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Finite Part of Spectrum and Isospectrality

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Abstract. We study geometric conditions under which finitely many eigenvalues are sufficient to determine all spectral data. We also discuss briefly the implication of this result to the local structure of moduli space of isopectral metrics.

1. Introduction

For a compact (smooth) manifold M^n, a Riemannian metric gives rise to a canonical differential operator, namely, the Laplacian Δ (acting on smooth functions). If $\partial M \neq \emptyset$, we put the Dirichlet boundary condition on the boundary. This makes Δ into a self-adjoint second order elliptic operator. Hence it has a discrete spectrum all consisting of eigenvalues of finite multiplicity:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

Here the eigenvalues are repeated according to their multiplicity.

Two Riemannian manifolds (M, g) and (M', g') are said to be isospectral if their corresponding eigenvalues are identical:

$$\lambda_i(M, g) = \lambda_i(M', g'), \ i = 1, 2, \cdots .$$

One of the main questions, popularized by Kac's question "can one hear the shape of the drum", in the (inverse) spectral geometry is how the geometries of two isospectral manifolds are related. The extensive work in this direction shows that isospectral manifolds exist in abundance (see Carolyn Gordon's survey lecture in this volume). In particular Kac's original question has been recently answered in the negative by Gordon-Webb-Wolpert [GWW]. On the other hand the beautiful work of Osgood-Phillips-Sarnak [OPS] exhibits the compactness of the isospectral surface, which was subsequently generalized to dimension 3 (with certain geometric assumptions) by [**BPY, CY, BPP**] and [**A1**].

The above question can be rephrased as how much geometric information can be determined by the spectral information. In the works mentioned above this spectral information consists of knowing *all* of the eigenvalues. But, in some sense, a more practical one would be the knowledge of finitely many eigenvalues. This

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question has actually been studied by P. Li, A. Treibergs and S. T. Yau [**LTY**]. They studied convex domains in R^n and showed that the volume of the domain can be determined to arbitrary accuracy from knowing sufficiently many eigenvalues. This result has been generalized to domains in more general Riemannian manifold by H. Donnelly and J. Lee (see [**L**]).

An interesting question relating this "finite inverse spectral geometry" and inverse spectral geometry is then: when would "finitely isospectral" metrics be actually isospectral? The following theorem of P. Buser and G. Courtois gives a partial answer for Riemann surfaces.

THEOREM 1.1. *([**BC**]) For a Riemann surface Σ of genus $g \geq 2$, let $R_g(\Sigma)$ denote the moduli space of metrics of constant curvature -1 and $R_g(\Sigma, i_0)$ the subset in which one also has an injectivity radius lower bound:*

$$inj \geq i_0 > 0.$$

Then there exists an integer $N = N(g, i_0)$ such that any $g,\ g' \in R_g(\Sigma, i_0)$ with

$$\lambda_i(g) = \lambda_i(g') \text{ for } i = 1, \cdots, N$$

must be isospectral.

The purpose of this note is to consider higher dimensional generalizations of the above result. Let's first introduce a few notations. We denote by $\mathcal{M}_1$ the moduli space of Riemannian metrics with volume = 1. $\mathcal{M}_1$ has a structure of real analytic (infinite dimensional) manifold.[1]

Definition. A subset $A \subset \mathcal{M}_1$ is called *thin* if for every point of A there exists a neighborhood U in $\mathcal{M}_1$, and a finite dimensional real analytic variety S[2] of $\mathcal{M}_1$ such that

$$A \cap U \subset S.$$

We can now state our result:

THEOREM 1.2. *For any compact, thin subset $C \subset \mathcal{M}_1$, there exists an integer $N = N(C)$ such that any $g,\ g' \in C$ with*

$$\lambda_i(g) = \lambda_i(g') \text{ for } i = 1, \cdots, N$$

must be isospectral.

As a corollary we have the following analog of Buser-Courtois' Theorem.

THEOREM 1.3. *Let $\mathcal{E}(i_0, D)$ be the (moduli) space of Einstein metrics (i.e. Ric $= \lambda g$ for $\lambda = 1, 0,$ or -1) with*

$$inj \geq i_0 > 0 \text{ and } \quad diam \leq D.$$

Then there exists an integer $N = N(i_0, D)$ such that any $g,\ g' \in \mathcal{E}(i_0, D)$ with

$$\lambda_i(g) = \lambda_i(g') \text{ for } i = 1, \cdots, N$$

must be isospectral.

