manuscripta math. 90, 49 - 61 (1996)

manuscripta mathematica

Smoothing Riemannian metrics with Ricci curvature bounds *

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Received August 15, 1995;
in revised form December 11, 1995

We prove that Riemannian metrics with an absolute Ricci curvature bound and a conjugate radius bound can be smoothed to having a sectional curvature bound. Using this we derive a number of results about structures of manifolds with Ricci curvature bounds.

1. Introduction

A central theme in global Riemannian geometry is to understand the global geometric and topological structures of manifolds with appropriate curvature bounds. In this regard, manifolds with sectional curvature bounds have been better understood than those with Ricci curvature bounds. It is natural to try to apply the results and methods in the former case to the latter. Hence we ask the following question: can one deform or "smooth" a metric with a Ricci curvature bound to a metric with a sectional curvature bound? In this paper we would like to address this question.

Previous work on smoothing mainly deals with metrics having already some sort of sectional curvature bounds. For example, one can smooth a metric with a sectional curvature bound to a metric with bounds on all the covariant derivatives of its Riemannian curvature tensor. Geometric applications of this smoothing result can be found in [8] and [7]. Its generalizations to weaker bounds on sectional curvatures can be found in e.g. [21, 22]. So far, two main methods of smoothing have been applied. The works [5], [4], [19] and [21, 22], among others, use the Ricci flow, while [9], [1] and [18] use an embedding method. The Ricci flow, as a heat flow type equation, tends to average out geometric quantities,

* The authors were supported in part by NSF Grant. The first author was also supported in part by Alfred P. Sloan Fellowship
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and hence is a natural tool for smoothing. The embedding method employs an elementary averaging process in a linear space in which a given Riemannian manifold is embedded nicely.

The following is a precise formulation of the problem of smoothing Riemannian metrics with Ricci curvature bounds.

Question: Given a positive integer n and any $\epsilon > 0$, does there exist a constant $C(n, \epsilon)$ such that for any closed Riemannian manifold (M^n, g) with $|\mathrm{Ric}(g)| \leq 1$ we can find another Riemannian metric $\tilde{g}$ on M satisfying
1) $|\tilde{g} - g| < \epsilon$ and
2) $|K(\tilde{g})| \leq C(n, \epsilon)$?

By examples in [2] among others, the answer to the question in this generality is actually negative. Consequently, additional geometric conditions are needed for smoothing metrics with Ricci curvature bounds. In this note we consider bounds on conjugate radius.

Definition. We define for $n \geq 2$ and $r_0 > 0$

$$\mathcal{M}(n, r_0) = \left\{ (M, g) \;\middle|\; \begin{array}{l} (M, g) \text{ is a compact Riemannian manifold,} \\ \dim M = n, \ |\mathrm{Ric}| \leq 1 \text{ and conj} \geq r_0 \end{array} \right\},$$

where "conj" denotes the conjugate radius of M.

Recall that the conjugate radius of a manifold M can be defined to be the maximal radius r such that for every $q \in M$, the exponential map $\exp_q$ has maximal rank in the open ball of radius r centered at the origin of the tangent space $T_q M$. In order to understand the geometric content of bounds on conjugate radius, it is good to compare with bounds on injectivity radius. As is well-known, the corresponding space $\tilde{\mathcal{M}}(n, r_0)$ concerning injectivity radius (replacing conj by injectivity radius in the above definition; or equivalently, adding an *additional* lower bound on volume in the above definition [10]) is $C^{1,\alpha}$ precompact [3]. Hence the situation is quite simple. On the other hand, under a conjugate radius bound, manifolds can collapse, and the space $\mathcal{M}(n, , r_0)$ contains an abundance of geometric and topological structures. Note also that if $K_M \leq K$ then conj $\geq \frac{\pi}{\sqrt{K}}$. This is the usual way of controlling conjugate radius. But we emphasize that conjugate radius is an important independent geometric invariant.

It seems that Ricci flow is more convenient than the embedding method in our situation, so we choose it. We have

Theorem 1.1. There exist $T(n, r_0) > 0$ and $C(n, r_0) > 0$ such that for any manifold $(M, g_0) \in \mathcal{M}(n, , r_0)$, the Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \quad g(0) = g_0 \tag{1.1}$$

has a unique smooth solution $g(t)$ for $0 \leq t \leq T(n, r_0)$ satisfying

$$|g(t) - g_0| \leq 4t, \tag{1.2}$$

$$|Rm(g(t))|_\infty \leq C(n, r_0) t^{-1/2}, \tag{1.3}$$

$$|\mathrm{Ric}(g(t))|_\infty \leq 2. \tag{1.4}$$

Remark 1. Here the conjugate radius condition is only used in the following way: it implies that for every $x \in M$ the lifted metric on $B_x(r_0) \subset T_xM$ has uniform $L^{2,p}$ harmonic coordinates, see Sections 3 and 4.

