Invent. math. 161, 151–176 (2005)
DOI: 10.1007/s00222-004-0424-x

*Inventiones mathematicae*

# On the stability of Riemannian manifold with parallel spinors

**Xianzhe Dai**[1], **Xiaodong Wang**[2,⋆], **Guofang Wei**[1,⋆⋆]

[1] Department of Mathematics, UCSB, Santa Barbara, CA 93106, USA
(e-mail: dai@math.ucsb.edu/wei@math.ucsb.edu)
[2] Department of Mathematics, MIT, Cambridge, MA 02139, USA
(e-mail: xwang@math.mit.edu)

Oblatum 10-XI-2003 & 4-XI-2004
Published online: 1 March 2005 – 

*Dedicated to Jeff Cheeger for his sixtieth birthday*

**Abstract.** Inspired by the recent work [HHM03], we prove two stability results for compact Riemannian manifolds with nonzero parallel spinors. Our first result says that Ricci flat metrics which also admit nonzero parallel spinors are stable (in the direction of changes in conformal structures) as the critical points of the total scalar curvature functional. Our second result, which is a local version of the first one, shows that any metric of positive scalar curvature cannot lie too close to a metric with nonzero parallel spinor. We also prove a rigidity result for special holonomy metrics. In the case of $SU(m)$ holonomy, the rigidity result implies that scalar flat deformations of Calabi-Yau metric must be Calabi-Yau. Finally we explore the connection with a positive mass theorem of [D03], which presents another approach to proving these stability and rigidity results.

## 1. Introduction

One of the most fruitful approaches to finding the 'best' (or canonical) metric on a manifold has been through the critical points of a natural geometric functional. In this approach one is led to the study of variational problems and it is important to understand the stability issue associated to the variational problem. Consider the space $\mathcal{M}$ of Riemannian metrics on a compact manifold $M$ (our manifolds here are assumed to have empty boundary).

⋆ Partially supported by NSF Grant # DMS 0202122.
⋆⋆ Partially supported by NSF Grant # DMS-0204187.

It is well known that the critical points of the total scalar curvature functional (also known as the Hilbert-Einstein action in general relativity) are Ricci flat metrics. It is also well known that the total scalar curvature functional behaves in opposite ways along the conformal deformations and its transversal directions (i.e., when the conformal structure changes). The variational problem in the conformal class of a metric (volume normalized) is the famous Yamabe problem, which was resolved by Aubin and Schoen. In this paper we study the stability for the total scalar curvature functional when we restrict to the transversal directions, that is, the space of conformal structures.

This has to do with the second variation of the total scalar curvature functional restricted to the traceless transverse symmetric 2-tensors, which is given in terms of Lichnerowicz Laplacian [Bes87]. Our first result shows that Riemannian manifolds with nonzero parallel spinors (which are necessarily Ricci flat) are stable in this sense.

**Theorem 1.1.** *If a compact Riemannian manifold* $(M, g)$ *has a cover which is spin and admits nonzero parallel spinors, then the Lichnerowicz Laplacian* $\mathcal{L}_g$ *is positive semi-definite.*

This makes essential use of a Bochner type formula relating the Lichnerowicz Laplacian to the square of a twisted Dirac operator for Riemannian manifolds with nonzero parallel spinors, which is a special case of a result in [Wa91, Proposition 2.4], where the general case of Killing spinor is discussed. Also, we note that the special case of Theorem 1.1 for K3 surfaces is proved in [GIK02].

Theorem 1.1 settles an open question raised in [KW75] about thirty years ago in the case when the Ricci flat manifold has a spin cover with nonzero parallel spinors. It is an interesting open question of how special our metric is compared to the general Ricci flat metric (Cf. Sect. 5) but we note that, so far, all known examples of compact Ricci flat manifolds are of this type, namely, they admit a spin cover with nonzero parallel spinors.

In a very recent work [CHI04], Cao-Hamilton-Ilmanen studied the stability problem for Ricci solitons and Ricci shrinking solitons using the functionals introduced by Perelman [P02] and showed that they are governed by the Lichnerowicz Laplacian. Thus, as an application of Theorem 1.1, Cao-Hamilton-Ilmanen deduce that compact manifolds with nonzero parallel spinors are also stable as Ricci soliton [CHI04].

We next prove that, in fact, there exists a neighborhood of the metric with nonzero parallel spinors, which contains no metrics with positive scalar curvature. This can be thought of as a local version of our previous (infinitesimal) stability result.

**Theorem 1.2.** *Let* $(M, g)$ *be a compact Riemannian manifold which admits a spin cover with nonzero parallel spinors. Then* $g$ *cannot be deformed to* *positive scalar curvature metrics. By this we mean that there exists no path of metrics $g_t$ such that $g_0 = g$ and the scalar curvature $S(g_t) > 0$ for $t > 0$. In fact, if $(M, g)$ is simply connected, then there is a neighborhood of g in the space of metrics which does not contain any metrics with positive scalar curvature.*

The existence of metrics with positive scalar curvature is a well studied subject, with important work such as [L63], [H74], [SY79-2], [GL80], culminating in the solution of the Gromov-Lawson conjecture in [S92] for simply connected manifolds. Thus, a $K3$ surface does not admit any metric of positive scalar curvature, but a simply connected Calabi-Yau 3-fold does. Of course, a Calabi-Yau admits nonzero parallel spinors (in fact, having nonzero parallel spinors is more or less equivalent to having special holonomy except quaternionic Kähler, cf. Sect. 3). Thus on a Calabi-Yau 3-fold there exist metrics of positive scalar curvature but they cannot be too close to the Calabi-Yau metric.

One should also contrast our result with an old result of Bourguignon, which says that a metric with zero scalar curvature but nonzero Ricci curvature can always be deformed to a metric with positive scalar curvature (essentially by Ricci flow).

The local stability theorem implies a rigidity result, Theorem 3.4, from which we deduce the following interesting application.

**Theorem 1.3.** *Any scalar flat deformation of a Calabi-Yau metric on a compact manifold must be Calabi-Yau. In fact, any deformation with nonnegative scalar curvature of a Calabi-Yau metric on a compact manifold is necessarily a Calabi-Yau deformation. The same is true for the other special holonomy metrics, i.e., hyperkähler, $G_2$, and Spin(7).*

Our result generalized a theorem in [Wa91] for Einstein deformations.

The proof of our second result, the local stability theorem, actually gives a very nice picture of what happens to scalar curvature near a metric with parallel spinor (we'll call it special holonomy metric): one has the finite dimensional moduli of special holonomy metrics; along the normal directions, the scalar curvature of the Yamabe metric in the conformal class will go negative. In other words, we have

**Theorem 1.4.** *The Calabi-Yau (and other special holonomy) metrics are local maxima for the Yamabe invariant.*

We also explore a remarkable connection between the stability problem for $M$ and positive mass theorem on $\mathbb{R}^3 \times M$ (One is reminded of Schoen's celebrated proof of the Yamabe problem which makes essential use of the positive mass theorem). This connection is pointed out in the recent work [HHM03] through physical considerations. It provides us with a uniform approach to the stability problem. The original positive mass theorem [SY79-1], [Wi81] is related to the stability of the Minkowski space as the vacuum (or minimal energy state). In superstring theory this vacuum is replaced by the product of the Minkowski space with a Calabi-Yau manifold [CHSW85]. The positive mass theorem for spaces which are asymptotic to $\mathbb{R}^k \times M$ at infinity is proved in [D03], see also [HHM03].

This paper is organized as follows. We first review some background on the variational problem. In fact, following [KW75], we use the first eigenvalue of the conformal Laplacian instead of the total scalar curvature, which is essentially equivalent but has the advantage of being conformally invariant in a certain sense (Cf. Sect. 2). Following [Wa91], we then recall how the parallel spinor enables us to identify symmetric 2-tensors with spinors twisted by the cotangent bundle. This leads us to our first main result, Theorem 1.1.

In Sect. 3 we prove the local stability theorem, Theorem 1.2. The essential ingredient here involves characterizing the kernel of Lichnerowicz Laplacian, which is done according to the holonomy group. In each case, we rely on the deformation theory of the special holonomy metric. For example, in the Calabi-Yau case, we use the Bogomolov-Tian-Todorov theorem [T86], [To89] about the smoothness of the universal deformation of Calabi-Yau manifold and Yau's celebrated solution of Calabi conjecture [Y77,Y78]. The deformation theory of $G_2$ metrics is discussed by Joyce [J00]. The kernel of Lichnerowicz Laplacian is also worked out in [Wa91] for the Einstein deformation problem (Theorem 3.1) with essentially the same techniques.

We then discuss the connection with positive mass theorem which presents a uniform approach. That is, one does not need separate discussions for each of the special holonomy. The positive mass theorem we use here is a special case of the result proved in [D03]. We show that, if a metric of positive scalar curvature is too close to the (Ricci flat) metric with nonzero parallel spinor, then one can construct a metric on $\mathbb{R}^3 \times M$ which has nonnegative scalar curvature, asymptotic to the product metric at infinity, but has negative mass. Hence contradictory to the positive mass theorem.

In the final section we make some remarks about compact Ricci flat manifolds and point out the existence of scalar flat metrics which are not Calabi-Yau on some Calabi-Yau manifolds. This existence result depends on Theorem 1.4, which gives a way to tell when a scalar flat metric is not Calabi-Yau (or special holonomy).

Our work is inspired and motivated by the recent work of Hertog-Horowitz-Maeda [HHM03].

*Acknowledgement.* The first and third authors are indebted to Gary Horowitz for numerous discussions on his work [HHM03]. The first author also thanks Thomas Hertog, John Lott, John Roe and Gang Tian for useful discussion. The second author wishes to thank Rick Schoen for stimulating discussions and encouragement. We thank McKenzie Wang for bringing his work [Wa91] to our attention. Finally, the authors thank the referees for many constructive comments and suggestions, and for clarifying the relation of the work [Wa91] with ours.

