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Integral pinching theorems

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Abstract. Using Hamilton's Ricci flow we shall prove several pinching results for integral curvature. In particular, we show that if $p > n/2$ and the L^p norm of the curvature tensor is small and the diameter is bounded, then the manifold is an infra-nilmanifold. We also obtain a result on deforming metrics to positive sectional curvature.

1. Introduction

The goal of this note is to prove several pinching results for manifolds with integral curvature bounds. Integral pinching has been studied extensively in [3], [2], [12], [13], [15], [8]. One distinct feature in our work is that assumptions on curvature are entirely in terms of integral bounds and no assumption on volume, injectivity radius or Sobolev constant is made.

Let us fix some notation before we state the results. For a Riemannian manifold (M, g), we will denote by sec : $M \to \mathbb{R}$ the minimum of the sectional curvature at each point, by $\mathfrak{R} : \Lambda^2 TM \to \Lambda^2 TM$ the curvature operator, by Ric the Ricci curvature, and by Scal the scalar curvature.

For functions and tensors we shall consistently use the *normalized* L^p norm defined by

$$\|u\|_p = \left(\frac{1}{\mathrm{vol} M}\int_M |u|^p\right)^{1/p}$$

$$\|\mathfrak{R} \pm I\|_p = \left(\frac{1}{\mathrm{vol} M}\int_M |\mathfrak{R} \pm I|^p\right)^{1/p}$$

Our notation for the integral bounds for the Ricci tensor is as follows. For each $x \in M$ let $r(x)$ denote the smallest eigenvalue for the Ricci tensor Ric : $T_x M \to T_x M$, and for any fixed number κ let

$$\rho_\kappa(x) = |\min\{0, r(x) - (n-1)\kappa\}|$$

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be the part of the Ricci tensor which lies below $(n-1)\kappa$. Then set

$$\bar{k}(p,\kappa,r)=\sup_{x\in M}\left(\frac{1}{\mathrm{vol}B(x,r)}\cdot\int_{B(x,r)}\rho_\kappa^p\right)^{\frac{1}{p}},$$
$$\bar{k}(p,\kappa)=\left(\frac{1}{\mathrm{vol}M}\cdot\int_M\rho_\kappa^p\right)^{\frac{1}{p}}=\|\rho_\kappa\|_p.$$

These curvature quantities evidently measure how much Ricci curvature lies below $(n-1)\kappa$ in the normalized integral sense. Moreover, $\bar{k}(p,\kappa,r)=0$ iff $\mathrm{Ric}\geq(n-1)\kappa$. Note that if we assume that M has a diameter bound of the form $\mathrm{diam}M\leq D$, then it suffices to bound $\bar{k}(p,\kappa)$ rather than $\bar{k}(p,\kappa,r)$.

Theorem 1.1. *Given $p>n/2$ and $K,D>0$, there exists an $\varepsilon=\varepsilon(n,p,D,K)$ such that if a closed Riemannian manifold (M^n,g) satisfies* $\mathrm{diam}M\leq D$*,* $\bar{k}(p,0)\leq K$ *and*

$$\|\mathfrak{R}\mp I\|_{n/2}\leq\varepsilon,$$

then M admits a metric of constant sectional curvature ±1.

Note that if $\bar{k}(p,\kappa)$ is bounded for some κ and in addition $\|\mathrm{Ric}\mp(n-1)g\|_{n/2}$ is small, then for any $q\in(n/2,p)$ it follows that $\bar{k}(q,\pm1)$ is small. Thus in the above theorem we implicitly have a smallness condition on $\bar{k}(p,\pm1)$. Also, note that both conditions on the integral curvature are satisfied if we impose L^p-curvature pinching ($p>n/2$). Thus we have:

Corollary 1.1. *Given $p>n/2$ and $D>0$, there exists an $\varepsilon=\varepsilon(n,p,D)$ such that if a closed Riemannian manifold (M^n,g) satisfies* $\mathrm{diam}M\leq D$ *and*

$$\|\mathfrak{R}\mp I\|_p\leq\varepsilon,$$

then M admits a metric of constant sectional curvature ±1.

It should also be remarked that in case we assume that $\bar{k}(p,1,r)\leq\varepsilon$ for some fixed r it follows from [11] that the manifold has diameter $\leq\pi+O(\varepsilon)$, thus the above results can be modified in a trivial way in the positive case.