[1] Strictly speaking, as we will see in the next section, locally $\mathcal{M}_1$ is the quotient of a real analytic manifold by a compact Lie group.

[2] Again, here S should be a quotient of a finite dimensional real analytic variety by a compact Lie group

Remarks.

1. If $\lambda = 1$, the upper bound for the diameter will follow from Myers' theorem.

2.If dim $= 2$, $\lambda = -1$ (the case considered by Buser-Courtois), the upper bound for the diameter will follow from the lower bound of the injectivity radius and Gauss-Bonnet Theorem.

3. The injectivity radius lower bound prevents the degeneration (or collapsing in the sense of Cheeger-Gromov) to a noncompact space where continuous spectrum appears.

4. Same result holds for p-spectrum (i.e. the spectrum of the Hodge-Laplacian acting on differential p-forms). In this regard we would like to mention the following interesting result of Patodi. A compact Riemannian manifold isospectral to a compact Einstein manifold on functions, 1-forms and 2-forms must be Einstein itself.

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2. Analytic structures of moduli spaces

Let $\widetilde{\mathcal{M}}$ denote the set of Riemannian metrics on M. In the compact open C^∞-topology $\widetilde{\mathcal{M}}$ is an open convex cone in the infinite dimensional affine space modelled on the space of smooth symmetric 2-tensors on M. Clearly then $\widetilde{\mathcal{M}}$ inherits a real analytic structure. Therefore the subset $\widetilde{\mathcal{M}}_1$ of Riemannian metrics with total volume 1 defines a real analytic variety in $\widetilde{\mathcal{M}}$ by the equation

$$\mathrm{vol}(M, g) = 1.$$

Now the infinite dimensional group $\mathrm{Diff}(M)$ acts on $\widetilde{\mathcal{M}}_1$ analytically. Thus one expects the analytic structure persists when one passes to the quotient space $\mathcal{M}_1 = \widetilde{\mathcal{M}}_1/\mathrm{Diff}(M)$. Indeed this is the case by the Slice theorem of D. Ebin (see [Be]).

THEOREM 2.1 (D. Ebin). *For any $[g] \in \mathcal{M}_1$, there exsits a real analytic submanifold $S_g \subset \widetilde{\mathcal{M}}_1$ such that*

a). S_g is invariant under the group $I(M, g)$ of isometries of g.

b). If $\varphi \in Diff(M)$ and $\varphi^* S_g \cap S_g \neq \emptyset$, then $\varphi \in I(M, g)$.

c). There is a local cross-section $\chi : Diff(M)/I(M, g) \to Diff(M)$ on a neighborhood of the coset I_g such that the local mapping

$$\begin{array}{ccc} Diff(M)/I(M,g) \times S_g & \to & \widetilde{\mathcal{M}}_1 \\ ([\varphi], \tilde{g}) & \to & \chi([\varphi])^* \tilde{g} \end{array}$$

is a diffeomorphism onto a neighborhood of g in $\widetilde{\mathcal{M}}_1$. In particular, the induced map $S_g/I(M, g) \to \mathcal{M}_1$ is a homeomorphism onto a neighborhood of Riemannian structures defined by g.

Thus, according to this theorem, locally $\mathcal{M}_1$ looks like a real analytic submanifold quotient out by a compact Lie group.

Remark. For a generic metric g, $\mathrm{I}(M, g)$ is trivial.

