit of exponentiated ξ-invariants. This is exactly the content of Theorem 3.16 in Ref. 3. [Again, since we use the inverse determinant line bundle the sign in (5.12) differs from Ref. 3.]

Corollary 5.11: Let $\bar{\gamma}:S^1\to Z$ be a loop and $Y_{\bar{\gamma}}\to S^1_{\text{nonbounding}}$ the corresponding fibered manifold. Then the holonomy around $\bar{\gamma}$ of the natural connection ∇ on the inverse determinant line bundle $L\to Z$ is

$$(-1)^{\text{index}\,D_Y}\text{a-lim}(e^{2\pi i\xi_{Y_{\bar{\gamma}}}}). \tag{5.12}$$

VI. REMARKS ON THE FAMILIES' INDEX THEOREM

Let $\pi:X\to Z$ be a spin map whose typical fiber is a compact even-dimensional manifold with boundary. When $\ker D_{\partial X_z}$ has constant rank, there is a well-defined index bundle $\text{Ind}\,D_{X/Z}\in K^0(Z)$. The families' index theorem of Bismut–Cheeger states that its Chern character $\text{ch}(\text{Ind}\,D_{X/Z})$ is represented in de Rham cohomology by (cf. Refs. 32, 35, and 36; a more general version has been proved in Ref. 37)

$$\int_{X/Z}\hat{A}(\Omega^{X/Z})-\tilde{\eta}, \tag{6.1}$$

where $\tilde{\eta}$ is a differential form on the base Z, defined as follows.

Consider a spin map $\pi:Y\to Z$ whose typical fiber is a closed manifold. (Our application takes $Y=\partial X$.) The associated Bismut superconnection A_t is

$$A_t=\tilde{\nabla}+t^{1/2}D_{Y/Z}-c(T)/4t^{1/2},$$

where $c(T)=\Sigma_{\alpha\leq\beta}\,dz^\alpha\,dz^\beta\,T(f_\alpha,f_\beta)$ with T the curvature form of the fibration, f_α a local orthonormal basis on Z, and dz^α the one-form dual to f_α. The asymptotics of heat kernels associated to the Bismut superconnection exhibit some remarkable cancellations. The first one is expressed in the local index theorem for families.[3,38] More essential to our discussion are two other cancellation results:[39]

$$tr_s[(D_{Y/Z}+c(T)/4t)e^{-A_t^2}]=O(t^{1/2})\quad\text{as } t\to 0,\quad\text{if }\dim Y/Z\text{ is even;} \tag{6.2}$$

$$tr^{\text{even}}[(D_{Y/Z}+c(T)/4t)e^{-A_t^2}]=O(t^{1/2})\quad\text{as } t\to 0,\quad\text{if }\dim Y/Z\text{ is odd.} \tag{6.3}$$

where tr^{even} indicates the even form part of tr. When $\ker D_{Y/Z}$ has constant rank, the expressions on the left-hand sides of (6.2), (6.3) are also well behaved for the large time. In fact, it is shown in Ref. 40 (in a more general setting) that

$$tr_s[(D_{Y/Z}+c(T)/4t)e^{-A_t^2}]=O(t^{-1}) \quad \text{as } t\to\infty, \quad \text{if dim } Y/Z \text{ is even;} \tag{6.4}$$

$$tr^{\text{even}}[(D_{Y/Z}+c(T)/4t)e^{-A_t^2}]=O(t^{-1}) \quad \text{as } t\to\infty, \quad \text{if dim } Y/Z \text{ is odd.} \tag{6.5}$$

By virtue of (6.2)–(6.5) we now define a differential form on Z, the $\hat{\eta}$ form:

$$\hat{\eta}=\begin{cases} \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^\infty tr_s\left[\left(D_{Y/Z}+\frac{c(T)}{4t}\right)e^{-A_t^2}\right]\frac{dt}{2t^{1/2}}, & \text{if dim } Y/Z \text{ is even;}\\ \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^\infty tr^{\text{even}}\left[\left(D_{Y/Z}+\frac{c(T)}{4t}\right)e^{-A_t^2}\right]\frac{dt}{2t^{1/2}}, & \text{if dim } Y/Z \text{ is odd.}\end{cases}$$

For example, the first integral is convergent at 0 because of (6.2), and convergent at ∞ because of (6.4). We normalize $\hat{\eta}$ by defining

$$\tilde{\eta}=\begin{cases} \Sigma \dfrac{1}{(2\pi i)^j}[\hat{\eta}]_{(2j-1)}, & \text{if dim } Y/Z \text{ is even;}\\ \Sigma \dfrac{1}{(2\pi i)^j}[\hat{\eta}]_{(2j)}, & \text{if dim } Y/Z \text{ is odd.}\end{cases}$$

Here we decompose the odd (resp. even) form $\hat{\eta}$ into its homogeneous components $[\hat{\eta}]_{(2j-1)}$ (resp. $[\hat{\eta}]_{(2j)}$). The $\tilde{\eta}$ form satisfies a transgression formula. If dim Y/Z is odd, then[32,33]

$$d\tilde{\eta}=-\int_{Y/Z}\hat{A}(\Omega^{Y/Z}). \tag{6.6}$$

If dim Y/Z is even and $\ker D_Y$ has constant rank, then[33]

$$d\tilde{\eta}=\text{ch}(\text{Ind } D_{Y/Z})-\int_{\partial X/Z}\hat{A}(\Omega^{Y/Z}). \tag{6.7}$$

Return now to a spin map $\pi: X\to Z$ whose typical fiber is a compact manifold with boundary. If $\dim X/Z$ is even, which is the case considered by Bismut–Cheeger, then (6.6) immediately implies that the differential form (6.1) is closed. We are interested in the case where $\dim X/Z$ is odd, and then (6.7) implies that unless $D_{\partial X/Z}$ is invertible, the differential form (6.1) is *not* closed. Thus in the odd-dimensional case one expects a correction term in the Bismut–Cheeger index formula from $\ker D_{\partial X/Z}$.