Now for the pinching around zero curvature,

Theorem 1.2. *Given $p>n/2$ and $K,D>0$, there exists an $\varepsilon=\varepsilon(n,p,D,K)$ such that if a closed Riemannian manifold (M^n,g) satisfies* $\mathrm{diam}M\leq D$*,* $\bar{k}(p,0)\leq K$ *and*

$$\|\mathfrak{R}\|_{n/2}\leq\varepsilon,$$

then M is diffeomorphic to an infra-nilmanifold.

Again, both conditions on integral curvature are satisfied if the L^p-curvature is already small ($p>n/2$), giving a very clean generalization of Gromov's almost flat manifold theorem [4] to integral curvature.

Corollary 1.2. *Given $p > n/2$ and $D > 0$, there exists an $\varepsilon = \varepsilon(n, p, D)$ such that if a closed Riemannian manifold (M^n, g) satisfies* $\mathrm{diam} M \le D$ *and*

$$\|\mathfrak{R}\|_p \le \varepsilon,$$

then M is diffeomorphic to an infra-nilmanifold.

The above pinching theorems were proved with similar techniques in [8], but there it was assumed that one has a pointwise curvature bound and L^2 pinching around ± 1 or 0. In [13] D. Yang has a very similar result about F-structures of manifolds with pinched integral curvature bounds.

Theorem 1.3. *Given $p > n/2$, there exists an $\varepsilon = \varepsilon(n, p)$ such that if a closed Riemannian manifold (M^n, g) satisfies either*

$$\int_M |\mathfrak{R}|^{n/2} \le \varepsilon,$$

$$(\mathrm{vol} M)^{(2p/n)-1} \int_M |\mathrm{Ric}|^p \le \varepsilon$$

or the stronger condition

$$(\mathrm{vol} M)^{(2p/n)-1} \int_M |\mathfrak{R}|^p \le \varepsilon,$$

then M admits an F-structure of positive rank.

It is curious that this result does not immediately yield an infranilmanifold theorem even if one assumes a diameter bound. The reason for this lies in the different type of integral curvature conditions that are used.

Note that the implicit smallness condition on $\bar{k}(p, \cdot)$ in Theorems 1 and 2 is neccessary, as there are manifolds with small $\frac{n}{2}$-norm of curvature tensor but with arbitrary large Betti numbers, cf. [2, Appendix]. It should also be noted that one can replace the bound on $\bar{k}(p, \cdot)$ by a bound on the Sobolev constant as we only use the smallness of $\bar{k}(p, \cdot)$ to obtain a bound on the Sobolev constant that appears in the inequality

$$\|f - \frac{1}{\mathrm{vol} M} \int f\|_{\frac{2n}{n-2}} \le C_S \|\nabla f\|_2.$$

As with previous integral pinching results the basic idea here is to use Hamilton's Ricci flow [7] to deform the metric. We are able to obtain better results because of the Sobolev constant bounds established by Petersen-Sprouse [11] for integral curvature. Also we observe, as in [14], that the Moser iteration for the Ricci flow (quadratic nonlinearity) still goes through in the borderline case, provided that we have certain smallness condition, which is satisfied in the pinching situation.

For metric deformation we also note the following result about deforming metrics to positive curvature metrics.

Theorem 1.4. *Given $q > 1$, $C > 1$, there exists an $\varepsilon = \varepsilon(n, p, C)$ such that if a Riemannian manifold (M^n, g) satisfies*

$$|\sec| \le C, \quad \|(\sec - 1)_-\|_q \le \varepsilon,$$

then M has a metric with positive sectional curvature.