## 2. The infinitesimal stability

Let $(M, g_0)$ be a compact Riemannian manifold with zero scalar curvature and $\mathrm{Vol}\,(g_0) = 1$. Our goal is to investigate the sign of the scalar curvature of metrics near $g_0$. For any metric $g$, we consider the conformal Laplacian $\Delta_g + c_n S_g$, where $c_n = (n-2)/4(n-1)$ and $S_g$ denotes the scalar curvature (and we use the nonnegative or the geometer's Laplacian). Let $\lambda(g)$ be its first eigenvalue and $\psi_g$ the first eigenfunction, normalized to satisfy $\int_M \psi_g dV_g = 1$, i.e.

$$\Delta_g \psi_g + c_n S_g \psi_g = \lambda(g)\psi_g, \tag{2.1}$$

$$\int_M \psi_g dV_g = 1. \tag{2.2}$$

So defined, $\psi_g$ is then uniquely determined and is in fact positive.

The functional $\lambda(g)$, studied first by Kazdan and Warner [KW75], has some nice properties. Though it is not conformally invariant, its sign is conformally invariant. By the solution of the Yamabe problem, any metric $g$ can be conformally deformed to have constant scalar curvature. The sign of the constant is the same as that of $\lambda(g)$. In fact the metric $\psi_g^{4/(n-2)} g$ has scalar curvature $c_n \lambda(g) \psi_g^{-4/(n-2)}$ whose sign is determined by $\lambda(g)$. Obviously $\lambda(g_0) = 0$ and the corresponding eigenfunction is $\psi_0 \equiv 1$. We present the variational analysis of the functional $\lambda$, essentially following [KW75] with some modification and simplification. (Our notations are also slightly different).

We first indicate that $\lambda(g)$ and $\psi_g$ are smooth in $g$ near $g_0$. Let $U$ be the space of metrics on $M$, $V = \{\psi \in C^\infty(M) | \int_M \psi dV_{g_0} = 1\}$ and $W = \{\psi \in C^\infty(M) | \int_M \psi dV_{g_0} = 0\}$. We define $F : U \times V \to W$ as follows

$$F(g, \psi) = \left[ \Delta_g \psi^g + c_n S_g \psi^g - c_n \left( \int S_g \psi^g dV_g \right) \psi^g \right] \frac{dV_g}{dV_{g_0}}, \tag{2.3}$$

where $\psi^g = \psi \frac{dV_{g_0}}{dV_g}$. Note $\int \psi^g dV_g = 1$. Obviously $F(g, \psi) = 0$ iff $\psi^g$ is an eigenfunction of $\Delta_g + c_n S_g$ and is the first eigenfunction iff $\psi > 0$. We have $F(g_0, \psi_0) = 0$ and the linearization in the second variable at $(g_0, \psi_0)$ is easily seen to be $\Delta_{g_0} : W \to W$. Since this is an isomorphism, we conclude by the implicit function theorem that $\lambda(g)$ and $\psi_g$ are smoothly dependent on $g$ in a neighborhood of $g_0$.

Let $g(t)$ for $t \in (-\epsilon, \epsilon)$ be a smooth family of metrics with $g(0) = g_0$. Before we analyze the variation of $\lambda$ near $g_0$, we collect a few formulas:

$$\dot{\mathrm{Ric}} = \frac{1}{2}(\nabla^*\nabla h - 2\overset{\circ}{R}h) - \delta^*\delta h - \frac{1}{2} D^2 \mathrm{tr}\, h + \mathrm{Ric} \circ h, \tag{2.4}$$

$$\dot{S} = -\langle h, \mathrm{Ric}\,\rangle + \delta^2 h + \Delta \mathrm{tr}\, h, \tag{2.5}$$

$$\dot{\Delta} f = \langle h, D^2 f\rangle - \langle \delta h + \frac{1}{2} d\mathrm{tr}\, h, df\rangle. \tag{2.6}$$

where $(\overset{\circ}{R}h)_{ij} = R_{ikjl}h_{kl}$ denotes the action of the curvature on symmetric 2-tensors, $D^2$ denotes the Hessian, and $k \circ h$ denotes the symmetric 2-tensor associated to the composition of $k$ and $h$ viewed as (1, 1)-tensors via the metric, i.e., as linear maps from $TM$ to itself. An "upperdot" denotes the derivative with respect to $t$ and $h = \dot{g}$. Differentiating (2.1) in $t$ gives

$$\dot{\lambda}\psi = \Delta\dot{\psi} + \dot{\Delta}\psi + c_n(\dot{S}\psi + S\dot{\psi}) - \lambda\dot{\psi}. \tag{2.7}$$

We integrate over $M$ and compute using the above formulas and integration by parts

$$\begin{aligned}
\dot{\lambda} &= \int_M \dot{\Delta}\psi + c_n(\dot{S}\psi + S\dot{\psi}) - \lambda \int_M \dot{\psi} \\
&= \int_M \langle h, D^2\psi\rangle - \langle \delta h + \frac{1}{2}d\mathrm{tr}\, h, d\psi\rangle \\
&\quad + c_n[-\langle h, \mathrm{Ric}\,\rangle\psi + \delta^2 h\psi + \Delta\mathrm{tr}\, h\psi + S\dot{\psi}] + \frac{\lambda}{2}\int_M \psi\mathrm{tr}\, h \\
&= c_n \int_M \langle h, -\psi\mathrm{Ric}\, + D^2\psi\rangle - \frac{n}{n-2}\mathrm{tr}\, h\Delta\psi + S\dot{\psi} + \frac{\lambda}{2}\int_M \psi\mathrm{tr}\, h.
\end{aligned}$$

Therefore the first variation formula is

$$\dot{\lambda} = c_n \int_M \langle h, -\psi\mathrm{Ric}\, + D^2\psi - \frac{n}{n-2}\Delta\psi g\rangle + S\dot{\psi} + \frac{\lambda}{2}\int_M \psi\mathrm{tr}\, h. \tag{2.8}$$

As $g_0$ is scalar flat, we have $\lambda(0) = 0$, $\psi_0 = 1$ and hence the elegant

$$\dot{\lambda}(0) = -c_n \int_M \langle \mathrm{Ric}\,(g_0), h\rangle dV_{g_0}. \tag{2.9}$$

This shows that $g_0$ is a critical point of $\lambda$ iff it is Ricci flat.

*Remark.* One can analyze the variation of the first eigenvalue of $\Delta_g + cS_g$ for any constant $c$ in the same fashion. It turns out that it has the most elegant first variation for general metric exactly for $c = \frac{1}{4}$, which corresponds to the Perelman's $\lambda$-functional [P02]. See also [CHI04].

As a corollary we have

**Proposition 2.1 (Bourguignon).** *If $g_0$ has zero scalar curvature but non-zero Ricci curvature, then it can be deformed to a metric of positive scalar curvature.*

*Proof.* Take $h = -\mathrm{Ric}\,(g_0)$ and $g(t) = g_0 + th$. Then $\dot{\lambda}(0) = c_n \int_M |\mathrm{Ric}\,(g_0)|^2 dV_{g_0} > 0$ and hence $\lambda(g(t)) > 0$ for $t > 0$ small. Then $\psi_t^{4/(n-2)} g(t)$ has positive scalar curvature with $\psi_t$ being the positive first eigenfunction of $g(t)$. □

Now, a natural question is then, what happens if $\mathrm{Ric}\,(g_0)$ is identically zero. This is exactly the question discussed by Hertog, Horowitz and Maeda

[HHM03] who, based on supersymmetry, argued that in a neighborhood of a Calabi-Yau metric there is no metric of positive scalar curvature. For this purpose we need to derive the second variation for $\lambda$.

We now assume that $g_0$ is Ricci flat. Differentiating (2.8) at $t = 0$ and using (2.4) and (2.5) we get

$$\begin{aligned}\ddot{\lambda}(0) &= c_n \int_M \langle h, -\dot{\mathrm{Ric}} + D^2\dot{\psi} - \frac{n}{n-2}\Delta\dot{\psi} g_0\rangle + \dot{S}\dot{\psi} \\ &= c_n \int_M \langle h, -\frac{1}{2}(\nabla^*\nabla h - 2\overset{\circ}{R}h) + \delta^*\delta h \\ &\qquad + \frac{1}{2}D^2\mathrm{tr}\, h + 2D^2\dot{\psi} - \frac{2}{n-2}\Delta\dot{\psi} g_0\rangle \\ &\overset{def}{=} c_n \int_M \langle h, \mathcal{F}(h)\rangle.\end{aligned}$$

And $\dot{\psi}$ at $t = 0$ appearing in the formula is determined by the equation

$$\Delta\dot{\psi} = -c_n\dot{S} = -c_n(\delta^2 h + \Delta \mathrm{tr}\, h). \tag{2.10}$$

The symmetric tensor $h$ can be decomposed as $h = \bar{h} + h_1 + h_2$, where $h_1 = L_X g_0$ for some vector field $X$, $h_2 = u g_0$ for some smooth function $u$ and $\bar{h}$ is transverse traceless, that is $\mathrm{tr}\, \bar{h} = 0$ and $\delta\bar{h} = 0$. Let $\phi_t$ be the flow generated by the vector field $X$. Let $g(t) = \phi_t^* g_0$ and note obviously its variation is $h_1 = L_X g_0$. Then $\mathrm{Ric}\,(g_t) = \phi_t^*\mathrm{Ric}\, g_0 = 0$. Differentiating in $t$ and using (2.4) we have

$$-\frac{1}{2}(\nabla^*\nabla h_1 - 2\overset{\circ}{R}h_1) + \delta^*\delta h_1 + \frac{1}{2}D^2\mathrm{tr}\, h_1 = 0. \tag{2.11}$$

Similarly by working with the scalar curvature we have

$$\delta^2 h_1 + \Delta \mathrm{tr}\, h_1 = 0. \tag{2.12}$$

These two equation shows that $h_1$ has no contribution to $\mathcal{F}(h)$. On the other hand it is easy to compute

$$\mathrm{tr}\, h_2 = nu, \quad \delta h_2 = -du, \quad \delta^*\delta h_2 = -D^2 u$$

$$\nabla^*\nabla h_2 - 2\overset{\circ}{R}h_2 = \Delta u g_0.$$

Therefore $\Delta\dot{\psi} = -c_n(\delta^2 h_2 + \Delta \mathrm{tr}\, h_2) = -\frac{n-2}{4}\Delta u$ and we may take $\dot{\psi}$ to be $-\frac{n-2}{4}u$. Putting all these identities together we get

$$\mathcal{F}(h) = -\frac{1}{2}(\nabla^*\nabla\bar{h} - 2\overset{\circ}{R}\bar{h}). \tag{2.13}$$

i.e. there is no contribution from $h_1$ and $h_2$. We summarize our calculations as

**Proposition 2.2.** *Let $(M, g_0)$ be a compact Ricci flat manifold and $g(t)$ a smooth family of metrics with $g(0) = g_0$ and $h = \frac{d}{dt}g(t)|_{t=0}$. The second* *variation of λ at a Ricci flat metric is given by*

$$\frac{d^2}{dt^2}\lambda(g(t))|_{t=0} = -\frac{n-2}{8(n-1)}\int_M \langle \nabla^*\nabla \bar{h} - 2\overset{\circ}{R}\bar{h}, \bar{h}\rangle dV_{g_0}, \tag{2.14}$$

*where $\bar{h}$ the orthogonal projection of $h$ in the space of transverse traceless symmetric 2-tensors.*

*Remark.* The variational analysis for $\lambda(g)$ parallels that for the total scalar curvature functional as discussed in Schoen [Sch89]. The two functionals are essentially equivalent for our purpose but $\lambda(g)$ has the slight advantage that the second variation on the space of conformal deformations is trivial.

