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A REMARK ON WEIGHTED BERGMAN KERNELS ON ORBIFOLDS

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Abstract. In this note, we explain that Ross–Thomas' result [4, Theorem 1.7] on the weighted Bergman kernels on orbifolds can be directly deduced from our previous result [1]. This result plays an important role in the companion paper [5] to prove an orbifold version of Donaldson theorem.

In two very interesting papers [4, 5], Ross–Thomas describe a notion of ampleness for line bundles on Kähler orbifolds with cyclic quotient singularities which is related to embeddings in weighted projective spaces. They then apply [4, Theorem 1.7] to prove an orbifold version of Donaldson theorem [5]. Namely, the existence of an orbifold Kähler metric with constant scalar curvature implies certain stability condition for the orbifold. In these papers, the result [4, Theorem 1.7] on the asymptotic expansion of Bergman kernels plays a crucial role.

In this note, we explain how to directly derive Ross–Thomas' result [4, Theorem 1.7] from Dai–Liu–Ma [1, (5.25)], provided Ross–Thomas condition [4, (1.8)] on c_i holds. Since in [1, Section 5], we state our results for general symplectic orbifolds, in what follows, we will just use the version from [2, Theorem 5.4.11], where Ma–Marinescu wrote them in detail for Kähler orbifolds. We will use freely the notation in [2, Section 5.4]. We assume also the auxiliary vector bundle E therein is $\mathbb{C}$.

Let (X, J, ω) be a compact n-dimensional Kähler orbifold with complex structure J, and with singular set X_{sing}. Let (L, h^L) be a holomorphic Hermitian proper orbifold line bundle on X. Let ∇^L be the holomorphic Hermitian connections on (L, h^L) with curvature $R^L = (\nabla^L)^2$.

We assume that (L, h^L, ∇^L) is a prequantum line bundle, i.e.,

$$R^L = -2\pi\sqrt{-1}\,\omega. \tag{0.1}$$

Let $g^{TX} = \omega(\cdot, J\cdot)$ be the Riemannian metric on X induced by ω. Let ∇^{TX} be the Levi–Civita connection on (X, g^{TX}). We denote by $R^{TX} = (\nabla^{TX})^2$ the curvature, by r^X the scalar curvature of ∇^{TX}. For $x \in X$, set $d(x, X_{\text{sing}}) := \inf_{y\in X_{\text{sing}}} d(x, y)$ the distance from x to X_{sing}.

For $p \in \mathbb{N}$, the Bergman kernel $P_p(x, x')$ $(x, x' \in X)$ is the smooth kernel of the orthogonal projection from $\mathscr{C}^\infty(X, L^p)$ onto $H^0(X, L^p)$, with respect to the Riemannian volume form $dv_X(x')$.

Theorem 0.1 ([1, Theorem 1.4], [2, Theorem 5.4.10])**.** *There exist smooth coefficients* $\boldsymbol{b}_r(x) \in \mathscr{C}^\infty(X)$ *which are polynomials in* R^{TX} *and its derivatives with order* $\leqslant 2r-2$

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at x, and $C_0 > 0$ such that for any $k, l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $M \in \mathbb{N}$ with

$$\left| \frac{1}{p^n} P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} \right|_{\mathscr{C}^l} \leqslant C_{k,l} \Big(p^{-k-1} + p^{l/2}(1+\sqrt{p}\, d(x, X_{\rm sing}))^M e^{-\sqrt{C_0 p}\, d(x, X_{\rm sing})} \Big), \tag{0.2}$$

for any $x \in X$, $p \in \mathbb{N}^*$. Moreover,

$$\boldsymbol{b}_0 = 1, \quad \boldsymbol{b}_1 = \frac{1}{8\pi} r^X. \tag{0.3}$$

In local coordinates, there is a more precise form [1, (5.25)], see also [2, Theorem 5.4.11]. Let $\{x_i\}_{i=1}^I \subset X_{\rm sing}$. For each point x_i we consider corresponding local charts $(G_{x_i}, \widetilde{U}_{x_i}) \to U_{x_i}$ with $\widetilde{U}_{x_i} \subset \mathbb{C}^n$, such that $0 \in \widetilde{U}_{x_i}$ is the inverse image of $x_i \in U_{x_i}$, and 0 is a fixed point of the finite stabilizer group G_{x_i} at x_i, which acts $\mathbb{C}$-linearly and effectively on $\mathbb{C}^n$ (cf. [2, Lemma 5.4.3]). We assume moreover that

$$B^{\widetilde{U}_{x_i}}(0, 2\varepsilon) \subset \widetilde{U}_{x_i}, \text{ and } X_{\rm sing} \subset W := \cup_{i=1}^I B^{\widetilde{U}_{x_i}}(0, \tfrac{1}{4}\varepsilon)/G_{x_i}.$$