2. Deformation of metrics

Given an initial metric g_0 on M, the Ricci flow

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}\,(g)\,,$$
$$g\,(0) = g_0$$

provides a very nice deformation of the initial metric. In this deformation the curvature operator $\mathfrak{R}$ evolves by the following parabolic equation [7] and [9]:

$$\frac{\partial \mathfrak{R}_t}{\partial t} = -\Delta \mathfrak{R}_t + Q\,(\mathfrak{R}_t)\,,$$

where $\Delta = \nabla^*\nabla$ denotes the connection Laplacian and $Q\,(\mathfrak{R}_t)$ is quadratic in the curvature tensor. By way of comparison recall that Lichnerowicz has shown that the curvature tensor for an Einstein metric satisfies

$$0 = -\Delta \mathfrak{R} + \tilde{Q}\,(\mathfrak{R})\,.$$

Thus the above evolution equation for the curvature is a parabolic analogue of the corrosponding elliptic equation for Einstein metrics. The parabolic equation obviously has the advantage that it works for all metrics. The two equations lead to very similar results (see e.g., [10] for a discussion on how the elliptic equation can be used).

Define the Sobolev constant $C_S = C_S(g)$ of the metric g on M to be the smallest constant such that

$$\|f - \frac{1}{\mathrm{vol}M}\int f\|_{\frac{2n}{n-2}} \leq C_S\|\nabla f\|_2.$$

Using Hölder's inequality this leads to

$$\|f\|_{\frac{2n}{n-2}} \leq C_S\|\nabla f\|_2 + \|f\|_2.$$

An essential ingredient in estimating the curvature evolution is Moser's weak maximum principle, which is a parabolic analogue of the more standard elliptic maximum principle. In the theorem we assume that g_t evolves according to the Ricci flow. The normalized volume form then satisfies an equation of the form

$$\frac{\partial}{\partial t}\frac{d\mathrm{vol}_{g_t}}{\mathrm{vol}M} = h\frac{d\mathrm{vol}_{g_t}}{\mathrm{vol}M}$$

for $h = \mathrm{Scal} - \frac{1}{\mathrm{vol}M}\int_M \mathrm{Scal}$. In particular, we have the universal estimates

$$\|h\|_p \leq 2\|\mathrm{Scal}\|_p \leq 2\,(n-1)\,\|\mathrm{Ric}\|_p$$

and a similar estimate in terms of the traceless Ricci.

Theorem 2.1. *Let f, b be smooth nonnegative functions on $M \times [0, T]$ and c a nonnegative constant satisfying the following equation on $M \times [0, T]$*

$$\frac{\partial f}{\partial t} \leq \Delta f + bf + cf, \tag{2.1}$$

where Δ is the Laplace-Beltrami operator of the metric g_t. Let $C_S = \max_{0 \leq t \leq T} C_S(g_t)$. Assume that either

1) *for some $p > n/2$,*

$$\max_{0 \leq t \leq T} (\|b\|_p + \|h\|_p) \leq \beta \ ,$$

or

2) *$b = f$ and*

$$\max_{0 \leq t \leq T} \left(\|f\|_{n/2} + \|h\|_{n/2}\right) \leq \frac{1}{(n+2)C_S^2},$$
$$\max_{0 \leq t \leq T} \|h\|_r \leq \beta, \ r > n/2.$$

Then given $q > 1$, there exsits a constant $C = C(n, p, q, \beta, c, C_S, T)$ such that for all $x \in M$ and $t \in (0, T]$

$$\|f_t\|_q \leq e^{Ct} \|f_0\|_q,$$
$$|f(x,t)| \leq C t^{-n/2q} \|f_0\|_q$$

where $f_t(x) = f(x, t)$.

Proof. In the presence of assumption 1) a fairly standard parabolic iteration argument can be used (see [12] and also [1]). Note, however, that since we have only integral bounds for $\frac{\partial}{\partial t} g_t$ we must be a little more careful. As we shall see below the roles of h and b are similar so no new problems actually occur. For case 2) we show below that it can be reduced to a slightly singular version of 1) but with $b = f$ (see also [14, Appendix A]).

First we need to check what the time derivative of $\|f^p\|_1$ is

$$\begin{aligned}\frac{\partial}{\partial t}\|f^p\|_1 &= \frac{\partial}{\partial t}\frac{1}{\mathrm{vol} M}\int_M f^p d\mathrm{vol}_{g_t} \\ &= \int_M \frac{\partial f^p}{\partial t}\frac{d\mathrm{vol}_{g_t}}{\mathrm{vol} M} + \int_M f^p \frac{\partial}{\partial t}\frac{d\mathrm{vol}_{g_t}}{\mathrm{vol} M} \\ &= \|\frac{\partial f^p}{\partial t}\|_1 + \|f^p h\|_1\end{aligned}$$