The operator acting on symmetric 2-tensors

$$\mathcal{L}_g h = \nabla^*\nabla h - 2\overset{\circ}{R}h, \tag{2.15}$$

appearing in the second variation formula for $\lambda(g)$ is the so called the Lichnerowicz Laplacian (when the metric is Ricci flat). To examine the nature of the critical points of $\lambda(g)$, it is important to determine if, for Ricci flat metric, the Lichnerowicz Laplacian is nonnegative. This was raised as an open question by Kazdan and Warner [KW75]. Infinitesimally, the aforementioned work of Hertog, Horowitz and Maeda indicates that it is expected to be so from physical point of view at least for Calabi-Yau metrics.

The Lichnerowicz Laplacian is also of fundamental importance in many other problems of Riemannian geometry (e.g. [CHI04]). In general, however, it is very difficult to study, for the curvature tensor is very complicated. For metrics that are sufficiently pinched, there are some results. See Besse [Bes87] 12.67. A very useful idea is to view $h$ as $T^*M$-valued 1-form and then we have

$$(\delta^\nabla d^\nabla + d^\nabla \delta^\nabla)h = \nabla^*\nabla h - \overset{\circ}{R}h + h \circ \mathrm{Ric}\,, \tag{2.16}$$

where $d^\nabla$ is the exterior differential operator on $T^*M$-valued differential forms and $\delta^\nabla$ its dual. Therefore the left hand side is apparently positive semi-definite. If $\mathrm{Ric} = 0$, the right hand side is different from $\mathcal{L}_g$ only in the coefficients of the second term. Though this formula does not help in general, it suggests that to prove $\mathcal{L}_g$ to be positive semi-definite, one should view $h$ as the section of some vector bundle with a differential operator $P$ such that $\mathcal{L}_g = P^*P$. We show this is possible for manifolds with parallel spinors.

We now assume $(M, g)$ is a compact spin manifold with the spinor bundle $\mathcal{S} \to M$. An excellent reference on spin geometry is Lawson and Michelsohn [LM89]. Let $E \to M$ be a vector bundle with a connection. The curvature is defined as

$$R_{XY} = -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}. \tag{2.17}$$

If $M$ is a Riemannian manifold, then for the Levi-Civita connection on $TM$, we have $R(X, Y, Z, W) = \langle R_{XY}Z, W\rangle$. We often work with an orthonormal frame $\{e_1, \dots, e_n\}$ and its dual frame $\{e^1, \dots, e^n\}$. Set $R_{ijkl} = R(e_i, e_j, e_k, e_l)$.

The spinor bundle has a natural connection induced by the Levi-Civita connection on $TM$. For a spinor $\sigma$, we have

$$R_{XY}\sigma = \frac{1}{4}R(X, Y, e_i, e_j)e_i e_j \cdot \sigma. \tag{2.18}$$

If $\sigma_0 \neq 0$ is a parallel spinor, then

$$R_{klij} e_i e_j \cdot \sigma_0 = 0. \tag{2.19}$$

It is well known this implies Ric $= 0$ by computation. From now on, we assume $M$ has a parallel spinor $\sigma_0 \neq 0$, which, without loss of generality, is normalized to be of unit length. We define a linear map $\Phi : S^2(M) \to \mathcal{S} \otimes T^*M$ by

$$\Phi(h) = h_{ij} e_i \cdot \sigma_0 \otimes e^j. \tag{2.20}$$

It is easy to check that the definition is independent of the choice of the orthonormal frame $\{e_1, \dots, e_n\}$.

**Lemma 2.3.** *The map $\Phi$ satisfies the following properties:*
1. Re $\langle \Phi(h), \Phi(\tilde{h})\rangle = \langle h, \tilde{h}\rangle$,
2. $\nabla_X \Phi(h) = \Phi(\nabla_X h)$.

*Proof.* We compute

$$\begin{aligned}
\langle \Phi(h), \Phi(\tilde{h})\rangle &= h_{ij}\tilde{h}_{kl}\langle e_i \cdot \sigma_0 \otimes e^j, e_k \cdot \sigma_0 \otimes e^l\rangle \\
&= h_{il}\tilde{h}_{kl}\langle e_i \cdot \sigma_0, e_k \cdot \sigma_0\rangle \\
&= -h_{il}\tilde{h}_{kl}\langle \sigma_0, e_i e_k \cdot \sigma_0\rangle \\
&= h_{kl}\tilde{h}_{kl} - \sum_{i\neq k} h_{il}\tilde{h}_{kl}\langle \sigma_0, e_i e_k \cdot \sigma_0\rangle \\
&= \langle h, \tilde{h}\rangle - \sum_{i\neq k} h_{il}\tilde{h}_{kl}\langle \sigma_0, e_i e_k \cdot \sigma_0\rangle.
\end{aligned}$$

Now, for $i \neq k$,

$$\langle \sigma_0, e_i e_k \cdot \sigma_0\rangle = \langle e_k e_i \cdot \sigma_0, \sigma_0\rangle = -\langle e_i e_k \cdot \sigma_0, \sigma_0\rangle = -\overline{\langle \sigma_0, e_i e_k \cdot \sigma_0\rangle}.$$

That is, $\langle \sigma_0, e_i e_k \cdot \sigma_0\rangle$ is purely imaginary. Taking the real part of the previous equation proves the first assertion.

To prove the second one, we choose our orthonormal frame such that $\nabla e_i = 0$ at $p$ and compute at $p$

$$\begin{aligned}
\nabla_X \Phi(h) &= X h_{ij} e_i \cdot \sigma_0 \otimes e^j \\
&= \nabla_X h(e_i, e_j) e_i \cdot \sigma_0 \otimes e^j \\
&= \Phi(\nabla_X h).
\end{aligned}$$

□

The following interesting Bochner type formula, taken from [Wa91, Proposition 2.4], plays an important role here.

**Proposition 2.4 (M. Wang).** *Let $h$ be a symmetric 2-tensor on $M$. Then*

$$\mathcal{D}^*\mathcal{D}\Phi(h) = \Phi(\nabla^*\nabla h - 2\overset{\circ}{R}h). \tag{2.21}$$

*Moreover, $\mathcal{L}_g h = 0$ iff $\mathcal{D}\Phi(h) = 0$.*

*Proof.* We present a lightly different proof here. Choose an orthonormal frame $\{e_1, \ldots, e_n\}$ near a point $p$ such that $\nabla e_i = 0$ at $p$. We compute at $p$

$$\begin{aligned}\mathcal{D}^*\mathcal{D}\Phi(h) &= \nabla_{e_k}\nabla_{e_l}h(e_i, e_j)e_k e_l e_i \cdot \sigma_0 \otimes e^j \\ &= -\nabla_{e_k}\nabla_{e_k}h(e_i, e_j)e_i \cdot \sigma_0 \otimes e^j - \frac{1}{2}R_{e_k e_l}h(e_i, e_j)e_k e_l e_i \cdot \sigma_0 \otimes e^j \\ &= \Phi(\nabla^*\nabla h) + \frac{1}{2}R_{kljp}h_{ip}e_k e_l e_i \cdot \sigma_0 \otimes e^j + \frac{1}{2}R_{klip}h_{pj}e_k e_l e_i \cdot \sigma_0 \otimes e^j.\end{aligned}$$

By using twice the Clifford relation $e_i e_j + e_j e_i = -2\delta_{ij}$ we have

$$\begin{aligned}\frac{1}{2}R_{kljp}h_{ip}e_k e_l e_i \cdot \sigma_0 &= \frac{1}{2}R_{kljp}h_{ip}e_i e_k e_l \cdot \sigma_0 + R_{kljp}h_{kp}e_l \cdot \sigma_0 - R_{kljp}h_{lp}e_k \cdot \sigma_0 \\ &= -2(\overset{\circ}{R}h)_{kj}e_k \cdot \sigma_0,\end{aligned}$$

where in the last equality we used $R_{kljp}e_k e_l \cdot \sigma_0 = 0$ by (2.19). Similarly (in fact easier) one can show using also the fact $\mathrm{Ric} = 0$

$$\frac{1}{2}R_{klip}h_{pj}e_k e_l e_i \cdot \sigma_0 = 0.$$

Thus we get

$$\mathcal{D}^*\mathcal{D}\Phi(h) = \Phi(\nabla^*\nabla h - 2\overset{\circ}{R}h).$$

By Lemma 2.3, $\Phi$ preserves the metrics. Hence, $\mathcal{L}_g h = 0$ iff $\mathcal{D}\Phi(h) = 0$. □

By using Lemma 2.3, Proposition 2.4, and working on a covering space, we obtain

**Theorem 2.5.** *If a compact Riemannian manifold $(M, g)$ has a cover which is spin and admits nonzero parallel spinors, then the Lichnerowicz Laplacian $\mathcal{L}_g$ is positive semi-definite.*