Theorem 3.3 suggests what the correction term should be, assuming that $\ker D_{\partial X/Z}$ has constant rank. To define the odd index bundle we need self-adjoint operators. In our case this amounts to a choice of a (smooth) family of isometries

$$T:\ \ker D^+_{\partial X/Z}\to\ker D^-_{\partial X/Z}.$$

The resulting family of self-adjoint operators $D_{X/Z}(T)$ gives rise to a well-defined index bundle $\text{Ind } D_{X/Z}(T)\in K^1(Z)$. On the other hand, $\text{ch}(\text{Ind } D_{\partial X/Z}) = \text{Tr}_s(e^{-(\nabla^a)^2})$, where a is chosen to be smaller than the smallest eigenvalue of $D_{\partial X/Z}$. Consider the superconnection $\nabla^a+\sqrt{t}V$ on $\ker D_{\partial X/Z}$, with V the symmetric endomorphism

$$V=\begin{pmatrix}0 & T^*\\ T & 0\end{pmatrix}.$$

One has the following transgression formula:

$$\frac{d}{dt}\,\mathrm{Tr}_s(e^{-(\nabla^a+\sqrt{t}V)^2}) = -d\left[\frac{1}{2\sqrt{t}}\,\mathrm{Tr}_s(Ve^{-(\nabla^a+\sqrt{t}V)^2})\right],$$

which, by the invertibility of V, yields

$$d\tilde{\eta}_T = \mathrm{ch}(\mathrm{Ind}\, D_{\partial X/Z}),$$

with $\tilde{\eta}_T$ defined by

$$\tilde{\eta}_T = \int_0^\infty \frac{1}{2\sqrt{t}}\,\mathrm{Tr}_s(Ve^{-(\nabla^a+\sqrt{t}V)^2})dt.$$

Conjecture 6.8: The (odd) Chern character of $\mathrm{Ind}\, D_{X/Z}(T)$ is represented in the de Rham cohomology by

$$\int_{X/Z} \hat{A}(\Omega^{X/Z}) - \tilde{\eta} - \tilde{\eta}_T.$$

We have the following evidence for this conjecture.

Theorem 6.9: The degree-one component of the odd Chern character of the index bundle $\mathrm{ch}_1(\mathrm{Ind}\, D_{X/Z}(T)) \in H^1(Z)$ is represented by

$$\left[\int_{X/Z} \hat{A}(\Omega^{X/Z}) - \tilde{\eta} - \tilde{\eta}_T\right]_{(1)}.$$

Proof: By the Duhamel principle

$$[\mathrm{Tr}_s(Ve^{-(\nabla^a+\sqrt{t}V)^2})]_{(1)} = -\sqrt{t}\,\mathrm{Tr}_s(V(\nabla^a V)e^{-tV^2}).$$

Therefore,

$$[\tilde{\eta}_T]_{(1)} = -\int_0^\infty \frac{1}{2}\,\mathrm{Tr}_s(V(\nabla^a V)e^{-tV^2})dt = -\tfrac{1}{2}\,\mathrm{Tr}_s(V^{-1}\nabla^a V) = -\mathrm{Tr}(T^{-1}\nabla^a T). \tag{6.10}$$

Similarly, we have

$$[\tilde{\eta}]_{(1)} = -\frac{1}{2}\int_0^\infty \mathrm{Tr}_s(D_{\partial X/Z}\tilde{\nabla} D_{\partial X/Z}e^{-tD^2_{\partial X/Z}})dt. \tag{6.11}$$

On the other hand, the degree-one component of $\mathrm{ch}(\mathrm{Ind}\, D_{X/Z}(T))$ is given by $d\xi_X(a,T)$, which, according to Theorem (3.3), gives

$$\mathrm{ch}_1(\mathrm{Ind}\, D_{X/Z}(T)) = \left[\int_{X/Z} \hat{A}(\Omega^{X/Z})\right]_{(1)} + \frac{1}{2\pi i}\, u^{-1}\nabla u.$$

From (3.24)–(3.26) and our choice of a we have

$$u^{-1}\nabla u = (\mathrm{Det}\ T)^{-1}\nabla^a(\mathrm{Det}\ T) + \mathrm{LIM}_{t\to 0}\ \mathrm{Tr}((D^+)^{-1}\tilde{\nabla} D^+ e^{-tD^2}) - \tfrac{1}{2}\,\mathrm{LIM}_{t\to 0}\ \mathrm{Tr}((D^2)^{-1}\tilde{\nabla}(D^2)e^{-tD^2}), \tag{6.12}$$

and the first term in (6.12) is exactly $-[\tilde{\eta}_T]_{(1)}$ by (6.10). For the remaining terms we note from (6.11) and (3.23)