Let $\widetilde{U}^g_{x_i}$ be the fixed point set of $g \in G_{x_i}$ in $\widetilde{U}_{x_i}$, and let $\widetilde{N}_{x_i,g}$ be the normal bundle of $\widetilde{U}^g_{x_i}$ in $\widetilde{U}_{x_i}$. For each $g \in G_{x_i}$, the exponential map $\widetilde{N}_{x_i,g,\widetilde{x}} \ni Y \to \exp^{\widetilde{U}_{x_i}}_{\widetilde{x}}(Y)$ identifies a neighborhood of $\widetilde{U}^g_{x_i}$ with $\widetilde{W}_{x_i,g} = \{Y \in \widetilde{N}_{x_i,g}, |Y| \leqslant \varepsilon\}$. We identify $L|_{\widetilde{W}_{x_i,g}}$ with $L|_{\widetilde{U}^g_{x_i}}$ by using the parallel transport along the above exponential map. Then the g-action on $L|_{\widetilde{W}_{x_i,g}}$ is the multiplication by $e^{\sqrt{-1}\theta_g}$, and θ_g is locally constant on $\widetilde{U}^g_{x_i}$.

Let $\nabla^{\widetilde{N}_{x_i,g}}$ be the connection on $\widetilde{N}_{x_i,g}$ induced by the Levi–Civita connection via projection. We trivialize $\widetilde{N}_{x_i,g} \simeq \widetilde{U}^g_{x_i} \times \mathbb{C}^{n_g}$ by the parallel transport along the curve $[0,1] \ni t \to t\widetilde{Z}_{1,g}$ for $\widetilde{Z}_{1,g} \in \widetilde{U}^g_{x_i}$, which identifies also the metric on $\widetilde{N}_{x_i,g}$ with the canonical metric on $\mathbb{C}^{n_g}$. If $\widetilde{Z} \in \widetilde{W}_{x_i,g}$, we will write $\widetilde{Z} = (\widetilde{Z}_{1,g}, \widetilde{Z}_{2,g})$ with $\widetilde{Z}_{1,g} \in \widetilde{U}^g_{x_i}$, $\widetilde{Z}_{2,g} \in \mathbb{C}^{n_g}$. We will denote by Z the corresponding point on the orbifold.

Theorem 0.2 ([1, (5.25)], [2, Theorem 5.4.11]). *On $\widetilde{U}_{x_i}$ as above, there exist polynomials $\mathscr{K}_{r,\widetilde{Z}_{1,g}}(\widetilde{Z}_{2,g})$ in $\widetilde{Z}_{2,g}$ of degree $\leqslant 3r$, of the same parity as r, whose coefficients are polynomials in R^{TX} and its derivatives of order $\leqslant r-2$, and a constant $C_0 > 0$ such that for any $k, l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $N \in \mathbb{N}$ such that*

$$\begin{aligned} \Bigg| \frac{1}{p^n} P_p(\widetilde{Z}, \widetilde{Z}) - \sum_{r=0}^{k} \boldsymbol{b}_r(\widetilde{Z}) p^{-r} & \\ - \sum_{r=0}^{2k} p^{-\frac{r}{2}} \sum_{1 \neq g \in G_{x_i}} e^{\sqrt{-1}\theta_g p} \mathscr{K}_{r,\widetilde{Z}_{1,g}}(\sqrt{p}\widetilde{Z}_{2,g}) e^{-2\pi p \langle (1-g^{-1})\widetilde{z}_{2,g}, \overline{\widetilde{z}_{2,g}} \rangle} \Bigg|_{\mathscr{C}^l} & \\ \leqslant C_{k,l} \left(p^{-k-1} + p^{-k+\frac{l-1}{2}} (1+\sqrt{p}\, d(Z, X_{\rm sing}))^N e^{-\sqrt{C_0 p}\, d(Z, X_{\rm sing})} \right)&, \end{aligned} \tag{0.4}$$

for any $|\widetilde{Z}| \leqslant \varepsilon/2$, $p \in \mathbb{N}$, with $\boldsymbol{b}_r(\widetilde{Z})$ as in Theorem 0.1 and $\mathscr{K}_{0,\widetilde{Z}_{1,g}} = 1$.