We now turn to the moduli space $\mathcal{E}$ of Einstein structures. By definition, the moduli space of Einstein structures on M is the subset of $\mathcal{M}_1$ consisting of all $[g]$ satisfying the Einstein equation

$$\mathrm{Ric}_g = \lambda g$$

for some constant λ. It turns out that the linearized operator for the Einstein equation is Fredholm and therefore one has (see [Be])

THEOREM 2.2 (N. Koiso). *Let $[g] \in \mathcal{E}$ be an Einstein metric on M. Then there exsits a finite dimensional real analytic submanifold Z in the slice S_g such that Z contains all Einstein metrics in the slice S_g.*

As an immediate corollary we have

COROLLARY 2.1. *$\mathcal{E} \subset \mathcal{M}_1$ is a thin subset.*

When $M = G/H$ is a homogenous space, where G is a Lie group and $H \subset G$ a closed subgroup, we have another interesting example of thin subset in $\mathcal{M}_1$. Namely the space $\mathcal{M}_1^{l.i.}$ of left invariant metrics on M. Since $\mathcal{M}_1^{l.i.}$ can be identified with the space of inner products on the vector space g/h satisfying certain (analytic) equation, $\mathcal{M}_1^{l.i.}$ is clearly thin.

Let S be a finite dimensional real analytic manifold contained in $\widetilde{\mathcal{M}}$. Then $S \ni g \to \Delta_g$ defines an analytic family of self-adjoint operators. Applying the analytic perturbation theory gives us (see [K])

PROPOSITION 2.1. *1). If L is not an eigenvalue of Δ_g then there exists a $\epsilon > 0$ such that $(L-\epsilon, L+\epsilon)$ contains no eigenvalues of $\Delta_{\bar{g}}$ for $\bar{g}$ in a small neighborhood U of g in S.*

2). In $U \times (L-\epsilon, L+\epsilon)$, the resolvent $(\Delta_{\bar{g}} - \lambda)^{-1}$ defines an analytic family of bounded linear operators.

3). Let $\lambda_n(g)$ be an eigenvalue of Δ_g such that

$$\lambda_n(g) < L < \lambda_{n+1}(g).$$

Then for all $\bar{g} \in U$

$$\lambda_n(\bar{g}) < L < \lambda_{n+1}(\bar{g}),$$

and for every symmetric function f of n variables the function $\bar{g} \to f(\lambda_1(\bar{g}), \cdots, \lambda_n(\bar{g}))$ is real analytic on U.

3. Proof of the theorem

The basic idea here, due to [BC] and [B], is as follows. First of all the compactness assumption reduces the consideration to a local one where by the thinness of C, we can, without the loss of generality, assume C to be a finite dimensional real analytic submanifold. Now by the analytic perturbation theory [K], the eigenvalues $\lambda_i(g)$ will then depend "analytically" in g. (This is not exactly true due to the change of multiplicity.) Assuming this, the equations

$$\lambda_1(g) = \lambda_1(g'), \cdots, \lambda_n(g) = \lambda_n(g')$$

for $n = 1, 2, \cdots$, will then cut out a nesting sequence of real analytic varieties, which, by the Noetherian property, must stabilize at finite stage, proving the theorem.

But of course, the problem with the change of multiplicity is a nontrivial one. Roughly speaking one has to set up an infinite sequence of "barriers" to prevent (locally) such a change of multiplicity. As a consequence one obtains the desired result only at the level of germs. To overcome this we then invoke a theorem of Bruhat-Cartan and the analyticity takes over the rest of the argument.

Proof of Theorem 1.2. We consider $C \times C \subset \mathcal{M}_1 \times \mathcal{M}_1$. By the compactness we just have to produce a neighborhood for each $(g, g') \in C \times C$ such that our claim holds. Since C is thin, we might as well assume that (near g and g') C is a finite dimensional real analytic submanifold. And the rest of our argument will be carried out in (the product of) this real analytic submanifold.

Now if g and g' are not isospectral, then

$$\lambda_n(g) \neq \lambda_n(g')$$

for some n. By the continuity of eigenvalues we can find a neighborhood U of (g, g') such that

$$\lambda_n(\bar{g}) \neq \lambda_n(\bar{g}') \ \forall (\bar{g}, \bar{g}') \in U.$$

Therefore, if we take this U and $N = n$, our claim is trivially satisfied. Thus we can assume that g and g' are actualy isospectral.

Choose an infinite sequence of positive numbers

$$0 < L_1 < L_2 < \cdots \to \infty$$

converging to $+\infty$ and none of them are eigenvalues of g (and g'). (This is our infinite sequence of "barriers".) Again the continuity of eigenvalues implies that we can find a nesting sequence of neighborhoods (with real analytic boundary)

$$U_1 \supset U_2 \supset \cdots$$

such that $L_1, \cdots, L_n$ are not eigenvalues of either $\bar{g}$ or $\bar{g}'$ for any $(\bar{g}, \bar{g}') \in U_n$.