$$\begin{aligned}[\tilde{\eta}]_{(1)} &= -\tfrac{1}{2}\operatorname*{LIM}_{t\to 0}\operatorname{Tr}_s[D^{-1}\tilde{\nabla}De^{-tD^2}]\\ &= -\tfrac{1}{2}\operatorname*{LIM}_{t\to 0}\operatorname{Tr}[(D^+)^{-1}\tilde{\nabla}D^+e^{-tD^2}]+\tfrac{1}{2}\operatorname*{LIM}_{t\to 0}\operatorname{Tr}[(D^-)^{-1}\tilde{\nabla}D^-e^{-tD^2}]\\ &= -\operatorname*{LIM}_{t\to 0}\operatorname{Tr}[(D^+)^{-1}\tilde{\nabla}D^+e^{-tD^2}]+\tfrac{1}{2}\operatorname*{LIM}_{t\to 0}\operatorname{Tr}[(D^2)^{-1}\tilde{\nabla}(D^2)e^{-tD^2}].\end{aligned}$$

This finishes the proof.

APPENDIX A: GENERALIZED APS BOUNDARY CONDITIONS

In this Appendix we discuss the analytical aspects of the generalized APS boundary conditions. For simplicity of notation we restrict ourself to the case of Dirac operators, although our discussion extends easily to the more general situation of Dirac-type operators.

Let X be an odd-dimensional compact oriented spin manifold with smooth boundary $\partial X = Y$. We shall always assume that the Riemannian metric on X is a product near the boundary. Let

$$D:\ C^\infty(X,S)\to C^\infty(X,S)$$

be the formally self-adjoint Dirac operator acting on the spinor bundle $S\to X$. Then in a collar neighborhood $[0,1)\times\partial X$ of the boundary, D takes the form

$$D=J(\partial_\mu+D_{\partial X}),$$

where $J=c(du)$ and

$$D_{\partial X}:\ C^\infty(\partial X,S|_{\partial X})\to C^\infty(\partial X,S|_{\partial X})$$

is the self-adjoint Dirac operator on ∂X under the identification $S|_{\partial X}\cong S(\partial X)$.

As an unbounded operator in $L^2(X,S)$ with domain $C_0^\infty(X,S)$, D is symmetric. (In other words, D is formally self-adjoint.) To obtain self-adjoint extensions of D, one has to impose boundary conditions. For our purpose, we would like to restrict our attention to boundary conditions of elliptic type. Appropriate boundary conditions that are of elliptic type are considered by Atiyah–Patodi–Singer.[1] Namely if we denote by Π_+ the orthogonal projection of $L^2(\partial X,S|_{\partial X})$ onto the subspace spanned by the eigensections of $D_{\partial X}$ with non-negative eigenvalues, then $D_+=D$ with domain

$$\operatorname{dom}(D_+)=\{\varphi\in H^1(X,S)\,|\,\Pi_+(\varphi|_{\partial X})=0\}$$

is an elliptic boundary value problem (in the generalized sense, see Refs. 1 and 41). Here D_+ is a closed symmetric extension of D, although, in general, D_+ is not self-adjoint. However, one can obtain elliptic self-adjoint boundary value problems by considering further self-adjoint extensions of D_+.

More generally, let $a\notin\operatorname{spec}D_{\partial X}^2$ be a positive number and Π_{-a} (resp. Π_a) denote the orthogonal projection of $L^2(\partial X,S|_{\partial X})$ onto the subspace spanned by eigensections of $D_{\partial X}$ with eigenvalues $>-\sqrt{a}$ (resp. $>\sqrt{a}$). Consider the operator $D_a=D$ with domain given by

$$\operatorname{dom}(D_a)=\{\varphi\in H^1(X,S)\,|\,\Pi_{-a}(\varphi|_{\partial X})=0\}.$$

Lemma A1: D_a is a closed symmetric extension of D, and its adjoint D_a^* is given by D with domain

$$\mathrm{dom}(D_a^*)=\{\varphi\in H^1(X,S)|\Pi_a(\varphi|_{\partial X})=0\}.$$

Proof: Proceeding in the same way as in Atiyah, Patodi, and Singer, Paper I,[1] (APS1), we can construct a two-sided parametrix

$$R:\ C^\infty(X,S)\to C^\infty(X,S;\Pi_{-a})$$

such that $DR-\mathrm{Id}$ and $RD-\mathrm{Id}$ are smoothing operators and

$$R:\ H^l(X,S)\to H^{l+1}(X,S)\quad (l\geqslant 0).$$

Thus if $\varphi_n\in\mathrm{dom}(D_a)$ such that $\varphi_n\to\varphi$, $D\varphi_n\to\psi$ in L^2, the existence of the paramatrix R shows that, in fact, $\varphi\in H^1(X,S)$ and $\varphi_n\to\varphi$ in $H^1(X,S)$. By the continuity of the restriction map

$$r:\ H^1(X,S)\to H^{1/2}(\partial X,S|_{\partial X})\to L^2(\partial X,S|_{\partial X}),$$

$\varphi\in\mathrm{dom}(D_a)$ and $D_a\varphi=\psi$. This shows that D_a is closed.

To show D_a is symmetric, it suffices to prove the statement about D_a^*. Integration by parts gives, for all $\varphi,\psi\in C^\infty(X,S)$,

$$(D\varphi,\psi)-(\varphi,D\psi)=\int_{\partial X}\langle J(\varphi|_{\partial X}),\psi|_{\partial X}\rangle\overset{\mathrm{def}}{=}(J(\varphi|_{\partial X}),\psi|_{\partial X})_{\partial X}. \tag{A2}$$

Again, the continuity of the restriction map r shows that (A2) actually holds for all $\varphi,\psi\in H^1(X,S)$.