Given a function $f(p,x)$ in $p\in\mathbb{N}$ and $x\in X$, we write $f=\mathcal{O}_{\mathscr{C}^j}(p^l)$ if the $\mathscr{C}^j$-norm of f is uniformly bounded by $C\,p^l$.

Theorem 0.3. *Let (X,ω) be a compact n-dimensional Kähler orbifold with cyclic quotient singularities (i.e., the stabilizer group G_x is a cyclic group for any $x\in X$), and L be a proper orbifold line bundle on X equipped with a Hermitian metric h^L whose curvature form is $-2\pi\sqrt{-1}\,\omega$, such that for any $x\in X$, the stabilizer group G_x acts on $L_{\widetilde{x}}$ as $\mathbb{Z}_{|G_x|}$-order cyclic group. Fix $N\geq 0$, and $r\geq 0$ and suppose c_i are a finite number of positive constants chosen so that if X has an orbifold point of order m then*

$$\frac{1}{m}\sum_i i^k c_i=\sum_{i\equiv u \bmod m} i^k c_i,\quad \textit{for all } u \textit{ and all } k=0,\dots,N+r. \tag{0.5}$$

Then the function

$$B_p^{\mathrm{orb}}(x):=\sum_i c_i P_{p+i}(x,x) \tag{0.6}$$

admits a global $\mathscr{C}^{2r}$-expansion of order N. That is, there exist smooth functions $b_0,\dots,b_N$ on X such that

$$B_p^{\mathrm{orb}}=\sum_{j=0}^N b_j p^{n-j}+\mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}). \tag{0.7}$$

Furthermore, b_j are universal polynomials in the constants c_i and the derivatives of ω; in particular

$$b_0=\sum_i c_i,\quad b_1=\sum_i c_i\Big(n\,i+\frac{1}{8\pi}r^X\Big). \tag{0.8}$$

Remark 0.1. Theorem 0.3 recovers [4, Theorem 1.7] of Ross–Thomas, where the remainder estimate is $\mathcal{O}_{\mathscr{C}^r}(p^{n-N-1})$.

We improve here their remainder estimate to $\mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1})$ and we get Theorem 0.3 directly from Theorems 0.1 and 0.2.

Remark 0.2. By Ma–Marinescu [3, (3.30), Remark 3.10], [2, Theorem 4.1.3, Remark 5.4.12], Theorem 0.3 generalizes to any J-invariant metric g^{TX} on TX. Set $\Theta:=g^{TX}(J\cdot,\cdot)$. The only change is that the coefficients in the expansion become

$$b_0=\frac{\omega^n}{\Theta^n}\sum_i c_i,\ b_1=\frac{\omega^n}{\Theta^n}\sum_i c_i\left[n\,i+\frac{r^X_\omega}{8\pi}-\frac{1}{4\pi}\Delta_\omega\log\left(\frac{\omega^n}{\Theta^n}\right)\right], \tag{0.9}$$

where r^X_ω, Δ_ω are the scalar curvature and the Bochner Laplacian associated to $g^{TX}_\omega=\omega(\cdot,J\cdot)$. Moreover, (0.7) can be taken to be uniform as (h^L,g^{TX}) runs over a compact set.

Proof of Theorem 0.3. Recall that now G is a cyclic group of order m. Let ζ be a generator of G. From the local condition for orbi-ample line bundles, ζ acts on L_{x_i} as a primitive mth root of unity λ. Thus in (0.4), $e^{\sqrt{-1}\theta_{\zeta^u}} = \lambda^u$. For $u \in \{1, \dots, m-1\}$, set

$$\begin{aligned}
\eta_u =& e^{-2\pi\langle(1-\zeta^{-u})\widetilde{z}_{2,\zeta^u}, \overline{\widetilde{z}}_{2,\zeta^u}\rangle}, \\
S_u(\widetilde{Z}) =& \sum_i c_i \sum_{j=0}^{2N+2r+1} (p+i)^{n-\frac{j}{2}} \mathscr{K}_{j,\widetilde{Z}_{1,\zeta^u}}(\sqrt{p+i}\widetilde{Z}_{2,\zeta^u})\lambda^{u(p+i)}\eta_u^{p+i}, \\
\mathcal{S}_2 =& \sum_{u=1}^{m-1} S_u, \qquad \mathcal{S}_1 = \sum_i c_i \sum_{j=0}^{N+r} \boldsymbol{b}_j(\widetilde{Z})(p+i)^{n-j}.
\end{aligned} \tag{0.10}$$