Remark 2. Although we can not expect that the Ricci curvature of $g(t)$ is uniformly close to that of g_0, using (1.3) one can show that the Ricci upper and lower bounds are arbitrarily close to that of original ones as t goes to 0. Namely, if we assume $C_1 < \mathrm{Ric}(g_0) < C_2$, then we obtain a metric $g(t)$ satisfying $C_1 - Ct^{1/2} < \mathrm{Ric}(g(t)) < C_2 + Ct^{1/2}$, where C is a uniform constant. In particular, negatively (or positively) curved metrics are still negatively (or positively) curved after smoothing.

Using Theorem 1.1, we are able to extend a number of results concerning manifolds with sectional curvature bounds to manifolds with Ricci curvature bounds. First, we have the following generalization of Gromov's almost flat manifold theorem [12].

Theorem 1.2. There exists an $\epsilon = \epsilon(n, r_0) > 0$ such that if a manifold M^n has a metric with $|\mathrm{Ric}_M| \leq 1$, conj $\geq r_0$ and diam $\leq \epsilon$, then M is diffeomorphic to an infranilmanifold.

Next we have a generalization of Fukaya's fibration theorem [11].

Theorem 1.3. There exists an $\epsilon = \epsilon(n, r_0, \mu) > 0$ such that if M^n and N^m $(m \leq n)$ are compact manifolds with $M^n \in \mathcal{M}(n, r_0)$, $|\mathrm{Ric}_N| \leq 1, \mathrm{inj}_N \geq \mu$, and if the Hausdorff distance $d_H(M, N) \leq \epsilon(n, r_0, \mu)$, then there exists a fibration $f : M \to N$ such that
1) The fiber of f is an almost flat manifold.
2) f is an almost Riemannian submersion.

Let Min Vol(M) be the infimum of the volumes of all the complete Riemannian metrics on M with sectional curvature $|K| \leq 1$. We can extend the gap result for minimal volume in dimension four in [16].

Theorem 1.4. There exists a real number $v(r_0) > 0$ such that if a 4-manifold M has a metric with $|\mathrm{Ric}_M| \leq 1$, conj $\geq r_0$ and Vol $\leq v(r_0)$, then Min Vol$(M) = 0$.

As an immediate corollary of Theorem 1.1 and Gromov's uniform betti number estimate [13], we have that the betti numbers of manifolds in $\mathcal{M}(n, r_0)$ are uniformly bounded. Indeed, such a uniform bound still holds if the absolute Ricci curvature bound is replaced by a lower bound [20]. Note however that $\mathcal{M}(n, r_0)$ contains infinitely many homotopy types. With Theorem 1.3 at disposal, we can also say something about the higher homotopy groups, generalizing Rong's results [17].

Theorem 1.5. For each $D > 0$ and each $q \geq 2$, the q-th rational homotopy group $\pi_q(M) \otimes Q$ has at most $N(n, D, q)$ isomorphism classes for manifolds in $\mathcal{M}(n, r_0)$ with diam$_M \leq D$.

Concerning the fundamental group we have

Theorem 1.6. For manifolds in $\mathcal{M}(n, r_0)$ with diam$_M \leq D$, $\pi_1(M)$ has a normal nilpotent subgroup G such that
1) The minimal number of generators for G is less than or equal to n,
2) $\pi_1(M)/G$ has at most $N(n, r_0, D)$ isomorphic classes up to a possible normal Z_2-extension.

Now we would like to say a few words about the strategy of proving Theorem 1.1. The Ricci flow exists at least for a very short time, and the main point is to show that it actually exists on a time interval of uniform size on which the desired estimates hold. A simple, but crucial idea is to lift the solution metrics to suitable domains of the tangent space via the initial exponential maps. The size of these domains can be made uniform by the virtue of the conjugate radius bound. Applying the results in [3] we obtain an initial L^p bounds on the sectional curvatures and an initial bound on the Sobolev constant for the lifted metrics. The lifted metrics still satisfy the Ricci flow equation, but they are only defined locally and no a priori control near the boundary is given for them. We apply Moser's weak maximum principle to estimate the sectional curvatures of the lifted metrics in a way similar to [21]. As in [21], we need to control the L^p bound of the sectional curvatures of the lifted metrics along the flow. The subtlety here is how to handle the difficulty caused by lack of control near boundary. Fortunately we found a covering argument to resolve this problem.