Let D_{-a} denote D with domain

$$\mathrm{dom}(D_{-a})=\{\varphi\in H^1(X,S)|\Pi_a(\varphi|_{\partial X})=0\}.$$

Then, for all $\varphi\in\mathrm{dom}(D_a)$, $\psi\in\mathrm{dom}(D_{-a})$,

$$J(\varphi|_{\partial X})=J(\mathrm{Id}-\Pi_{-a})(\varphi|_{\partial X})=\Pi_aJ(\varphi|_{\partial X}),\qquad \psi|_{\partial X}=(\mathrm{Id}-\Pi_a)(\psi|_{\partial X}).$$

Thus $(J(\varphi|_{\partial X}),\psi|_{\partial X})_{\partial X}=0$ and (A2) shows that $D_{-a}\subset D_a^*$.

The equality $D_a^*=D_{-a}$ requires considerably more effort. Let

$$L^2_{int}(X,S)=\{\varphi\in L^2(X,S)|\mathrm{dist}(\mathrm{supp}\ \varphi,\partial X)\geqslant \tfrac{1}{3}\}$$

and

$$L^2_{bd}(X,S)=\{\varphi\in L^2(X,S)|\mathrm{supp}\ \varphi\subset[0,\ \tfrac{2}{3}]\times\partial X\}.$$

Then $L^2(X,S)=L^2_{int}(X,S)+L^2_{bd}(X,S)$ and we just have to specify D_a^* restricted to each of the subspaces.

Clearly for $\psi\in L^2_{int}(X,S)\cap\mathrm{dom}(D_a^*)$, we have $D_a^*\psi=D\psi$ and

$$L^2_{int}(X,S)\cap\mathrm{dom}(D_a^*)=L^2_{int}(X,S)\cap H^1(X,S).$$

The subspace $L^2_{bd}(X,S)$ splits further:

$$L^2_{bd}(X,S)=L^2([0,\ \tfrac{2}{3}],K_{\partial X}(a))\oplus L^2([0,\ \tfrac{2}{3}],H_{\partial X}(a)),$$

where $K_{\partial X}(a)$, $H_{\partial X}(a)$ are defined in (1.1). Moreover, D_a is diagonal with respect to this splitting. Now restricted to $L^2([0, 2/3],K_{\partial X}(a))$, $D_a = J(\partial_u + A)$, with A a symmetric endomorphism of $K_{\partial X}(a)$ which anticommutes with J, and the boundary condition at $u=0$ is $\varphi|_{u=0}=0$. Clearly then, $D_a^* = D_{-a}$ on $L^2([0, 2/3],K_{\partial X}(a))$.

On the other hand, for D_a restricted to $L^2([0, 2/3],H_{\partial X}(a))$, the construction in APS1[1] actually gives bounded inverse R_a for D_a and R_{-a} for D_{-a}. From

$$(D_a\varphi, \psi) = (\varphi, D_{-a}\psi)$$

for $\varphi \in \mathrm{dom}(D_a)$, $\psi \in \mathrm{dom}(D_{-a})$, we obtain, by continuity, $R_a^* = R_{-a}$. Since adjoints commute with inverses, the lemma is established, for the discussion above shows that $D_a^* \subset D_{-a}$.

From the lemma it is clear that D_a is, in general, not self-adjoint, so we need to consider self-adjoint extensions of D_a. Suppose D_s is such a self-adjoint extension, then $D_a \subset D_s \subset D_a^*$, i.e., $D_s = D$ with

$$\mathrm{dom}(D_a) \subset \mathrm{dom}(D_s) \subset \mathrm{dom}(D_a^*). \tag{A3}$$

Recall our notation from Sec. I. We have $K_{\partial X}(a) = \mathrm{Im}(\Pi_{-a} - \Pi_a)$ splits into the $(\pm i)$-eigenspace of J [cf. (1.1)]:

$$K_{\partial X}(a) = K_{\partial X}^+(a) \oplus K_{\partial X}^-(a).$$

Lemma A4: We have $\dim K_{\partial X}^+(a) = \dim K_{\partial X}^-(a)$.

Proof: This is a consequence of the cobordism invariance of index. Alternatively, it follows from the Atiyah–Patodi–Singer index formula, as follows. First of all, by the symmetry of spec $D_{\partial X}$, we just need to show that for a less than the smallest nonzero eigenvalue of $D_{\partial X}^2$. Namely, $\dim K_{\partial X}^+ = \dim K_{\partial X}^-$, where $K_{\partial X}^{\pm}$ are the $\pm i$-eigenspace of J restricted to $\ker D_{\partial X}$. Applying the APS index formula to D_a yields

$$\dim L = \frac{\dim \ker D_{\partial X}}{2}, \tag{A5}$$

where $L \subset \ker D_{\partial X}$ is the subspace of limiting values of the extended L^2-solutions of D (see APS1[1]). Alternatively, $L = \Pi r(\ker D_a^*) = \Pi r(\ker D_{-a})$, where Π is the orthogonal projection onto $\ker D_{\partial X}$. From (A2), together with (A5), we see that L is a "Lagrangian" subspace of $(\ker D_{\partial X}, (\cdot,\cdot)_{\partial X}, J)$: $(J\alpha,\beta)_{\partial X} = 0$ for all $\alpha,\beta \in L$. This shows that the $(+i)$-eigenspace of J has the same dimension as the $(-i)$-eigenspace.