Using this estimate we can now multiply the partial differential inequality (2.1) by f^{p-1} and integrate to get (for $p > 1$)

$$\begin{aligned}\frac{1}{p}\frac{\partial}{\partial t}\|f^p\|_1 &\leq \frac{1}{\mathrm{vol} M}\int_M (f^{p-1}\Delta f + (b-h) f^p + cf^p) \\ &\leq -\frac{4(p-1)}{p^2}\|\nabla f^{p/2}\|_2^2 + \|(b-h) f^p\|_1 + c\|f^p\|_1 \qquad (2.2)\end{aligned}$$

We can use Hölder's inequality to get

$$\|(b-h)\, f^p\|_1 \le \left(\|b\|_{n/2} + \|h\|_{n/2}\right) \|f^p\|_{n/(n-2)},$$

The Sobolev inequality can be used on the first term on the right-hand of (2.2) as follows

$$\begin{aligned}
-\frac{4(p-1)}{p^2}\|\nabla f^{p/2}\|_2^2 &\le -\frac{4(p-1)}{p^2}\frac{1}{C_S^2}\left(\|f^{p/2}\|_{2n/(n-2)} - \|f\|_2\right)^2 \\
&= -\frac{4(p-1)}{p^2}\frac{1}{C_S^2}\|f^p\|_{n/(n-2)} - \frac{4(p-1)}{p^2}\frac{1}{C_S^2}\|f^{p/2}\|_2^2 \\
&\quad +2\frac{4(p-1)}{p^2}\frac{1}{C_S^2}\|f^{p/2}\|_{2n/(n-2)}\|f^{p/2}\|_2 \\
&\le -\frac{3(p-1)}{p^2}\frac{1}{C_S^2}\|f^p\|_{n/(n-2)} + \frac{60(p-1)}{p^2}\frac{1}{C_S^2}\|f^p\|_1.
\end{aligned}$$

Inserting this in (2.2) we get

$$\begin{aligned}
\frac{1}{p}\frac{\partial}{\partial t}\|f^p\|_1 \le &\left(-\frac{3(p-1)}{p^2}\frac{1}{C_S^2} + \|b\|_{n/2} + \|h\|_{n/2}\right)\|f^p\|_{n/(n-2)} \\
&+ \left(\frac{60}{pC_S^2} + c\right)\|f^p\|_1.
\end{aligned}$$

Therefore, when $p \ge 2$ and

$$\|b\|_{n/2} + \|h\|_{n/2} \le \frac{1}{2pC_S^2}$$

we obtain

$$\frac{\partial}{\partial t}\|f^p\|_1 + \frac{1}{S^2}\|f^p\|_{n/(n-2)} \le \left(\frac{60}{C_S^2} + cp\right)\|f^p\|_1 = C_1\,(p)\,\|f^p\|_1 \tag{2.3}$$

This implies in particular that

$$\frac{\partial}{\partial t}\|f^p\|_1 \le C_1\,(p)\,\|f^p\|_1 \tag{2.4}$$

and thus

$$\|f_t^p\|_1 \le \|f_{t_0}^p\|_1 e^{C_1(p)(t-t_0)} = C_2\,(p, T)\,\|f_{t_0}^p\|_1 \tag{2.5}$$

Integrating (2.3) with $p = n/2$ we obtain

$$\|f_t^{n/2}\|_1 - \|f_0^{n/2}\|_1 + \int_0^t \frac{1}{C_S^2}\|f^{n/2}\|_{n/(n-2)} \le \int_0^t C_1\,(n/2)\,\|f^{n/2}\|_1$$

Using the estimate (2.5) for $p = n/2$ and $t_0 = 0$ this leads to

$$\|f_\tau^{n/2}\|_{n/(n-2)} \le t^{-1} C_3\,(n/2, T)\,\|f_0^{n/2}\|_1$$

for some $\tau \in (0, t)$. We can then combine this with the estimate (2.5) for $p = n^2/(2(n-2))$ and $t_0 = \tau$ to get the estimate

$$\| f_t^{n^2/(2(n-2))} \|_1 \leq C_2 \left(n^2/(2(n-2)), T \right) \left(t^{-1} C_3 (n/2, T) \| f_0^{n/2} \|_1 \right)^{n/(n-2)}.$$