It is possible to obtain part of the smoothing result of Theorem 1.1 by the embedding method, see [15] for more details. There smoothing in a more general setting is established.

2. Moser's weak maximum principle

Let N be a compact n-dimensional manifold with non-empty boundary, $n \geq 3$. For a given metric g on N the Sobolev constant C_S of g (or better, of the Riemannian manifold (N,g)) is defined to be the supremum of $(\int |f|^{\frac{2n}{n-2}})^{\frac{n-2}{2n}}$ over all C^1 functions f on N with $\int |\nabla f|^2 = 1$, which vanish along ∂N. So for such f we have

$$||f||_{\frac{2n}{n-2}} \leq C_S ||\nabla f||_2. \tag{2.1}$$

Theorem 2.1. Let f, b be smooth nonnegative functions on $N \times [0,T]$ which satisfy the following:

$$\frac{\partial f}{\partial t} \leq \Delta f + bf \text{ on } N \times [0,T], \tag{2.2}$$

where Δ is the Laplace-Beltrami operator of the metric $g(t)$, b is assumed to satisfy:

$$\sup_{0\leq t\leq T} \left(\int_N b^{q/2}\right)^{2/q} \leq \beta\ ,$$

for some $q > n$. Put $l = \max\limits_{0\leq t\leq T} \left\| \frac{\partial g}{\partial t} \right\|_{C^0}$ (here the norm is measured in $g(t)$) and $C_S = \max\limits_{0\leq t\leq T} C_S(g(t))$. Then, given any $p_0 > 1$, there exists a constant $C = C(n, q, p_0, \beta, C_S, l, T, R)$ such that for any x in the interior of N and $t \in (0,T]$,

$$|f(x,t)| \leq C t^{-\frac{n+2}{2p_0}} \left(\int_0^T \int_{B_R} f^{p_0}\right)^{1/p_0}, \tag{2.3}$$

where $R = \frac{1}{2} dist_{g(0)}(x, \partial N)$ and $B_R = B_R(x)$ denotes the geodesic ball of radius R and center x defined in terms of the metric $g(0)$.

Proof. We follow closely [23], [21]. Let η be a non-negative Lipschitz function vanishing along ∂N. The partial differential inequality (2.2) implies for $p \geq 2$

$$\frac{1}{p}\frac{\partial}{\partial t}\int f^p\eta^2 dv_t \leq \int \eta^2 f^{p-1}\Delta f\, dv_t + \int bf^p\eta^2 dv_t + \frac{1}{p}\int f^p\eta^2\frac{\partial}{\partial t}dv_t\ ,$$

where dv_t is the volume form of $g(t)$. Integration by parts yields

$$\begin{aligned}\int \eta^2 f^{p-1}\Delta f\, dv_t &= -\frac{4(p-1)}{p^2}\int |\nabla(\eta f^{p/2})|^2 dv_t + \frac{4}{p^2}\int |\nabla\eta|^2 f^p dv_t \\ &\quad + \frac{4(p-2)}{p^2}\int \nabla(\eta f^{p/2})f^{p/2}\nabla\eta\, dv_t \\ &\leq -\frac{2}{p}\int |\nabla(\eta f^{p/2})|^2 + \frac{2}{p}\int |\nabla\eta|^2 f^p,\end{aligned}$$

where the gradient ∇ refers to $g(t)$. By Hölder inequality

$$\begin{aligned}\int bf^p\eta^2 dv_t &\leq (\int b^{q/2}dv_t)^{2/q}\left(\int (f^p\eta^2)^{\frac{q}{q-2}}dv_t\right)^{\frac{q-2}{q}} \\ &\leq \beta(\epsilon^{-\frac{n-2}{q}}\int f^p\eta^2 dv_t)^{1-\frac{n}{q}}\left(\epsilon^{1-\frac{n}{q}}\int (f^p\eta^2)^{\frac{n}{n-2}}dv_t\right)^{\frac{n-2}{n}\frac{n}{q}} \\ &\leq \beta\left(\epsilon^{-\frac{n-2}{q}}\int f^p\eta^2 dv_t\right)^{1-n/q}\left(\epsilon^{(1-\frac{n}{q})(\frac{n-2}{n})}C_S^2\int |\nabla(\eta f^{p/2})|^2 dv_t\right)^{n/q} \\ &\leq (1-\frac{n}{q})\beta\epsilon^{-\frac{n-2}{q}}\int f^p\eta^2 dv_t + \frac{n}{q}\beta\epsilon^{(1-\frac{n}{q})(\frac{n-2}{n})}C_S^2\int |\nabla(\eta f^{p/2})|^2 dv_t.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{p}\frac{\partial}{\partial t}\int f^p\eta^2 &\leq \left(-\frac{2}{p} + \frac{n}{q}\epsilon^{(1-\frac{n}{q})(\frac{n-2}{n})}\beta C_S^2\right)\int |\nabla(\eta f^{p/2})|^2 \\ &\quad + \frac{2}{p}\int |\nabla\eta|^2 f^p + \left(\beta(1-\frac{n}{q})\epsilon^{-\frac{n-2}{q}} + \frac{\sqrt{n}l}{p}\right)\int f^p\eta^2\ .\end{aligned}$$