We now denote $h^+(a) = \dim K_{\partial X}^+(a)$.

Proposition A6: There is a one–one correspondence

$$\{\text{self-adjoint extensions of } D_a\} \leftrightarrow \{\text{unitary maps } T\colon K_{\partial X}^+(a) \to K_{\partial X}^-(a)\}.$$

For a unitary map T, its corresponding self-adjoint extension $D(a,T)$ is given by D with

$$\mathrm{dom}(D(a,T)) = \{\varphi \in H^1(X,S) \mid (\Pi_a + \Pi_T)(\varphi|_{\partial X}) = 0\},$$

where Π_T is the orthogonal projection onto the graph of T in $K_{\partial X}(a)$.

Proof: Any self-adjoint extension of D_a is given by $D_s = D$ with domain satisfying (A3). Thus

$$r(\mathrm{dom}(D_a)) \subset r(\mathrm{dom}(D_s)) \subset r(\mathrm{dom}(D_a^*))$$

or

$$r(\mathrm{dom}(D_a)) \subset r(\mathrm{dom}(D_s)) \subset r(\mathrm{dom}(D_a)) \oplus K_{\partial X}(a).$$

From (A2),

$$(J(\varphi|_{\partial X}),\psi|_{\partial X})_{\partial X}\equiv 0 \tag{A7}$$

for all $\varphi,\psi\in\mathrm{dom}(D_s)$, or, equivalently, for all $\varphi|_{\partial X},\psi|_{\partial X}\in r(\mathrm{dom}(D_s))$. Since D_a is symmetric, (A7) is automatically satisfied on $r(\mathrm{dom}(D_s))$. Let $L=r(\mathrm{dom}(D_s))\cap K_{\partial X}(a)$ be a subspace of $K_{\partial X}(a)$. Then (A7) shows that L is an "isotropic" subspace of $(K_{\partial X}(a),J)$. Since D_s is self-adjoint, L must be maximal isotropic, hence "Lagrangian." Now it is a little linear algebra to show that there is a one–one correspondence

$$\{\text{Lagrangian subspace } L \text{ of } (K_{\partial X}(a),J)\leftrightarrow\{\text{unitary map } T\colon\ K^+_{\partial X}(a)\to K^-_{\partial X}(a)\}$$

given by $L=$the graph of T. This shows one way of the correspondence. However, the other direction is completely similar to the proof of Lemma A1.

Remark: This is very similar to von Neumann's theory of deficiency indexes, which completely characterizes self-adjoint extensions of a closed symmetric operator.

Remark: Formally, for D with domain $C_0^\infty(X,S)$, there is also a one–one correspondence

$$\{\text{self-adjoint extensions of } D\}\leftrightarrow\{\text{unitary maps: } H^+_{\partial X}\to H^-_{\partial X}\}$$
$$\leftrightarrow\{\text{Lagrangian subspaces of } H_{\partial X}=L^2(\partial X,S|_{\partial X})\}.$$

However, one loses the ellipticity in this generality.

Thus, given $a\notin\mathrm{spec}\, D^2_{\partial X}$ positive and $T\colon K^+_{\partial X}(a)\to K^-_{\partial X}(a)$ an isometry (unitary), the operator $D(a,T)$ is self-adjoint, and, as we mentioned earlier, elliptic in a generalized sense. We will not, however, go into the discussion of the ellipticity of $D(a,T)$, but, instead, derive some of its consequences from the study of the heat kernel, $e^{-tD^2(a,T)}$.

For this purpose, we first consider the situation on the infinite half-cylinder $R_+\times\partial X$. In this case, $D=J(\partial_u+D_{\partial X})$ and we have a global decomposition.

$$L^2(R_+\times\partial X,S)=L^2(R_+,L^2(\partial X,S|_{\partial X}))=L^2(R_+,K_{\partial X}(a))\oplus L^2(R_+,H_{\partial X}(a)).$$

Since both D and the boundary condition are diagonal with respect to this decomposition, $e^{-tD^2(a,T)} = E_{<a}(t) + E_{>a}(t)$ splits into two pieces as well. As the boundary condition on $L^2(R_+,H_{\partial X}(a))$ is completely analogous to the APS boundary condition, $E_{>a}(t)$ can be given an explicit formula. Let $\{\varphi_\lambda;\lambda\in\mathrm{spec}\, D_{\partial X},\lambda>\sqrt{a}\}$ be an orthonormal basis for $\mathrm{Im}\,\Pi_a$ consisting of eigensections of $D_{\partial X}$. Then the same construction in Ref. 1 gives

$$E_{>a}(t)=\sum_{\lambda>\sqrt{a}}\frac{e^{-\lambda^2t}}{\sqrt{4\pi t}}\left(e^{-(u-v)^2/4t}-e^{-(u+v)^2/4t}\right)\varphi_\lambda\otimes\varphi^*_\lambda+\left\{\frac{e^{-\lambda^2t}}{\sqrt{4\pi t}}\left(e^{-(u-v)^2/4t}+e^{-(u+v)^2/4t}\right)\right.$$
$$\left.-\lambda e^{\lambda(u+v)}\,\mathrm{erfc}\left(\frac{u+v}{2\sqrt{t}}+\lambda\sqrt{t}\right)\right\}J\varphi_\lambda\otimes J\varphi^*_\lambda.$$

On the other hand, there is no explicit formula for $E_{<a}(t)$. However, it is reduced to a heat kernel on the half-line R_+, with L^2 boundary condition at ∞ and local elliptic condition at 0:

$$(\partial_t-\partial_u^2+A^2)E_{<a}(t,u,v)=0,$$

$$E_{<a}|_{t=0}=\mathrm{Id},\quad \Pi_T E_{<a}|_{u=0}=0,\quad J\Pi_T J(\partial_u+A)E_{<a}|_{u=0}=0,$$

with $A = D_{\partial X}|_{K_{\partial X}(a)}$ a finite-dimensional symmetric endomorphism.