*Proof.* Let $\pi : (\hat{M}, \hat{g}) \to (M, g)$ be the cover. Clearly the following diagram commutes

$$\begin{array}{rccc}\mathcal{L}_g : & S^2(M) & \to & S^2(M) \\ & \pi^* \downarrow & & \pi^* \downarrow \\ \mathcal{L}_{\hat{g}} : & S^2(\hat{M}) & \to & S^2(\hat{M}).\end{array}$$

Now if we denote by $\langle \cdot\ ,\ \cdot \rangle$ the pointwise inner product on symmetric 2-tensors and $(\cdot\ ,\ \cdot)$ the $L^2$ inner product, i.e., for example,

$$(h, h')_g = \int_M \langle h, h' \rangle_g dvol(g),$$

then we have

$$\langle \mathcal{L}_g h, h \rangle_g = \langle \mathcal{L}_{\hat{g}} \pi^*(h), \pi^*(h) \rangle_{\hat{g}}. \tag{2.22}$$

Thus, for a fundamental domain $F$ of $M$ in $\hat{M}$, one has

$$(\mathcal{L}_g h, h)_g = \int_F \langle \mathcal{L}_{\hat{g}} \pi^*(h), \pi^*(h) \rangle_{\hat{g}} dvol(\hat{g}). \tag{2.23}$$

Since $(\hat{M}, \hat{g})$ has nonzero parallel spinor, we have $\Phi(\mathcal{L}_{\hat{g}} \pi^* h) = \mathcal{D}^* \mathcal{D} \Phi(\pi^* h)$ by (2.21) where the map $\Phi$ is defined as in (2.20).

Without the loss of generality we take $\hat{M}$ to be the universal cover. Since $M$ is Ricci flat, its fundamental group has polynomial growth and therefore is amenable [M68]. Now we choose $F$ as in [Br81]. Namely, we pick a smooth triangulation of $M$, and for each $n$-simplex in this triangulation, we pick one simplex in $\hat{M}$ covering this simplex. We then let $F$ be the union of all these simplices thus chosen in $\hat{M}$. Thus defined, $F$ is a union of finitely many smooth $n$-simplices, but $F$ may not be connected. With this choice of $F$, by Folner's theorem [Br81], for every $\epsilon > 0$, there is a finite subset $E$ of the fundamental group such that the union of translates of $F$ by elements of $E$,

$$H = \bigcup_{g \in E} gF$$

satisfies

$$\frac{\text{area}(\partial H)}{\text{vol}(H)} < \epsilon.$$

Hence by Lemma 2.3 and Proposition 2.4

$$\begin{aligned}
(\mathcal{L}_g h, h)_g &= \frac{1}{\# E} \int_H \text{Re}\, \langle \Phi(\mathcal{L}_{\hat{g}} \pi^*(h)), \Phi(\pi^*(h)) \rangle_{\hat{g}} dvol(\hat{g}) \\
&= \frac{\text{vol}(M)}{\text{vol}(H)} \Bigg[ \int_H \langle \mathcal{D}\Phi(\pi^* h), \mathcal{D}\Phi(\pi^* h) \rangle_{\hat{g}} dvol(\hat{g}) \\
&\qquad + \text{Re} \int_{\partial H} \langle \nu \cdot \mathcal{D}\Phi(\pi^* h), \Phi(\pi^* h) \rangle_{\hat{g}}\, int(\nu) dvol(\hat{g}) \Bigg] \\
&\geq -\frac{C \text{vol}(M) \text{area}(\partial H)}{\text{vol}(H)}.
\end{aligned}$$

Here we denote $\nu$ the outer unit normal of $\partial H$, and $C$ some constant depending on the $C^1$ norm of $h$ on $M$. Since the right hand side of the last inequality above can be taken to be arbitrarily small by appropriate choice of $E$, we obtain

$$(\mathcal{L}_g h, h)_g \geq 0.$$

□

## 3. The local stability theorem

In this section we prove the following local stability theorem.

**Theorem 3.1.** *Let $(M, g_0)$ be a compact, simply connected Riemannian spin manifold of dimension $n$ with a parallel spinor. Then there exists a neighborhood $\mathcal{U}$ of $g_0$ in the space of smooth Riemannian metrics on $M$ such that there exists no metric of positive scalar curvature in $\mathcal{U}$.*

The key here is the identification of the kernel space

$$W_g = \{h | \mathrm{tr}_g h = 0, \delta h = 0, \mathcal{D}\Phi(h) = 0\} \tag{3.1}$$

of $\mathcal{L}_g$ on the space of transverse traceless symmetric 2-tensors, according to the infinitesimal stability theorem. It should be pointed out that this kernel space is also studied in the proof of Theorem 3.1 in [Wa91]. Our proof turns out to be similar to his.

For this purpose, we need to understand the geometry of $(M, g_0)$ better. According to [Wa89] (cf. [J00] 3.6), if $(M, g_0)$ is a compact, simply connected, irreducible Riemannian spin manifold of dimension $n$ with a parallel spinor, then one of the following holds

1. $n = 2m, m \geq 2$, the holonomy group is $SU(m)$,
2. $n = 4m, m \geq 2$, the holonomy group is $Sp(m)$,
3. $n = 8$, the holonomy group is $Spin(7)$,
4. $n = 7$, the holonomy group is $G_2$.

In cases 2 and 3, it is further shown in [Wa89] that the index of the Dirac operator is nonzero, hence by Lichnerowicz's theorem there is no metric of positive scalar curvature. Therefore the theorem is "trivial" except in cases 1 and 4.

Suppose $(M, g_0)$ is a compact Riemannian manifold of dimension $n = 2m$ with holonomy $SU(m)$. This is a Calabi-Yau manifold. By Yau's solution of Calabi conjecture [Y77,Y78] and the theorem of Bogomolov-Tian-Todorov [Bo78], [T86], [To89], the universal deformation space $\Sigma$ of Calabi-Yau metrics is smooth of dimension $h^{1,1} + 2h^{m-1,1} - 1$ (it is one less than the usual number because we normalize the volume and hence discount the trivial deformation of scaling). Its tangent space at $g_0$ must be a subspace of $W_{g_0}$ for $W_{g_0}$ is the Zariski tangent space of the moduli space of Ricci flat metrics. In fact we have

**Lemma 3.2.** $T_{g_0}\Sigma = W_{g_0}$.

*Proof.* This follows from a theorem of Koiso [Ko80] which says Einstein deformations of a Kähler-Einstein metric are also Kähler, provided that first Chern class is nonpositive and the complex deformation are unobstructed, which is guaranteed by Bogomolov-Tian-Todorov theorem [T86], [To89]. It can also be easily seen from our approach. For a Calabi-Yau manifold its spinor bundle is

$$\mathcal{S}^+(M) = \bigoplus_{k\ even} \wedge^{0,k}(M), \quad \mathcal{S}^-(M) = \bigoplus_{k\ odd} \wedge^{0,k}(M). \tag{3.2}$$

The Clifford action at a point $p \in M$ is defined by

$$X \cdot \alpha = \sqrt{2}(\pi^{0,1}(X^*) \wedge \alpha - \pi^{0,1}(X) \lrcorner \alpha) \tag{3.3}$$

for any $X \in T_pM$ and $\alpha \in \mathcal{S}_p(M)$ and the parallel spinor $\sigma_0 \in C^\infty(\mathcal{S}^+(M))$ can be taken as the function which is identically 1.

Let $J$ be the complex structure. Then we have $W_{g_0} = W^+ \oplus W^-$, where

$$W^+ = \{h \in W_{g_0} | h(J, J) = h\}, \quad W^- = \{h \in W_{g_0} | h(J, J) = -h\}. \tag{3.4}$$

We choose a local orthonormal $(1, 0)$ frame $\{X_1, \ldots, X_m\}$ for $T^{1,0}M$ and its dual frame $\{\theta^1, \ldots, \theta^m\}$. By straightforward computation we have for $h \in W^+$

$$\Phi(h) = h(\bar{X}_i, X_j)\bar{\theta}^i \otimes \theta^j \tag{3.5}$$

which can be identified with the real $(1, 1)$ form $\sqrt{-1}h(\bar{X}_i, X_j)\bar{\theta}^i \wedge \theta^j$. The Dirac operator is then identified as $\sqrt{2}(\bar{\partial} - \bar{\partial}^*)$ (cf. Morgan [M96]). Therefore $W^+$ is identified with the space of harmonic $(1, 1)$-forms orthogonal to the Kähler form $\omega$. Similarly $W^-$ can be identified as $H^1(M, \Theta) - H^{0,2}(M)$, where $\Theta$ is the holomorphic tangent bundle. As $H^{0,2}(M) = 0$ and $H^1(M, \Theta) \cong H^{m-1,1}(M)$ by the Hodge theory, we have $\dim W_{g_0} = h^{1,1} + 2h^{m-1,1} - 1$. This is exactly the dimension of the moduli space of Calabi-Yau metrics. □

We now turn to the proof of our local stability theorem in the case of Calabi-Yau manifold. Let $\mathcal{M}$ be the space of Riemannian metric of volume 1. By Ebin's slice theorem, there is a real submanifold $\mathcal{S}$ containing $g_0$, which is a slice for the action of the diffeomorphism group on $\mathcal{M}$. The tangent space

$$T_{g_0}\mathcal{S} = \{h | \delta_{g_0} h = 0, \int_M \operatorname{tr}_{g_0} h dV_{g_0} = 0.\} \tag{3.6}$$

Let $\mathcal{C} \subset \mathcal{S}$ be the submanifold of constant scalar curvatures metrics.If $g \in \mathcal{M}$ is a metric of positive scalar curvature very close to $g_0$, then by the solution of the Yamabe problem there is a metric $\tilde{g} \in \mathcal{C}$ conformal to $g$ and with constant positive scalar curvature. Moreover as $g$ is close to $g_0$ which is the unique Yamabe solution in its conformal class, $\tilde{g}$ is also close to $g_0$. Therefore to prove the theorem, it suffices to work on $\mathcal{C}$. It is easy to see

$$T_{g_0}\mathcal{C} = \{h | \delta_{g_0} h = 0, \operatorname{tr}_{g_0} h = 0.\} \tag{3.7}$$

It contains the finite dimensional submanifold of Calabi-Yau metrics $\mathcal{E}$. We now restrict our function $\lambda$ to $\mathcal{C}$. It is identically zero on $\mathcal{E}$. Moreover, by Lemma 3.2 and Proposition 2.2, $D^2\lambda$ is negative definite on the normal bundle. Therefore there is a possibly smaller neighborhood of $\mathcal{E} \subset \mathcal{C}$, still denoted by $\mathcal{U}$, such that $\lambda$ is negative on $\mathcal{U} - \mathcal{E}$.