Let $\lambda_{m(n)}$ be the largest eigenvalue less than L_n. By Proposition 2.1, for any symmetric polynomial f of $\lambda_1, \cdots, \lambda_{m(n)}$, the function

$$\begin{array}{ccc} U_n & \to & R \\ (\bar{g}, \bar{g}') & \to & F(\bar{g}, \bar{g}') = f(\lambda_1(\bar{g}), \cdots, \lambda_{m(n)}(\bar{g})) - f(\lambda_1(\bar{g}'), \cdots, \lambda_{m(n)}(\bar{g}')) \end{array}$$

is real analytic. On the other hand the set

$$A_n = \{(\bar{g}, \bar{g}') | \lambda_1(\bar{g}) = \lambda_1(\bar{g}'), \cdots, \lambda_{m(n)}(\bar{g}) = \lambda_{m(n)}(\bar{g}')\}$$

coincides with the level set $\{F_f(\bar{g}, \bar{g}') = 0$ for all symmetric polynomials $f\}$[3] and thus is a real analytic variety in U_n. Now

$$A_1 \cap U_1 \supset A_2 \cap U_2 \supset \cdots$$

is a nesting sequence of real analytic variety. Since the germ of analytic functions is Notherian we can found a large integer K such that $n \geq K$ implies

$$A_n \cap U_n = (A_K \cap U_K) \cap U_n.$$

What this equation is saying is that when the first $(K+1)$ eigenvalues agree, the first $(n+1)$ eigenvalues will also agree, provided we restrict to (a probably smaller neighborhood) U_n. This is not our theorem yet, but we note that $(g, g') \in A_n \cap U_n$ for all n.

We now take U_0 to be the component of U_K that contains (g, g') and we show that for any $(\bar{g}, \bar{g}') \in U_0$,

$$\lambda_i(\bar{g}) = \lambda(\bar{g}') \text{ for } i = 1, \cdots, K$$

implies equality for the rest of eigenvalues.

[3]Since $\prod_{i=1}^{n}(\lambda - \lambda_i) = \sum_{k=0}^{n} \sigma_k(\lambda_1, \cdots, \lambda_n)\lambda^{n-k}$, with $\sigma_k(\lambda_1, \cdots, \lambda_n)$ elementary symmetric functions of $\lambda_1, \cdots, \lambda_n$.

By a theorem of Bruhat-Cartan [BCa] we can connect $(\bar{g}, \bar{g}')$ to (g, g') by a real analytic path, $(g(t), g(t)')$, with

$$(g(0), g(0)') = (g, g'), \ (g(1), g(1)') = (\bar{g}, \bar{g}').$$

The idea here is that for any n,

$$\lambda_i(g(t)) = \lambda_i(g(t)'), \ i = 1, \cdots, n$$

for all t sufficiently small. By the analyticity, this should then hold for all $t \in [0, 1]$. But once again we have to deal with the multiplicity problem.

For this purpose, we are going to rearrange the order of the eigenvalues. we start at (g, g'). By the analytic perturbation theory we can arrange the eigenvalues of $g(t), \ g(t)'$ near $t = 0$ into analytic functions $\lambda_\alpha(g(t)), \ \lambda_\alpha(g(t)')$. (Here the self-adjointness of Laplacian is essential.) Near $t = 0$, the index α can be arranged so that it actually represents the increasing order for the eigenvalues. But then we use the analytic perturbation theory to analytic continue these functions $\lambda_\alpha(g(t)), \ \lambda_\alpha(g(t)')$ to all of $[0, 1]$. (Note that when we pass a point where the change of multiplicity occurs the index α may no longer represent the increasing order of the eigenvalues.) In this way we can arrange the eigenvalues of the families $g(t), \ g(t)'$ into analytic functions $\lambda_\alpha(g(t)), \ \lambda_\alpha(g(t)')$. Therefore the above argument applies and the theorem is proved. □

Theorem 1.3 is an immediate consequence of Theorem 1.2, Corollary 2.1 and the following compactness theorem of M. Anderson [A2].