Hence

$$\| f_t \|_{n^2/(2(n-2))} \leq t^{-2/n} C_4 \| f_0 \|_{n/2}$$

Since $f = b$ and $\|h\|_r, r > n/2$ is bounded, we are now reduced to a singular version of 1) (with $p = n^2/(2(n-2)) = 1 + n/2 + 2/(n-2)$). But this can be dealt with in a similar way—the point is that the cut off procedure introduces the singular factor in t anyway. More precisely, an assumption on b of the type

$$\|b\|_p \leq \beta t^{-\alpha}$$

for $p > \frac{n}{2}$ and some nonnegative constant α will produce

$$|f(x,t)| \leq \left(C t^{-\frac{\alpha p}{p - n/2}} + C(n,p) t^{-1} \right)^{\frac{n+2}{2q}} \left(\int_0^t \| f \|_q^q \right)^{\frac{1}{q}}$$

for $q > 1$. Taking $p = 1 + n/2 + 2/(n-2)$, and $\alpha = 2/n$, we obtain

$$|f(x,t)| \leq C t^{-\frac{n}{2q}} \max \| f \|_q .$$

But from (2.4) we have

$$\| f_t \|_q \leq e^{Ct} \| f_0 \|_q .$$

Alternatively one can also use 1) but start the flow at time $t_0 > 0$ rather than 0 and obtain the desired estimate. □

3. Proof of theorems

Using Moser's weak maximum principle from Theorem 4 we are now in a position to prove the results mentioned in the introduction.

Applying Moser iteration to the curvature evolution equation we will prove the following

Theorem 3.1. *Given $p > n/2$, $K, D > 0$ and $\kappa \in \mathbb{R}$ there exist $\varepsilon(n, p, \kappa, D) > 0$, $T(n, p, \kappa, D, K) > 0$, and $C(n, p, \kappa, D)$ such that for any manifold satisfying*

$$\operatorname{diam} M \leq D, \ \|\mathfrak{R}\|_p \leq K, \ \bar{k}(p, \kappa) \leq \varepsilon$$

the Ricci flow has a unique smooth solution g_t for $t \in [0, T]$ satifying

$$\|\mathfrak{R}_t\|_p \leq 2K, \ \|\mathfrak{R}_t\|_\infty \leq C(n, p, \kappa, D)\, t^{-n/2p} K$$

Proof. We apply the continuity method as in [12] and [1]. For the initial metric one has from [11] a bound for the Sobolev constant in terms of n, p, κ, D provided that $\bar{k}(p,\kappa)$ is sufficiently small. Thus we inially have bounds of the type

$$\|\mathfrak{R}\|_p \le K, \quad C_S \le H(n,p,\kappa,D).$$

Since the Ricci flow exists for a short time, we can assume that for some maximal interval $[0,T),\ T>0$,

$$\begin{aligned}\|\mathfrak{R}_t\|_p &\le 2K\\ C_S(g_t) &\le 2H(n,p,\kappa,D).\end{aligned}$$

We need to check how T depends on the assumptions. By Moser's weak maximum principle

$$\begin{aligned}|\mathfrak{R}_t|_\infty &\le C(n,p,\kappa,D)t^{-\frac{n}{2p}}K,\\ \|\mathfrak{R}_t\|_p &\le Ke^{C(n,p,\kappa,D)t}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{1}{C_S(g_t)} &\ge \inf_{u\in C^\infty(M)} \frac{\int_M(|\nabla u|^2_{g_t}+u^2)d\mathrm{vol}g_t}{\left(\int_M |u|^{2n/(n-2)}d\mathrm{vol}g_t\right)^{(n-2)/n}}\\ &= \inf_{u\in C^\infty(M)} E_t(u),\end{aligned}$$

and $E_t(u)$ satisfies (see [12])

$$\frac{d}{dt}E_t(u) \ge -c(n)\|\mathrm{Ric}_{g_t}\|_\infty E_t(u).$$

It follows then that

$$\begin{aligned}E_t(u) &\ge H(n,p,\kappa,D)\exp\left(-C(n,p,\kappa,D)\int_0^t |\mathfrak{R}_t|\,ds\right)\\ &\ge H(n,p,\kappa,D)\exp\left(-C(n,p,\kappa,D)\,t^{1-n/2p}\right).\end{aligned}$$