Next we consider the case 4, that is, the case of $G_2$ manifold. Our basic references are Bryant [B89,B03] and Joyce [J00]. Let $(M, g_0)$ be a compact Riemannian manifold with holonomy group $G_2$. We denote the fundamental 3-form by $\phi$. With a local $G_2$-frame $\{e_1, e_2, \ldots, e_7\}$ and the dual frame $\{e^1, e^2, \ldots, e^7\}$ we have

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \tag{3.8}$$

$$*\phi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{3.9}$$

We also define the cross product $P : TM \times TM \to TM$ by

$$\langle P(X, Y), Z\rangle = \phi(X, Y, Z). \tag{3.10}$$

The cross product has many wonderful properties. We list what we need in the following lemma.

**Lemma 3.3.** *For any tangent vectors $X, Y, Z$*

1. $P(X, Y) = -P(Y, X)$.
2. $\langle P(X, Y), P(X, Z)\rangle = |X|^2\langle Y, Z\rangle - \langle X, Y\rangle\langle X, Z\rangle$,
3. $P(X, P(X, Y)) = -|X|^2 Y + \langle X, Y\rangle X$,
4. $X \lrcorner (Y \lrcorner * \phi) = -P(X, Y) \lrcorner \phi + X^* \wedge Y^*$.

*Proof.* The first three identities are proved in Bryant [B89]. The fourth can be proven by the same idea: it is obviously true for $X = e_1$, $Y = e_2$ and the general case follows by the transitivity of $G_2$ on orthonormal pairs. □

The spinor bundle is $\mathcal{S}(M) = \mathbb{R} \oplus TM$ with the first factor being the trivial line bundle. The Clifford action at $p \in M$ is defined by

$$X \cdot (a, Y) = (-\langle X, Y\rangle, aX + P(X, Y)) \tag{3.11}$$

for any $X, Y \in T_pM$. The parallel spinor $\sigma_0 = (1, 0)$. One easily check $\phi(X, Y, Z) = -\langle X \cdot Y \cdot Z \cdot \sigma_0, \sigma_0\rangle$. It is also obvious that $\mathcal{S}(M) \otimes T^*M = T^*M \oplus (TM \otimes T^*M)$ and for any symmetric 2-tensor

$$\Phi(h) = (0, h_{ij} e_i \otimes e^j). \tag{3.12}$$

We compute

$$\begin{aligned}\mathcal{D}\Phi(h) &= h_{ij,k}e_k \cdot (0, e_i) \otimes e^j \\ &= h_{ij,k}(-\delta_{ik}, P(e_k, e_i)) \otimes e^j \\ &= (\delta h, -h_{ij,k}P(e_i, e_k) \otimes e^j).\end{aligned} \tag{3.13}$$

The $G_2$ structure gives rise to orthogonal decomposition of the vector bundle of exterior differential forms. We are only concerned with

$$\wedge^3(M) = \wedge^3_1(M) \oplus \wedge^3_7(M) \oplus \wedge^3_{27}(M) \tag{3.14}$$

where

$$\wedge^3_1 = \{a\phi \mid a \in \mathbb{R}\} \tag{3.15}$$

$$\wedge^3_7 = \{*(\phi \wedge \alpha) \mid \alpha \in T^*M\} \tag{3.16}$$

$$\wedge^3_{27} = \{\alpha \in \wedge^3(M) \mid \alpha \wedge \phi = 0, \alpha \wedge *\phi = 0\}. \tag{3.17}$$

This also leads to the decomposition of the cohomology group

$$H^3(M, \mathbb{R}) = \mathbb{R} \oplus H^3_7(M, \mathbb{R}) \oplus H^3_{27}(M, \mathbb{R}). \tag{3.18}$$

In fact one can show $H^3_7(M, \mathbb{R}) = 0$. Therefore the Betti number $b^3 = 1 + b^3_{27}$, where $b^3_{27} = \dim H^3_{27}(M, \mathbb{R})$.

There is another natural isomorphism from the bundle of traceless symmetric 2-tensors to $\wedge^3_{27}(M)$

$$\Psi : S^2_0(M) \to \wedge^3_{27}(M) \tag{3.19}$$

defined by

$$\Psi(h) = h_{ij}e^i \wedge (e_j \lrcorner \phi). \tag{3.20}$$

It is proved by Joyce (Theorem 10.4.4 in [J00]) that the moduli space of $G_2$ metrics is smooth and its tangent space at $g_0$ can be identified with

$$V_{g_0} = \{h \mid \operatorname{tr}_{g_0} h = 0, \delta h = 0, \Psi(h) \text{ is harmonic}\}. \tag{3.21}$$

Moreover $\dim V_{g_0} = b_3 - 1$. We have $V_{g_0} \subset W_{g_0}$. To show they are equal, let $h \in W_{g_0}$. By (3.13) we have

$$h_{ij,k}P(e_i, e_k) = 0. \tag{3.22}$$

It suffices to show $\Psi(h)$ is harmonic. We compute

$$\begin{aligned}d^*\Psi(h) &= -e_k \lrcorner \nabla_{e_k}\Psi(h) \\ &= -h_{ij,k}e_k \lrcorner \left(e^i \wedge (e_j \lrcorner \phi)\right) \\ &= -h_{kj,k}e_j \lrcorner \phi + h_{ij,k}e^i \wedge \left(e_k \lrcorner (e_j \lrcorner \phi)\right) \\ &= (\delta h)^\sharp \lrcorner \phi + h_{ij,k}e^i \wedge P(e_j, e_k)^* \\ &= 0,\end{aligned}$$

where in the last step we used $\delta h = 0$ and (3.22). Similarly

$$\begin{aligned} *d\Psi(h) &= *\big(e^k \wedge \nabla_{e_k}\Psi(h)\big) \\ &= h_{ij,k} * \big(e^k \wedge e^i \wedge (e_j \lrcorner \phi)\big) \\ &= -h_{ij,k} e_k \lrcorner e_i \lrcorner (e^j \wedge *\phi) \\ &= -h_{ii,k} e_k \lrcorner * \phi + h_{ik,k} e_i \lrcorner * \phi - h_{ij,k} e^j \wedge (e_k \lrcorner (e_i \lrcorner * \phi)) \\ &= -h_{ij,k} e^j \wedge (e_k \lrcorner (e_i \lrcorner * \phi)) \,, \end{aligned}$$

where in the last step we used $\operatorname{tr} h = 0$ and $\delta h = 0$. Then by Lemma 3.3 we continue

$$\begin{aligned} *d\Psi(h) &= -h_{ij,k} e^j \wedge (e_k \lrcorner (e_i \lrcorner * \phi)) \\ &= -h_{ij,k} e^j \wedge (P(e_i, e_k) \lrcorner \phi) + h_{ij,k} e^j \wedge e^i \wedge e^k \\ &= 0, \end{aligned}$$

where in the last step we used (3.22) and the fact that $h$ is symmetric. Therefore for any $h \in W_{g_0}$, the 3-form $\Psi(h)$ is harmonic. This proves $W_{g_0} = V_{g_0}$ is the tangent space to the moduli space of $G_2$ metrics. The rest of argument is the same as in the Calabi-Yau case.

By the above argument we also obtain the following rigidity theorem which generalize the Einstein deformation result, Theorem 3.1, of [Wa91].

**Theorem 3.4.** *Let $(M, g_0)$ be a compact, simply connected Riemannian spin manifold of dimension $n$ with a parallel spinor. Then there exists a neighborhood $\mathcal{U}$ of $g_0$ in the space of smooth Riemannian metrics on $M$ such that any metric with nonnegative scalar curvature in $\mathcal{U}$ must in fact admit a parallel spinor (and hence Ricci flat in particular).*

*Proof.* We give the proof in the Calabi-Yau case. As shown in the proof of Theorem 3.1 $\lambda \leq 0$ on $\mathcal{U}$ and is negative on $\mathcal{U} - \mathcal{E}$, where $\mathcal{E}$ is the moduli space of Calabi-Yau metrics. If $g \in \mathcal{U}$ has nonnegative scalar curvature, then $\lambda(g) \geq 0$. Therefore $\lambda(g) = 0$ and $g \in \mathcal{E}$. □

Note that our proof actually gives a very nice picture of what happens to scalar curvature near a metric with a parallel spinor (let's call it special holonomy metric): one has the finite dimensional smooth moduli of special holonomy metrics; along the normal directions, the scalar curvature of the Yamabe metric in the conformal class will go negative. That is, we have

**Theorem 3.5.** *Let $(M, g_0)$ be a compact, simply connected Riemannian spin manifold of dimension $n$ with a parallel spinor. Then $g_0$ is a local maximum of the Yamabe invariant.*

*Proof.* Recall that the Yamabe invariant of a metric $g$ is

$$\mu(g) = \inf_{f \in C^\infty(M),\ f>0} \frac{\int_M \left(4\frac{n-1}{n-2}|df|_g^2 + S(g) f^2\right) dvol(g)}{\left(\int_M f^{2n/(n-2)} dvol(g)\right)^{(n-2)/n}} \tag{3.23}$$

and it is a conformal invariant. The corresponding Euler-Lagrange equation is

$$4\frac{n-1}{n-2}\Delta_g f + S(g)f = \mu(g) f^{(n+2)/(n-2)}. \tag{3.24}$$

Its nontrivial solution, whose existence guaranteed by the solution of Yamabe problem, gives rise to the so-called Yamabe metric $f^{4/(n-2)}g$ which has constant scalar curvature $\mu(g)$. Note that the left hand side of (3.24) is (up to the positive multiple of $4\frac{n-1}{n-2}$) the conformal Laplacian and the numerator of the quotient in (3.23) is the quadratic form defined by the conformal Laplacian (again up to the positive multiple of $4\frac{n-1}{n-2}$). Thus, $\mu(g) < 0$ if the first eigenvalue of the conformal Laplacian $\lambda(g) < 0$. Now $\mu(g_0) = \lambda(g) = 0$ since $g_0$ is scalar flat. Since we have shown that $g_0$ is a local maximum of $\lambda(g)$, our result follows. □

## 4. A uniform approach: connection with the positive mass theorem

In this section we explore a remarkable connection between the stability of the Riemannian manifold $M$ and the positive mass theorem on $\mathbb{R}^3 \times M$. It provides us with a uniform approach to the stability problem, rather than the case by case treatment of the previous section. The original positive mass theorem [SY79-1], [Wi81] is related to the stability of the Minkowski space as the vacuum (or minimal energy state). In superstring theory this vacuum is replaced by the product of the Minkowski space with a Calabi-Yau manifold [CHSW85]. The positive mass theorem for spaces which are asymptotic to $\mathbb{R}^k \times M$ at infinity is proved in [D03]. The following result is a special case of what is considered in [D03].