THEOREM 3.1 (M. Anderson). *The space $\mathcal{M}(\lambda, i_0, D)$ of metrics with*

$$|Ric| \leq \lambda, \ inj \geq i_0 (> 0) \text{ and } diam \leq D$$

is compact in $C^{1,\alpha}$-topology. Moreover the space $\mathcal{E}(\lambda, i_0, D)$ of Einstein metrics is compact in C^∞-topology.

4. Final remarks

A very interesting result on the finite part of spectrum of Laplacian is a result of Colin de Verdiere [C], which says that the finite part can be arbitrarily prescribed.

THEOREM 4.1 (Colin de Verdiere). *Let M^n be compact and connected, $n \geq 3$. Then any finite sequence*

$$0 = a_0 < a_1 \leq a_2 \leq \cdots \leq a_N$$

can be realized as the first $N + 1$ eigenvalues of (M, g) for some g.

Thus if one defines, for each $N \geq 1$,

$$\begin{array}{rcl} S_N : \mathcal{M} & \to & R^N \\ g & \to & (\lambda_1(g), \cdots, \lambda_N(g)), \end{array}$$

then the image of S_N is precisely the cone in R^N defined by

$$\{0 < x_1 \leq x_2 \leq \cdots \leq x_N\}.$$

The level set $S_N^{-1}(a), \ a = (a_1, \cdots, a_N)$ can be thought of as the moduli space of "N-isospectral" metrics:

$$S_N^{-1}(a) = \{[g] | \ \lambda_i(g) = a_i, i = 1, \cdots, N\}.$$

One can define similarly

$$S_\infty : \mathcal{M} \to R^\infty$$

and the level sets of S_∞ can be thought of as the moduli spaces of isospectral metrics. Note that $S_\infty^{-1}(a) = \cap_{N=1}^\infty S_N^{-1}(a)$. In general, one expects the inclusion

$$S_\infty^{-1}(a) \subset S_N^{-1}(a)$$

to be strict for any N. The reason is that $S_N^{-1}(a)$ should be infinite dimensional while $S_\infty^{-1}(a)$ is conjectured to be compact.

However our result indicates that when restricting to a finite dimensional family of metrics, such as those of Einstein metrics or homogenous metrics (satisfying additional geometric restrictions) the moduli space of isospectral metrics will actually coincide with the moduli space of N-isospectral metrics for N sufficiently large. Whether or not this helps understand the (local) structure of the moduli space of isospectral metrics is not clear to us. However we note the following

Proposition 4.1. *For any $a \in R^\infty$ the intersection of $S_\infty^{-1}(a)$ with any finite dimensional real analytic family of metrics is locally a real analytic manifold.*

Proof. Since this is a local problem, when restrcited to a finite dimensional family, we clearly have compactness. Therefore by Theorem 1.2 we only have to consider $S_N^{-1}(a)$ for N sufficiently large. Now for suitable N we can make sure that $\lambda_N < \lambda_{N+1}$. Therefore the same argument as in the proof of Theorem 1.2 shows that $S_N^{-1}(a)$ is locally a real analytic manifold. □

We would also like to make a few remarks about these additional geometric restrictions. It should be possible to eliminate some of these restriction by enlarging the class of "admissble" metrics, as is often required in compactifying these spaces of special metrics. For example in dimension 4, the space

$$\mathcal{E}^{+1} = \{[g] \in \mathcal{M} | \ \mathrm{Ric} = g\}$$

can be compactified by adding orbifold singular Einstein metrics [A3]. Thus if we consider our question in the larger space of orbifold singular metrics we would prove the same result without the restriction on the injectivity radius.

If one is mainly conserned with the moduli space of N-isospectral metrics, more can be said about the additional geometric restrictions. In fact a theorem of S. Y. Cheng states that the diameter can be estimated in terms of the lower bound on the Ricci, say Ric ≥ -1, and the number of eigenvalues lying in the interval $[0, \frac{(n-1)^2}{4} + \epsilon)$. Thus for appropriate N and $a \in R^N$ the metrics in $S_N^{-1}(a)$ with Ric ≥ -1 will also be bounded in diameter. Similarly, in dimension 2, the injectivity radius of hyperbolic metrics can be estimated by the number of eigenvalues lying in $[1/4, 1]$.

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