To discuss the heat kernel on X, we use the patching construction of APS1.[1] More precisely, let $\rho(a,b)$ be an increasing C^∞ function on R such that $\rho=0$ for $u\leqslant a$ and $\rho=1$ for $u\geqslant b$. Define

$$\phi_1=\rho(\tfrac{1}{6},\tfrac{2}{6}),\quad \psi_1=\rho(\tfrac{3}{6},\tfrac{4}{6}),\quad \phi_2=1-\rho(\tfrac{5}{6},1),\quad \psi_2=1-\psi_1.$$

These extend to smooth functions on X in an obvious way. Let $\tilde{D}$ be the Dirac operator on the double of X. Then

$$e=\phi_1 e^{-t\tilde{D}^2}\psi_1+\phi_2(E_{<a}(t)+E_{>a}(t))\psi_2$$

is a parametrix for the heat operator $\partial_t+D^2(a,T)$, and

$$e^{-tD^2(a,T)}=e+\sum_{m=1}^{\infty}(-1)^m c_m * e, \tag{A8}$$

where $*$ denotes the convolution of kernels, $c_1=(\partial_t+D^2(a,T))e$, and $c_m = c_{m-1}*c_1$, $m\geqslant 2$. It follows that for $t>0$, $e^{-tD^2(a,T)}$ is a C^∞ kernel which differs from e by an exponentially small term as $t\to 0$.

Lemma A9: (i) Both $e^{-tD^2(a,T)}$ and $D(a,T)e^{-tD^2(a,T)}$ are trace class for $t>0$.

(ii) As $t\to 0$,

$$\mathrm{Tr}(e^{-tD^2(a,T)})\sim\sum_{j=0}^{\infty} a_j(D(a,T))t^{(j-n)/2},$$

and

$$\mathrm{Tr}(D(a,T)e^{-tD^2(a,T)})\sim\sum_{j=0}^{\infty} b_j(D(a,T))t^{(j-n-1)/2},$$

with a_j,b_j given by integral of local densities computable from the (total) symbol of D and boundary conditions.

Proof: (i) Since for $t>0$, $e^{-tD^2(a,T)}$ is smooth, it is Hilbert–Schmidt. Now the semigroup properties show that $e^{-tD^2(a,T)} = e^{-(t/2)D^2(a,T)}\circ e^{-(t/2)D^2(a,T)}$ is a product of Hilbert–Schmidt operators, hence trace class. Similarly for $D(a,T)e^{-tD^2(a,T)}$.

(ii) From (i) and Lidskii's theorem

$$\mathrm{Tr}(e^{-tD^2(a,T)})=\int_X \mathrm{tr}(e^{-tD^2(a,T)})(x,x)dx.$$

For the asymptotic expansion we may replace $e^{-tD^2(a,T)}$ by its parametrix e. The asymptotic expansion for e follows from its explicit construction, as in APS1.[1]

Corollary A10: The spectrum of $D(a,T)$ consists of eigenvalues of finite multiplicities satisfying Weyl's asymptotic law:

$$N(\lambda)=\#\{\lambda_j||\lambda_j|\leqslant\lambda\}=\frac{\mathrm{vol}(X)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^n+o(\lambda^n)\quad\text{as }\lambda\to\infty.$$

Thus, the eta function

$$\eta(s,D(a,T))=\sum_{\lambda_j\neq 0} \text{sign } \lambda_j|\lambda_j|^{-s}$$

is well defined for Re $s>n$. Further, by Mellin transform,

$$\eta(s,D(a,T))=\frac{1}{\Gamma((s+1)/2)}\int_0^\infty t^{(s-1)/2}\,\mathrm{Tr}(D(a,T)e^{-tD^2(a,T)})dt. \tag{A11}$$

Lemma A9 shows that $\eta(s,D(a,T))$ admits a meromorphic continuation to the complex plane with only simple poles.

Proposition A12: $\eta(s,D(a,T))$ is actually holomorphic in Re $s>-\frac{1}{2}$. Therefore the eta invariant $\eta(a,T)=\eta(0,D(a,T))$ is well defined. Moreover,

$$\eta(a,T)=\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{Tr}(D(a,T)e^{-tD^2(a,T)})dt.$$

Proof: It suffices to show that

$$\mathrm{Tr}(D(a,T)e^{-tD^2(a,T)})=O(1) \quad \text{as } t\to 0.$$

The same argument as in the proof of Lemma A9 shows that

$$\begin{aligned}\mathrm{Tr}(D(a,T)e^{-tD^2(a,T)})&=\int_X \mathrm{tr}(D_x e(t,x,x')|_{x=x'})dx+O(e^{-c/t})\\&=\int_X \mathrm{tr}(D_x e^{-t\tilde{D}^2}(x,x')|_{x=x'})\psi_1(x)dx\\&\quad+\int_{R_+\times\partial X}\mathrm{tr}(D_x(E_{<a}(t)+E_{>a}(t))|_{x=x'})\psi_2(x)dx+O(e^{-c/t}).\end{aligned} \tag{A13}$$

The local cancellation result for closed manifold gives

$$\mathrm{tr}(D_x e^{-t\tilde{D}^2}(x,x')|_{x=x'})=O(t^{1/2})$$

uniformly in x. Therefore the first term in (A13) is $O(t^{1/2})$.