Therefore,

$$C_S(g_t) \le H(n,p,\kappa,D)\exp\left(C(n,p,\kappa,D)\,t^{1-n/2p}\right).$$

Consequently, there exists $T(n,p,\kappa,D,K)>0$ such that if $t<\min(T, T(n, p,\kappa,D,K))$, then the metric g_t will have a uniformly bounded curvature as one approaches T. In particular, if $T<T(n,p,\kappa,D,K)$, then a theorem of Hamilton [7] shows that the solution to the Ricci flow equation extends smoothly beyond T (while preserving the required bounds), contradicting with the maximality of T. □

We note that by using 2) in the Moser's weak maximum principle the above argument goes through with only notational change if we initially have

$$\|\mathfrak{R}\|_{n/2} \leq \frac{1}{2nC\,(n)\,H(n,\,p,\kappa,\,D)^2}.$$

Note that the necessary bounds on h come from the curvature bounds. Thus when we are on the interval where we assume $\|\mathfrak{R}\|_{n/2} \leq \frac{1}{nC(n)H(n,p,\kappa,D)^2}$ we also have $\|h\|_{n/2} \leq \frac{1}{nH(n,p,\kappa,D)^2}$. From this we derive the L^p estimates for $\mathfrak{R}$ and hence h as in the proof of Theorem 4. We are then in a positition to use Theorem 5.

Our estimate then shows that if the ε in Theorem 2 is chosen sufficiently small, the metric $g(T)$ is a smooth Riemannian metric with small pointwise curvature bound. Therefore we can apply Gromov's almost flat manifold theorem [4]. This finishes the proof of Theorem 2.

The proof of Theorem 1 follows a similar scheme. Assume that a compact Reimannian manifold M satisfies the assumption of Theorem 1, with ε small. Let $g(t)$ be the unique solution to the Ricci flow. Now according to [8], the reduced curvature tensor

$$\tilde{\mathfrak{R}} = \mathfrak{R} \pm I$$

satisfies the parabolic inequality

$$(\frac{\partial}{\partial t} - \Delta)|\tilde{\mathfrak{R}}|^2 + |\nabla\tilde{\mathfrak{R}}|^2 \leq \frac{4}{n}|\tilde{\mathfrak{R}}|^2 + c(n)|\tilde{\mathfrak{R}}|^3.$$

Making use of Kato's inequality one derives

$$\frac{\partial}{\partial t}|\tilde{\mathfrak{R}}| \leq \Delta|\tilde{\mathfrak{R}}| + \frac{4}{n}|\tilde{\mathfrak{R}}| + c(n)|\tilde{\mathfrak{R}}|^2.$$

Once again we can apply the continuity method together with Moser's weak maximum principle. Therefore, if ε is chosen sufficiently small, the metric $g(T)$ is a smooth Riemannian metric with small uniform curvature pinching. We can then sply appeal to the pinching results of Grove-Karcher-Ruh (see [6]) and Gromov (see [5]) to finish the proof.

We finally establish Theorem 3. Once again we run the Ricci flow. Let $\sec_t = \sec(t,x)$ denote the pointwise minimum of the sectional curvature for g_t. This function satisfies

$$\frac{\partial}{\partial t}\sec \geq \Delta\sec - c\,(n)\,|\mathfrak{R}|^2\,,$$

which implies

$$\frac{\partial}{\partial t}(\sec - 1)_- \leq \Delta(\sec - 1)_- + c\,(n)\,C^2.$$

Let $f = ((\sec - 1)_- - c\,(n)\,C^2 t)_+$. Then

$$\frac{\partial}{\partial t} f \leq \Delta f.$$

Therefore applying Moser's iteration (in a much simpler situation) we arrive at

$$\|(\sec_t - 1)_-\|_\infty \leq C_1 t^{-n/2p} \|(\sec_0 - 1)_-\|_p + c\,(n)\,C^2 t.$$

We now choose t so that $c\,(n)\,C^2 t < 1/3$ and then determine ε so that

$$C_1 t^{-n/2p} \|(\sec_0 - 1)_-\|_p < 1/3.$$

Then g_t will have positive sectional curvature.

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