**Theorem 4.1.** *Let $M$ be a compact Riemannian spin manifold with nonzero parallel spinors. If $\tilde{g}$ is a complete Riemannian metric on $\mathbb{R}^3 \times M$ which is asymptotic of order $> 1/2$ to the product metric at infinity and with nonnegative scalar curvature, then its mass $m(\tilde{g}) \geq 0$. (Moreover, $m(\tilde{g}) = 0$ iff $\tilde{g}$ is isometric to the product metric.)*

*Remark.* The result of [D03] is stated for simply connected compact Riemannian spin manifolds $M$ with special holonomy with the exception of quaternionic Kähler but the proof only uses the existence of a nonzero parallel spinor, which is equivalent to the holonomy condition by a result of [Wa89]. Also, the simply connectedness of $M$ is not needed here since the total space is the product $\mathbb{R}^3 \times M$.

Using the positive mass theorem, we prove the following deformation stability for compact Riemannian spin manifold with nonzero parallel spinors. This is a special case of our previous local stability theorem. However, as we point out earlier, this approach treats all cases of special holonomy manifolds at the same time.

**Theorem 4.2.** *Let $M$ be a compact spin manifold and $g$ a Riemannian metric which admits nonzero parallel spinors. Then $g$ cannot be deformed to metrics with positive scalar curvature. Namely, there is no path of metrics $g_s$ such that $g_0 = g$ and $S(g_s) > 0$ for $s > 0$.*

*Remark.* The condition on the deformation can be relaxed to $S(g_s) \geq 0$ and $S(g_{s_i}) > 0$ for a sequence $s_i \to 0$.

This has the following interesting consequence, which, once again, is a special case of the rigidity result of the previous section, Theorem 3.4.

**Theorem 4.3.** *Any scalar flat deformation of Calabi-Yau metric on a compact manifold must also be Calabi-Yau. Indeed, any deformation of Calabi-Yau metric on a compact manifold with nonnegative scalar curvature must be a Calabi-Yau deformation.*

*Proof.* Let $(M, g)$ be a compact Calabi-Yau manifold and $g_s$ be a scalar flat deformation of $g$, i.e., $S(g_s) = 0$ for all $s$. We will show by Theorem 4.2 that $g_s$ must also be Ricci flat. It follows then by a theorem of Koiso and the Bogomolov-Tian-Todorov theorem, $g_s$ must be in fact Calabi-Yau.

We first show that for $s$ sufficiently small, $g_s$ must be Ricci flat. If not, there is a sequence $s_i \to 0$ such that $\mathrm{Ric}(g_{s_i}) \neq 0$. Hence this will also be true in a small neighborhood of $s_i$. By Bourguignon's theorem, we can deform $g_s$ slightly in such neighborhood so that it will now have positive scalar curvature. Moreover we can do this while keeping the metrics $g_s$ unchanged outside this neighborhood. This shows that we have a path of metrics satisfying the conditions in the Remark above. Hence contradictory to Theorem 4.2.

Once we know that $g_s$ is Ricci flat for $s$ sufficiently small, we then deduce that $g_s$ is Calabi-Yau for $s$ sufficiently small, by using Koiso's Theorem [Bes87], [Ko80] and Bogomolov-Tian-Todorov Theorem [T86], [Bo78], [To89]. Then the above argument applies again to extend to the whole deformation. The case of deformation with nonnegative scalar curvature is dealt with similarly. □

To prove Theorem 4.2, we show that if there is such a deformation of $g$, then one can construct a complete Riemannian metric $\tilde{g}$ on $\mathbb{R}^3 \times M$ which is asymptotic of order 1 to the product metric at infinity and with nonnegative scalar curvature, but with negative mass, hence contradicting Theorem 4.1.

Thus, let $g(r)$, $r \geq 0$ be a one-parameter family of metrics on $M$. Consider $\tilde{g}$ a warped product metric on $\mathbb{R}^3 \times M^k$ defined by

$$\tilde{g} = \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 ds^2 + g(r), \tag{4.1}$$

where $m(r) < r/2, m(0) = m'(0) = m''(0) = 0$, $ds^2$ is the unit sphere metric on $S^2$. The following lemma seems a folklore.

**Lemma 4.4.** *If $g(r)$ is independent of $r$ for $r$ sufficiently large and $\lim_{r\to\infty} m(r) = m_\infty$ exists and finite, then $\tilde{g}$ on $\mathbb{R}^3 \times M$ is asymptotically flat of order 1 with its mass $m(\tilde{g}) = m_\infty$.*

We now turn to the scalar curvature of $\tilde{g}$. Given a point $(r, p, q) \in \mathbb{R}^3 \times M^k$, choose a basis as follows: $U_0 = \left(1 - \frac{2m(r)}{r}\right)^{\frac{1}{2}} \frac{\partial}{\partial r}$, $\{U_i\}$ an orthonormal basis of $ds^2$ at $p$, and $\{Y_\alpha\}$ a coordinate basis of $M$ at $q$.

Note that all brackets vanish except $[U_i, U_j]$ which belongs to $TS^2$, and $\langle U_0, U_0\rangle_{\tilde{g}} = 1$, $\langle U_i, U_j\rangle_{\tilde{g}} = r^2\delta_{ij}$. Denote $\langle Y_\alpha, Y_\beta\rangle_{\tilde{g}} = g_{\alpha\beta}(r)$.

By [HHM03, Formula (3.2)]

**Lemma 4.5.** *The scalar curvature of $\tilde{g}$ at $(r, p, q)$ is given by*

$$\tilde{S} = S_M(g(r)) + m'\left(\frac{4}{r^2} + \frac{g'}{r}\right) - \frac{m}{r^2}g' - \left(1 - \frac{2m(r)}{r}\right)\left[g'' + \frac{2}{r}g' + \frac{1}{4}(g')^2 - \frac{1}{4}\partial_r g_{\alpha\beta}\partial_r g^{\alpha\beta}\right], \tag{4.2}$$

*where $g' = g^{\alpha\beta}\partial_r g_{\alpha\beta}$, $g'' = \partial_r g'$, $S_M(g(r))$ is the scalar curvature of $(M, g(r))$ at $q$.*

This can be verified in local coordinates using Christoffel symbols. A simpler way is to view $S^2 \times M^k$ with the product metric $r^2 ds^2 + g(r)$ as a Riemannian submanifold of $(\mathbb{R}^3 \times M^k, \tilde{g})$ and use Gauss equation to compute some of the curvature tensors of $\tilde{g}$. It seems this view point is useful whenever one has one parameter family of metrics warping with $\mathbb{R}$. This type of warped metric appears quite often, Cf. (for example) [W88,BW01], [P02]. For this reason we present a proof here.

*Proof.* Given tangent vectors $X, Y, Z, T$ at $(p, q) \in S^2 \times M^k$, recall the Gauss equation [DoC92]

$$\langle \tilde{R}(X, Y)Z, T\rangle = \langle R(X, Y)Z, T\rangle - \langle B(Y, T), B(X, Z)\rangle + \langle B(X, T), B(Y, Z)\rangle, \tag{4.3}$$

where $B$ is the second fundamental form. Now $\langle B(U_i, Y_\beta), U_0\rangle = 0$ and

$$\langle B(U_i, U_j), U_0\rangle = -r\left(1 - \frac{2m(r)}{r}\right)^{\frac{1}{2}} \delta_{ij},$$

$$\langle B(Y_\alpha, Y_\beta), U_0\rangle = -\frac{1}{2}\left(1 - \frac{2m(r)}{r}\right)^{\frac{1}{2}} g_{\alpha\beta,r}, \tag{4.4}$$

where $g_{\alpha\beta,r} = \frac{\partial g_{\alpha\beta}(r)}{\partial r}$. Hence, applying (4.3), we get

$$\langle \tilde{R}(U_1, U_2)U_1, U_2\rangle = r^2 - r^2\left(1 - \frac{2m(r)}{r}\right) = 2mr \tag{4.5}$$

$$\langle \tilde{R}(U_i, Y_\alpha)U_i, Y_\beta\rangle = 0 - \frac{1}{2}r\left(1 - \frac{2m(r)}{r}\right) g_{\alpha\beta,r}, \tag{4.6}$$

$$\begin{aligned}\langle \tilde{R}(Y_\alpha, Y_\beta)Y_s, Y_l\rangle &= \langle R(Y_\alpha, Y_\beta)Y_s, Y_l\rangle_{(M,g(r))} \\ &\quad - \frac{1}{4}\left(1 - \frac{2m(r)}{r}\right)\left[g_{\beta l,r}g_{\alpha s,r} - g_{\alpha l,r}g_{\beta s,r}\right].\end{aligned} \tag{4.7}$$