Here $Z = z + \overline{z}$, and $z = \sum_i z_i \frac{\partial}{\partial z_i}$, $\overline{z} = \sum_i \overline{z}_i \frac{\partial}{\partial \overline{z}_i}$ when we consider them as vector fields, and $\left|\frac{\partial}{\partial z_i}\right|^2 = \left|\frac{\partial}{\partial \overline{z}_i}\right|^2 = \frac{1}{2}$. Similarly for $\widetilde{Z}$ (and those with subscripts).

Applying (0.4) for $k = N + r + 1$ we obtain for $|\widetilde{Z}| \le \varepsilon/2$,

$$\begin{aligned}
&|B_p^{\rm orb}(\widetilde{Z}) - \mathcal{S}_1 - \mathcal{S}_2|_{\mathscr{C}^{l'}} \\
&\quad\leqslant C_{l'}\, p^{n-N-r-2}(1 + p^{\frac{l'+1}{2}}(1+\sqrt{p}d(Z, X_{\rm sing}))^M e^{-\sqrt{C_0\, p}\, d(Z,X_{\rm sing})}) \\
&\qquad + \sum_i c_i (p+i)^{n-N-r-1} \\
&\qquad \times \left(\sum_{u=1}^{m-1} \left| \mathscr{K}_{2N+2r+2,\widetilde{Z}_{1,\zeta^u}}(\sqrt{p+i}\widetilde{Z}_{2,\zeta^u})\eta_u^{p+i} \right|_{\mathscr{C}^{l'}} + \left| \boldsymbol{b}_{N+r+1}(\widetilde{Z}) \right|_{\mathscr{C}^{l'}} \right).
\end{aligned} \tag{0.11}$$

In what follows, we write for simplicity $\widetilde{Z}_{1,\zeta^u}$ as $Z_{1,u}$ and $\widetilde{Z}_{2,\zeta^u}$ as $Z_{2,u}$. For a function $f(p, Z)$ with $p \in \mathbb{N}$ and $|Z| \le \varepsilon/2$ we write $f = \mathcal{O}_{\mathscr{C}^j}(g(p, Z))$ if the $\mathscr{C}^j$-norm of f in Z can be uniformly controlled by $C\,|g(p, Z)|$.

Note that $\mathscr{K}_{j,Z_{1,u}}(Z_{2,u})$ is a polynomial in $Z_{2,u}$ with the same parity as j and $\deg \mathscr{K}_{j,Z_{1,u}} \le 3j$. Denote by $\mathscr{K}_{j,Z_{1,u},l}$ the l-homogeneous part of $\mathscr{K}_{j,Z_{1,u}}$. Then $\mathscr{K}_{j,Z_{1,u},l} = 0$ if l and j are not in the same parity or $l > 3j$. By (0.10),

$$\begin{aligned}
S_u(Z) =& \sum_i c_i \sum_{j=0}^{2N+2r+1} \sum_l (p+i)^{n-\frac{j-l}{2}} \mathscr{K}_{j,Z_{1,u},l}(Z_{2,u})\lambda^{u(p+i)}\eta_u^{p+i} \\
=& \lambda^{up} \sum_{j=0}^{2N+2r+1} \left\{ \left(\sum_{l \ge j-2n} \sum_{q=0}^{n-\frac{j-l}{2}} + \sum_{l<j-2n} \sum_{q=0}^{N+r} \right) \mathscr{K}_{j,Z_{1,u},l}(\sqrt{p}Z_{2,u}) \right. \\
& \left. \times\ p^{n-\frac{j}{2}-q} \binom{n-\frac{j-l}{2}}{q} \sum_i c_i i^q \lambda^{ui}\eta_u^{p+i} \right\} + \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}).
\end{aligned} \tag{0.12}$$