For the second term, a straightforward calculation shows that

$$\int_{\partial X}\mathrm{tr}(D_x E_{>a}(t)|_{x=x'})\equiv 0.$$

Also $\mathrm{tr}(JAE_{<a}(t))\equiv 0$ since $JA=-AJ$. Thus

$$\mathrm{Tr}(D(a,T)e^{-tD^2(a,T)})=\int_{R_+\times\partial X}\mathrm{tr}[J\partial_u E_{<a}(t)|_{u=v}]\psi_2(u)du\,dy+O(t^{1/2}).$$

Since $E_{<a}(t)$ is the heat kernel of an elliptic local boundary value problem on R_+, we have

$$E_{<a}(t,u,v)=\frac{e^{-(u-v)^2/4t}}{\sqrt{4\pi t}}(1+b_1(T,u,v)t^{1/2}+O(t))$$

uniformly in u,v. Therefore

$$J\partial_u E_{<a}(t)|_{u=v}=\frac{1}{\sqrt{4\pi}}J\partial_u b_1(T,u,v)|_{u=v}+O(t^{1/2}),$$

and our claim follows.

We now turn to the variation of eta invariants. For our purpose we are going to work in complete generality. So let $P(z)$ be a family of operators satisfying:

(Ha) $P(z)$ is a smooth family of (unbounded) self-adjoint operators on $L^2(X,S)$ with dom $(P(z))$ independent of the parameter z;

(Hb) the heat semigroup $e^{-tP^2(z)}$ $(t>0)$ is a smooth family of smoothing operators, i.e., the heat kernel is given by smooth functions on X depending smoothing on z.

Lemma A14: For a family satisfying (Ha) and (Hb), we have

$$\frac{\partial}{\partial z}\operatorname{Tr}(P(z)e^{-tP^2(z)})=\left(1+2t\frac{\partial}{\partial t}\right)\operatorname{Tr}(\dot{P}(z)e^{-tP^2(z)}).$$

Proof: First of all,

$$\frac{\partial}{\partial z}\operatorname{Tr}(P(z)e^{-tP^2(z)})=\operatorname{Tr}(\dot{P}(z)e^{-tP^2(z)})+\operatorname{Tr}\left(P(z)\frac{\partial}{\partial z}e^{-tP^2(z)}\right).$$

To compute $(\partial/\partial z)e^{-tP^2(z)}$, we apply the heat operator:

$$\left(\frac{\partial}{\partial t}+P^2(z)\right)\frac{\partial}{\partial z}e^{-tP^2(z)}=\left[P^2(z),\frac{\partial}{\partial z}\right]e^{-tP^2(z)}.$$

Now, with the initial condition of the heat equation and dom$(P(z))$ independent of z, Duhamel's principle gives

$$\frac{\partial}{\partial z}e^{-tP^2(z)}=\int_0^t e^{-(t-s)P^2(z)}\left[P^2(z),\frac{\partial}{\partial z}\right]e^{-sP^2(z)}\,ds.$$

Consequently,

$$\operatorname{Tr}\left(P(z)\frac{\partial}{\partial z}e^{-tP^2(z)}\right)=-2t\operatorname{Tr}(\dot{P}(z)P^2(z)e^{-tP^2(z)})=2t\frac{\partial}{\partial t}\operatorname{Tr}(\dot{P}(z)e^{-tP^2(z)}).$$

This finishes the proof.

We now consider the variation of eta function $\eta(s,P(z))$ defined by (A11). For it to be well defined we make the following additional assumption.

(Hc) There is a uniform asymptotic expansion of $\operatorname{Tr}(P(z)e^{-tP^2(z)})$ at $t=0$:

$$\operatorname{Tr}(P(z)e^{-tP^2(z)})\sim\sum_{j\geqslant -N}a_j(P(z))t^{j/d},$$

and $a_j(P(z))$ are smooth in z.

Lemma A15: Let $P(z)$ be a family of operators satisfying (Ha), (Hb), and (Hc). Furthermore, assume that dim ker $P(z)$ is constant. Then for Re $s>N$, we have

$$\frac{\partial}{\partial z}\eta(s,P(z))=-\frac{s}{\Gamma((s+1)/2)}\int_0^\infty t^{(s-1)/2}\,\mathrm{Tr}(\dot{P}(z)e^{-tP^2(z)})dt.$$

Proof: By (Hb), $P(z)$ all have discrete spectrum. It follows from the assumption on dim ker $P(z)$ that $\mathrm{Tr}(P(z)e^{-tP^2(z)})$ is exponentially decaying, uniformly in z, as $t\to\infty$. (Hc) implies that $\eta(s,P(z))$ analytically continues to a meromorphic function smooth in z.