The curvature tensors involving the $U_0$ direction can be computed from the metric:

$$\langle \tilde{R}(U_0, U_i)U_0, U_i\rangle = -\frac{m}{r} + m'. \tag{4.8}$$

$$\begin{aligned}\langle \tilde{R}(U_0, Y_\alpha)U_0, Y_\beta\rangle &= \frac{1}{2}\frac{m'r - m}{r^2}g_{\alpha\beta,r} \\ &\quad -\frac{1}{2}\left(1 - \frac{2m(r)}{r}\right)\sum_{s,l}\frac{d}{dr}\left[g_{\alpha s,r}g^{ls}\right]g_{l\beta} \\ &\quad -\frac{1}{4}\left(1 - \frac{2m(r)}{r}\right)\sum_{s,l}g_{\alpha s,r}g^{ls}g_{l\beta,r}.\end{aligned} \tag{4.9}$$

Therefore, using (4.8) (4.9),

$$\begin{aligned}\tilde{R}_{00} &= 2\left(-\frac{m}{r^3} + \frac{m'}{r^2}\right) + \frac{1}{2}\frac{m'r - m}{r^2}g' - \frac{1}{2}\left(1 - \frac{2m(r)}{r}\right)g'' \\ &\quad -\frac{1}{4}\left(1 - \frac{2m(r)}{r}\right)\sum_{\alpha,\beta,s,l}g_{\alpha s,r}g_{l\beta,r}g^{ls}g^{\alpha\beta}.\end{aligned} \tag{4.10}$$

Similarly by (4.8) (4.5) (4.6) and (4.9) (4.6) (4.7)

$$\tilde{R}_{ii} = -\frac{m}{r} + m' + \frac{2m}{r} - \frac{r}{2}\left(1 - \frac{2m(r)}{r}\right)g', \tag{4.11}$$

$$\begin{aligned}\tilde{R}_{\alpha\beta} &= \frac{1}{2}\frac{m'r - m}{r^2}g_{\alpha\beta,r} - \frac{1}{2}\left(1 - \frac{2m(r)}{r}\right)\sum_{s,l}\frac{d}{dr}\left[g_{\alpha s,r}g^{ls}\right]g_{l\beta} \\ &\quad -\frac{1}{4}\left(1 - \frac{2m(r)}{r}\right)\sum_{s,l}g_{\alpha\beta,r}g_{sl,r}g^{ls} \\ &\quad -\frac{1}{r}\left(1 - \frac{2m(r)}{r}\right)g_{\alpha\beta,r} + R_{\alpha\beta}(M, g(r)).\end{aligned} \tag{4.12}$$

Finally, using (4.10) (4.11) (4.12), we have

$$\tilde{S} = S_M(g(r)) + m'\left(\frac{4}{r^2} + \frac{g'}{r}\right) - \frac{m}{r^2}g' \\ - \left(1 - \frac{2m(r)}{r}\right)\left[g'' + \frac{2}{r}g' + \frac{1}{4}(g')^2 - \frac{1}{4}\partial_r g_{\alpha\beta}\partial_r g^{\alpha\beta}\right].$$

□

We now turn to the construction of asymptotically flat metrics of the type (4.1) with nonnegative scalar curvature and negative mass.

**Proposition 4.6.** *Let $g_s$, $s \in [0, 1]$, be a one-parameter family of metrics on M such that*

$$S_M(g_1) \geq a_0 > 0, \quad S_M(g_0) \geq 0.$$

*If in addition*

$$\max\left|\partial_s(g_s)_{\alpha\beta}\partial_s g_s^{\alpha\beta}\right| \leq \frac{1}{200}, \quad \max|g_s'| \leq \frac{1}{200}, \quad \max|g_s''| \leq \frac{1}{200} \quad (4.13)$$

*(here we did not try to get the optimal constants), and*

$$S_M^-(g_s) \stackrel{def}{=} \max(0, -S_M(g_s)) \leq \frac{a_0}{10}$$

*for all $s \in [0, 1]$, then there exists $m(r)$ with $m(0) = m'(0) = m''(0) = 0$, $\lim_{r\to\infty} m(r) = m_\infty < 0$ and $g(r)$ with $g(r) = g_0$ for r sufficiently large, such that the metric $\tilde{g}$ in (4.1) has scalar curvature $\tilde{S} \geq 0$.*

*Proof.* We note first that by reparametrizing the interval [0, 1] we can arrange the family $g_s$ so that $g_s = g_0$ for $s$ sufficiently small and $g_s = g_1$ for $s$ sufficiently close to 1 and the assumption on $g_s$ still holds (with different constants). This will ensure that the following gluing construction will produce a smooth metric. Now, the function $m(r)$ and the metrics $g(r)$ will be constructed separately on the intervals $[0, r_1]$, $[r_1, r_2]$, $[r_2, r_3]$ and $[r_3, \infty)$, with $r_1, r_2, r_3$ to be chosen appropriately.

First of all, define

$$g(r) = g_1 \quad \text{for } 0 \leq r \leq r_2. \quad (4.14)$$

For the function $m(r)$, we let $m(r) = -\frac{a_0}{12}r^3$ when $0 \leq r \leq r_1$ and just require that $m'(r) \geq -\frac{a_0}{4}r^2$ for $r_1 \leq r \leq r_2$, $m(r_2) = m(r_1)$ and $m'(r_2) = -m'(r_1) = \frac{a_0}{4}r_1^2$. We note that there are many such choices for $m(r)$ on the interval $[r_1, r_2]$ and further $r_2$ can be taken arbitrarily close to $r_1$. For definiteness we put $r_2 = r_1 + 1$.

In the region $0 \leq r \leq r_2$, since the metrics $g(r) = g_1$ does not change with $r$, one easily sees from the scalar curvature formula, Lemma 4.5, that

$$\tilde{S} = S_M(g_1) + m'(r)\frac{4}{r^2} \geq 0. \quad (4.15)$$

For the interval $[r_2, r_3]$, we define $m(r)$ to be the linear function

$$m(r) = m(r_2) + \frac{a_0}{4} r_1^2 (r - r_2)$$

and

$$g(r) = g_{\frac{r_3 - r}{r_3 - r_2}}.$$

Note that by our remark at the beginning of the proof, $g(r)$ glues smoothly in $r$ at $r_2$ (similar remark applies to $r_3$ below). To make sure that the scalar curvature $\tilde{S}$ of the total space remains nonnegative in this region, we need to choose the parameters $r_1, r_3$ appropriately. Let $C_1 = \max |\partial_s (g_s)_{\alpha\beta} \partial_s g_s^{\alpha\beta}|$, $C_2 = \max |g_s'|$, $C_3 = \max |g_s''|$. Then we have

$$\begin{aligned}\tilde{S} \geq\ & S_M(g(r)) + \frac{a_0}{4} r_1^2 \left( \frac{4}{r^2} - \frac{C_2}{r(r_3 - r_2)} \right) \\ & - \frac{a_0}{12} \frac{r_1^2}{r^2} \left| -r_1 + 3(r - r_2) \right| \frac{C_2}{r_3 - r_2} - \left( 1 + \frac{a_0}{6} \frac{r_1^2}{r} | - r_1 + 3(r - r_2)| \right) \\ & \times \left( \frac{C_1}{4(r_3 - r_2)^2} + \frac{C_3}{(r_3 - r_2)^2} + \frac{2C_2}{r(r_3 - r_2)} + \frac{C_2^2}{4(r_3 - r_2)^2} \right).\end{aligned}$$

This yields

$$\tilde{S} \geq -S_M^-(g_s) + \frac{a_0}{4} \frac{r_1^2}{r^2} (4 - A(r)) - \frac{1}{(r_3 - r_2)^2} B(r), \tag{4.16}$$

where

$$A(r) = C_2 \frac{r}{r_3 - r_2} + \left( \frac{5C_2}{3} + \frac{4C_3 + C_1 + C_2^2}{6} \frac{r}{r_3 - r_2} \right) \frac{| - r_1 + 3(r - r_2)|}{r_3 - r_2},$$

and

$$B(r) = \frac{C_1 + 4C_3 + C_2^2}{4} + 2C_2 \frac{r_3 - r_2}{r}.$$

We now set $r_3 = r_2 + \frac{1}{7} r_1$. By the given bounds on $C_1, C_2, C_3$ we have, for $r_2 \leq r \leq r_3,\ r_1 \geq 7$,

$$|A(r)| \leq 3, \quad |B(r)| \leq 1.$$

We then take $r_1$ sufficiently large so that

$$\frac{1}{(r_3 - r_2)^2} \leq \frac{a_0}{4} \left( \frac{1}{9} \right)^2,$$

With these choices we find that $\tilde{S} \geq 0$ on the region $r_2 \leq r \leq r_3$. Note that

$$m(r_3) = m(r_2) + \frac{a_0}{2} r_1^2 (r_3 - r_2) = -\frac{1}{84} r_1^3 a_0$$

is still negative.

Finally, for $r \geq r_3$, we put $g(r) = g_0$ and choose $m(r)$ to be increasing and limit to (say) $-\frac{1}{168} r_1^3 a_0$. Clearly, $\tilde{S} \geq 0$ here. □

As a consequence, one has the following result.

**Corollary 4.7.** *If $g_s$, $s \in [0, 1]$ is a one-parameter family of metrics on $M$ such that*

$$S_M(g_0) \geq 0, \quad S_M(g_s) > 0 \quad \textit{for } s \in (0, 1),$$

*then there exists $m(r)$ with $m(0) = m'(0) = m''(0) = 0$, $\lim_{r\to\infty} m(r) = m_\infty < 0$ and $g(r)$ with $g(r) = g_0$ for $r$ sufficiently large, such that the metric $\tilde{g}$ in (4.1) has scalar curvature $\tilde{S} \geq 0$.*

*Proof.* Let $\epsilon \in (0, 1)$ be a small constant. By taking $\epsilon$ sufficiently small, the hypothesis in Proposition 4.6 can be obviously satisfied for the one-parameter family of metrics $g_{\epsilon s}$ $(s \in [0, 1])$. □

Theorem 4.2 follows from the corollary above and the Positive Mass Theorem.