Here we used $(p+i)^\gamma = \sum_{q=0}^{N+r} p^{\gamma-q}\binom{\gamma}{q} i^q + \mathcal{O}(p^{\gamma-N-r-1})$ for $\gamma < 0$ and the following relations for $r', r'' \in \mathbb{N}$, $r'' \le l$,

$$\begin{aligned}
&\mathscr{K}_{j,Z_{1,u},l}(\sqrt{p}Z_{2,u})\eta_u^p = \mathcal{O}_{\mathscr{C}^{r'}}(p^{\frac{r'}{2}}\eta_u^{p/2}), \\
&\mathscr{K}_{j,Z_{1,u},l}(\sqrt{p}Z_{2,u}) = \mathcal{O}_{\mathscr{C}^{r''}}(p^{\frac{l}{2}}|Z_{2,u}|^{l-r''}).
\end{aligned} \tag{0.13}$$

In order to prove (0.7) it is sufficient to show that for $0 \le l \le N+r$, $r' \le 2r$,

$$w_{l,p} := \sum_i c_i i^l \lambda^{ui} \eta_u^{p+i} = \mathcal{O}_{\mathscr{C}^{r'}}(p^{l-N-r-1+\frac{r'}{2}} \eta_u^{p/2}). \tag{0.14}$$

In fact, we will prove that $w_{l,p} = \mathcal{O}_{\mathscr{C}^{r'}}(p^{l-N-r-1+\frac{r'}{2}} \eta_u^{(\frac{3}{4}-\frac{r'}{8r})p})$ for $r' \le 2r$.

Since $dw_{l,p} = \frac{d\eta_u}{\eta_u}(pw_{l,p} + w_{l+1,p})$, and $\frac{d\eta_u}{\eta_u}$ has a term $z_{2,u}$ or $\overline{z}_{2,u}$ which can be absorbed by $\eta_u^{\frac{1}{8r}p}$ to get a factor $p^{-1/2}$, we see by induction that it is sufficient to prove $w_{l,p} = \mathcal{O}_{\mathscr{C}^0}(p^{l-N-r-1}\eta_u^{\frac{3}{4}p})$. To this end, write

$$w_{l,p} = \left[\frac{\sum_i c_i i^l \lambda^{ui}\eta_u^i}{(\eta_u - 1)^{N+r-l+1}}\right](\eta_u - 1)^{N+r-l+1}\eta_u^p. \tag{0.15}$$

Since λ is a primitive mth root of unity, $\lambda^u \neq 1$ if $u \in \{1, \dots, m-1\}$. From [4, Lemma 3.5], under the condition (0.5), the function $\eta \to \sum_i c_i i^l \lambda^{ui}\eta^i$ has a root of order $N+r-l+1$ at $\eta = 1$ and so the term in square brackets is bounded.

For $|z_{2,u}| \le \varepsilon$, we have by (0.10),

$$|\eta_u - 1| \le C|z_{2,u}|^2. \tag{0.16}$$

By using (0.10) and (0.16) and the fact that $[0,\infty) \ni x \mapsto x^s e^{-x}$ is bounded for any $s \ge 0$, we get

$$(\eta_u - 1)^s \eta_u^{p/4} = \mathcal{O}(p^{-s}), \quad \text{for } s \ge 1. \tag{0.17}$$

Thus, $w_{l,p} = \mathcal{O}_{\mathscr{C}^0}(p^{l-N-r-1}\eta_u^{\frac{3}{4}p})$ and (0.14) follows.

Back in (0.12), for $q > N+r$, the corresponding contribution is certainly $p^{n-\frac{j}{2}-q} \cdot \mathcal{O}_{\mathscr{C}^{2r}}(p^r \eta_u^{p/2}) = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}\eta_u^{p/2})$, by (0.13). On the other hand, if $0 \le q \le N+r$, then, by (0.13) and (0.14), the corresponding contribution is $p^{n-\frac{j}{2}-q}\mathcal{O}_{\mathscr{C}^{2r}}(p^{q-N-r-1+r}(1+\sqrt{p}|Z_{2,u}|)^{6N+6r+3}\eta_u^{p/2}) = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}\eta_u^{p/4})$ again. Thus $S_u = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1})$.

From (0.10) and the above argument, $\mathcal{S}_2 = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1})$. Combining with (0.10), (0.11) and (0.13), we get (0.7) and (0.8). □

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