Let $T>0$ and Re $s>N$. By Lemma A14,

$$\frac{\partial}{\partial z}\int_0^T t^{(s-1)/2}\,\mathrm{Tr}(P(z)e^{-tP^2(z)})dt=2T^{(s+1)/2}\,\mathrm{Tr}(\dot{P}(z)e^{-TP^2(z)})$$
$$-s\frac{\partial}{\partial z}\int_0^T t^{(s-1)/2}\,\mathrm{Tr}(\dot{P}(z)e^{-tP^2(z)})dt. \tag{A16}$$

Denote by $H(z)$ the orthogonal projection of $L^2(X,S)$ onto ker $P(z)$. Since dim ker $P(z)$ is constant, $H(z)$ depends smoothly on z. Furthermore, the self-adjointness of $P(z)$ implies that

$$P(z)H(z)=H(z)P(z)=0.$$

Therefore

$$P(z)=(\mathrm{Id}-H(z))P(z)(\mathrm{Id}-H(z)),$$

and hence

$$\dot{P}(z)=-\dot{H}(z)P(z)(\mathrm{Id}-H(z))+(\mathrm{Id}-H(z))\dot{P}(z)(\mathrm{Id}-H(z))-(\mathrm{Id}-H(z))P(z)\dot{H}(z).$$

Since $(\mathrm{Id}-H(z))e^{-tP^2(z)}$ is given by a smooth kernel decaying exponentially in t as $t\to\infty$, it follows that the right-hand side of (A16) is absolutely convergent so we can take the limit of (A16) as $T\to\infty$ and exchange the limit with the differentiation. The same discussion applies to the left-hand side of (A16) and we obtain the lemma.

An immediate consequence of the lemma is that when $\eta(s,P(z))$ are all regular at $s=0$,

$$\frac{\partial}{\partial z}\eta(P(z))=-\frac{2}{\sqrt{\pi}}\operatorname*{LIM}_{t\to 0}t^{1/2}\,\mathrm{Tr}(\dot{P}(z)e^{-tP^2(z)}),$$

where $\mathrm{LIM}_{t\to 0}$ means taking the constant term in the asymptotic expansion at $t=0$.

Now define

$$\xi(P(z))=\frac{\eta(P(z))+\dim\ker P(z)}{2}.$$

Proposition A17: Let (Ha), (Hb), and (Hc) hold for $P(z)$. Then $\xi(P(z))$ (mod 1) defines a smooth function and

$$\frac{d}{dz}\xi(P(z))=-\frac{1}{\sqrt{\pi}}\operatorname*{LIM}_{t\to 0}t^{1/2}\,\mathrm{Tr}(\dot{P}(z)e^{-tP^2(z)}).$$

Proof: Choose a $c>0$ such that c is not in the spectrum of $P(z)$ for all z in a small neighborhood. Let $\Pi_c(z)$ be the orthogonal projection onto the space spanned by all eigensections with eigenvalues λ satisfying $|\lambda|<c$. Define a new family

$$P^c(z)=P(z)(\mathrm{Id}-\Pi_c(z))+\Pi_c(z).$$

Namely one replaces by 1 all eigenvalues λ of $P_B(z)$ satisfying $|\lambda|<c$ and leaves the rest unchanged. Therefore $P^c(z)$ is clearly invertible, $\xi(P^c(z))=\frac{1}{2}\eta(P^c(z))$ is smooth, and

$$\frac{d}{dz}\,\xi(P^c(z))=-\frac{1}{\sqrt{\pi}}\operatorname*{LIM}_{t\to 0} t^{1/2}\,\mathrm{Tr}(\dot{P}^c(z)e^{-t(P^c(z))^2}).$$

Now

$$\xi(P(z))=\xi(P^c(z))+\frac{\sum_{\lambda\in \mathrm{spec}\,P(z),|\lambda|<c}(\mathrm{sign}\,\lambda-1)}{2},$$

here

$$\mathrm{sign}\,\lambda=\begin{cases}1, & \text{if } \lambda\geqslant 0;\\ -1, & \text{if } \lambda<0.\end{cases}$$

Clearly then

$$\xi(P_B(z))\equiv\xi(P_B^c(z))\,\mathrm{mod}\ \mathbb{Z}.$$

On the other hand,

$$e^{-t(P^c(z))^2}=e^{-tP^2(z)}+\text{finite rank}$$

and

$$\dot{P}^c(z)=\dot{P}(z)+\text{finite rank},$$

which implies that

$$\mathrm{Tr}(\dot{P}^c(z)e^{-t(P^c(z))^2})=\mathrm{Tr}(P(z)e^{-tP^2(z)})+O(1).$$

Therefore

$$\operatorname*{LIM}_{t\to 0} t^{1/2}\,\mathrm{Tr}(\dot{P}^c(z)e^{-t(P^c(z))^2})=\operatorname*{LIM}_{t\to 0} t^{1/2}\,\mathrm{Tr}(\dot{P}(z)e^{-tP^2(z)}).$$

Finally, we point out that although the L^2-norm on $L^2(X,S)$ depends on the metric, a smooth family of metrics gives rise to a smooth family of equivalent norms. Therefore the resulting trace functional on $L^2(X,S)$ is independent of the metric change.

ACKNOWLEDGMENTS

The first author is supported by NSF Grant No. DMS-9204267 and Alfred P. Sloan Fellowship. He would also like to thank MSRI for its support and hospitality while some of this work was completed. The second author is supported by NSF Grant No. DMS-8805684, a Presidential Young Investigators award DMS-9057144, and by the O'Donnell Foundation. He would also like to thank the Geometry Center at the University of Minnesota for its hospitality while some of this work was completed. We thank Graeme Segal for sharing his ideas with us. We also benefitted from conversations with Ulrich Bunke and John Lott.

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