In view of Proposition 4.6 one also conclude that if $M$ is a compact spin manifold and $g$ a Riemannian metric which admits nonzero parallel spinors, then there is no metrics of positive scalar curvature in $U(g)$, where

$$U(g) = \Big\{ g' \in \mathcal{M} \mid \exists g_s \text{ such that } g_0 = g, g_1 = g' \text{ and (4.13) holds;} \\ S^-(g_s) \leq \frac{\min S(g')}{10} \text{ if } S(g') > 0. \Big\}$$

Condition (4.13) essentially defines a scale invariant $C^2$ neighborhood of $g$. The restriction on the negative part of the scalar curvature of the path above is more subtle. Although we believe that $U(g)$ is a neighborhood of $g$, we have not been able to prove it.

Finally we remark that, just as in the previous section, the results here still hold if $M$ may not be spin but there is a spin cover of $M$ which admits nonzero parallel spinor. To see this, we note that by Cheeger-Gromoll splitting theorem [CG71] the universal cover $\hat{M}$ of $M$ is the product of a Euclidean factor with a simply connected compact manifold. It is not hard to see that the positive mass theorem as used here extends to this setting.

## 5. Remarks on compact Ricci flat manifolds

By Cheeger-Gromoll's splitting theorem [CG71], we only have to look at simply connected compact Ricci flat manifolds. Also, without loss of generality, we can assume that they are irreducible. Since all Ricci flat symmetric spaces are flat, by Berger's classification of holonomy group [Bes87], the possible holonomy groups for a simply connected compact Ricci flat $n$-manifold are: $SO(n)$; $U(m)$ with $n = 2m$; $SU(m)$; $Sp(k) \cdot Sp(1)$ with $n = 4k$; $Sk(k)$; $G_2$; and $Spin(7)$. However, a simply connected Ricci flat manifold with holonomy group contained in $U(m)$ must in fact have holonomy group $SU(m)$; similarly, a simply connected Ricci flat manifold with holonomy group contained in $Sp(k) \cdot Sp(1)$ must in fact have holonomy group $Sp(k)$ [Bes87]. Thus,

**Proposition 5.1.** *A simply connected irreducible compact Ricci flat $n$-manifold either has holonomy $SO(n)$ or admits a nonzero parallel spinor.*

It is an interesting open question of what, if any, restriction the Ricci flat condition will impose on the holonomy. All know examples of simply connected compact Ricci flat manifolds are of special holonomy and hence admit nonzero parallel spinors.

From our local stability theorem, scalar flat metrics sufficiently close to a Calabi-Yau metric are necessarily Calabi-Yau themselves. One is thus led to the question of whether there are any scalar flat metrics on a compact Calabi-Yau manifold which are not Calabi-Yau.

**Proposition 5.2.** *If $(M, g)$ is a compact simply connected $G_2$ manifold or Calabi-Yau $n$-fold (i.e. $\dim_{\mathbb{C}} = n$) with $n = 2k + 1$, then there are scalar flat metrics on $M$ which are not $G_2$ or Calabi-Yau, respectively.*

*Proof.* By the theorem of Stolz [S92], $M$ carries a metric of positive scalar curvature (because of the dimension in the case of $G_2$ manifold, and $\hat{A} = 0$ in the case of Calabi-Yau [Wa89], [J00]). Let $g'$ be such a metric. Then, its Yamabe invariant $\mu(g') > 0$. On the other hand, every compact manifold carries a metric of constant negative scalar curvature [Bes87], say $g''$. Then $\mu(g'') < 0$. Now consider $g_t = tg'' + (1 - t)g'$. Since the Yamabe invariant depends continuously on the metric [Bes87], there is a $t_0 \in (0, 1)$ such that $\mu(g_{t_0}) = 0$ and $\mu(g_t) > 0$ for $0 \leq t < t_0$. The Yamabe metric in the conformal class of $g_{t_0}$ is scalar flat but cannot be a $G_2$ (Calabi-Yau, resp.) by Theorem 3.5. □

*Remark.* For the other special holonomy metric (including the Calabi-Yau $SU(2k)$), the index $\hat{A} \neq 0$ [Wa89], [J00]. Thus there is no positive scalar curvature metric on such manifold. Scalar flat metrics on such manifolds are classified by Futaki [Fu93]: they are all products of the special holonomy metrics (with nonzero index).

One is also led naturally to the following question: are there any scalar flat but not Ricci flat metrics on a compact Calabi-Yau (or $G_2$) manifold? A more interesting and closely related question is:

*Question.* Is any Ricci flat metric on a compact Calabi-Yau manifold necessarily Kähler (and hence Calabi-Yau)?

We note that the deformation analogue of this question has a positive answer by the work of Koiso, Bogomolov, Tian, Todorov. Moreover, in (real) dimension four, the answer is positive by the work of Hitchin [H74.2] on the rigidity case of Hitchin-Thorpe inequality.

## References


[BW01] Belegradek, I., Wei, G.: Metrics of positive Ricci curvature on bundles. Int. Math. Res. Not. **2004**, 56–72 (2004)

[Bes87] Besse, A.L.: Einstein manifolds. Berlin: Springer 1987

[Bo78] Bogomolov, F.A.: Hamiltonian Kähler manifolds. Dokl. Akad. Nauk SSSR **243**, 1101–1104 (1978)

[Br81] Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. **56**, 581–598 (1981)

[B89] Bryant, R.: Metrics with exceptional holonomy. Ann. Math. (2) **126**, 525–576 (1987)

[B03] Bryant, R.: Some remarks on $G_2$ manifolds. math.DG/0305124

[CHSW85] Candelas, P., Horowitz, G., Strominger, A., Witten, E.: Vacuum configurations for superstrings. Nucl. Phys. B **258**, 46–74 (1985)

[CHI04] Cao, H.-D., Hamilton, R.S., Ilmanen, T.: Gaussian densities and stability for some Ricci solitons. math.DG/0404165

[CG71] Cheeger, J., Gromoll, D.: The splitting theorem for manifolds of non-negative Ricci curvature. J. Differ. Geom. **6**, 119–128 (1971)

[D03] Dai, X.: A Positive Mass Theorem for Sapces with Asymptotic SUSY Compactifcation. Commun. Math. Phys. **244**, 335–345 (2004)

[DoC92] do Carmo, M.: Riemannian Geometry. Birkhäuser 1992

[FM75] Fischer, A., Marsden, J.: Linearization stability of nonlinear partial differential equations. In: Differential geometry (Proc. Symp. Pure Math., vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 2, pp. 219–263. Providence, R.I.: Amer. Math. Soc. 1975

[Fu93] Futaki, A.: Scalar-flat closed manifolds not admitting positive scalar curvature metrics. Invent. Math. **112**, 23–29 (1993)

[GL80] Gromov, M., Lawson Jr., H.B.: The classification of simply connected manifolds of positive scalar curvature. Ann. Math. **111**, 423–434 (1980)

[GIK02] Guenther, C., Isenberg, J., Knopf, D.: Stability of the Ricci flow at Ricci-flat metrics. Commun. Anal. Geom. **10**, 741–777 (2002)

[HHM03] Hertog, T., Horowitz, G., Maeda, K.: Negative energy density in Calabi-Yau compactifications. JHEP **0305**, 060 (2003). hep-th/0304199

[H74] Hitchin, N.: Harmonic spinors. Adv. Math. **14**, 1–55 (1974)

[H74.2] Hitchin, N.: Compact four-dimensional Einstein manifolds. J. Differ. Geom. **9**, 435–441 (1974)

[J00] Joyce, D.: Compact manifolds with special holonomy. Oxford: Oxford Univ. Press 2000

[KW75] Kazdan, J.L., Warner, F.W.: Prescribing curvatures. In: Differential geometry (Proc. Symp. Pure Math., vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 2, pp. 309–319. Providence, R.I.: Amer. Math. Soc. 1975

[Ko80] Koiso, N.: Rigidity and stability of Einstein metrics. The case of compact symmetric spaces. Osaka J. Math. **17**, 51–73 (1980)

[LM89] Lawson Jr., H.B., Michelsohn, M.-L.: Spin geometry. Princeton, NJ: Princeton University Press 1989

[L63] Lichnerowicz, A.: Spineurs harmonique. C. R. Acad. Sci., Paris, Sér. A-B **257**, 7–9 (1963)

[M68] Milnor, J.: A note on curvature and fundamental group. J. Differ. Geom. **2**, 1–7 (1968)

[M96] Morgan, J.: The Seiberg-Witten equations and applications to the topology of smooth four-manifolds. Princeton, NJ: Princeton Univ. Press 1996

[P02] Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. math.DG/0211159

[Sch84] Schoen, R.M.: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differ. Geom. **20**, 479–495 (1984)

[Sch89] Schoen, R.M.: Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In: Topics in calculus of variations (Montecatini Terme, 1987), pp. 120–154. Berlin: Springer 1989

[SY79-1] Schoen, R.M., Yau, S.T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. **65**, 45–76 (1979)

[SY79-2] Schoen, R.M., Yau, S.T.: On the structure of manifolds with posiitve scalar curvature. Manuscr. Math. **28**, 159–183 (1979)

[S92] Stolz, S.: Simply connected manifolds of positive scalar curvature. Ann. Math. **136**, 511–540 (1992)

[T86] Tian, G.: Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In: Mathematical aspects of string theory, pp. 629–646. World Scientific 1986

[To89] Todorov, A.: The Weil-Petersson geometry of the moduli space of SU($n \geq 3$) (Calabi-Yau) manifolds. I. Commun. Math. Phys. **126**, 325–346 (1989)

[Wa89] Wang, M.: Parallel spinors and parallel forms. Ann. Global Anal. Geom. **7**, 59–68 (1989)

[Wa91] Wang, M.: Preserving parallel spinors under metric deformations. Indiana Univ. Math. J. **40**, 815–844 (1991)

[W88] Wei, G.: Examples of complete manifolds of positive Ricci curvature with nilpotent isometry groups. Bull. Am. Math. Soc., New Ser. **19**, 311–313 (1988)

[Wi81] Witten, E.: A new proof of the positive energy theorem. Commun. Math. Phys. **80**, 381–402 (1981)

[Y77] Yau, S.T.: Calabi's conjecture and some new results in algebraic geometry. Proc. Natl. Acad. Sci. USA **74**, 1798–1799 (1977)

[Y78] Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Commun. Pure Appl. Math. **31**, 339–411 (1978)