(The volume form is omitted.)

Setting

$$\epsilon = (\frac{q}{np\beta C_S^2})^{\frac{nq}{(q-n)(n-2)}},$$

we obtain the following basic estimate:

$$\frac{\partial}{\partial t}\int f^p\eta^2 + \int |\nabla(\eta f^{p/2})|^2 \leq 2\int |\nabla\eta|^2 f^p + C_1(n,p,\beta,C_S,q,l)\int f^p\eta^2. \quad (2.4)$$

Now given $0 < \tau < \tau' < T$, let

$$\psi(t) = \begin{cases} 0 & 0 \leq t \leq \tau\ , \\ (t-\tau)/(\tau'-\tau) & \tau \leq t \leq \tau'\ , \\ 1 & \tau' \leq t \leq T\ . \end{cases}$$

Multiplying (2.4) by ψ, we obtain

$$\frac{\partial}{\partial t}\left(\psi\int f^p\eta^2\right)+\psi\int|\nabla(\eta f^{p/2})|^2\le 2\psi\int|\nabla\eta|^2 f^p+(C_1\psi+\psi')\int f^p\eta^2 .$$

Integrating this with respect to t we get

$$\int_t f^p\eta^2+\int_{\tau'}^t\int|\nabla(\eta f^{p/2})|^2 \le 2\int_\tau^T\int|\nabla\eta|^2 f^p+\left(C_1+\frac{1}{\tau'-\tau}\right)\int_\tau^T\int f^p\eta^2 , \quad \tau'\le t\le T.$$

Applying this estimate and the Sobolev inequality we deduce

$$\begin{aligned}\int_{\tau'}^T\int f^{p(1+\frac{2}{n})}\eta^{2+\frac{1}{n}} &\le \int_{\tau'}^T\left(\int f^p\eta^2\right)^{2/n}\left(\int f^{\frac{pn}{n-2}}\eta^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\ &\le C_S^2\Big(\sup_{\tau'\le t\le T}\int f^p\eta^2\Big)^{2/n}\int_{\tau'}^T\int\left|\nabla(\eta f^{p/2})\right|^2\\ &\le C_S^2\left[2\int_\tau^T\int|\nabla\eta|^2 f^p+\left(C_1+\frac{1}{\tau'-\tau}\right)\int_\tau^T\int f^p\eta^2\right]^{1+\frac{2}{n}} \quad (2.5)\end{aligned}$$

For a given x in the interior of N, $p\ge p_0$ and $0\le\tau\le T$, we put

$$H(p,\tau,R)=\int_\tau^T\int_{B_R} f^p ,$$

where B_R is the geodesic ball of center x and radius R measured in $g(0)$. Choosing a suitable cut-off function η (and noticing $|\nabla\eta|_t\le|\nabla\eta|_0 e^{\frac{1}{2}lt}$) we derive from (2.5)

$$H\left(p\left(1+\frac{2}{n}\right),\tau',R\right)\le C_S^2\left[C_1+\frac{1}{\tau'-\tau}+\frac{2e^{lT}}{(R'-R)^2}\right]^{1+\frac{2}{n}} H(p,\tau,R')^{1+\frac{2}{n}} , \quad (2.6)$$

where $0<R<R'\le dist_{g(0)}(x,\partial N)$. To proceed we set $\mu=1+\frac{2}{n}$, $p_k=p_0\mu^k$, $\tau_k=(1-\frac{1}{\mu^{k+1}})t$ and $R_k=\frac{R}{2}(1+\frac{1}{\mu^{k/2}})$ with $R=\frac{1}{2}dist_{g(0)}(x,\partial N)$. Then it follows from (2.6) that

$$H(p_{k+1},\tau_{k+1},R_{k+1})^{1/p_{k+1}}\le C_S^{2/p_{k+1}}\left[C_1+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}+\frac{4e^{lT}}{R^2}\cdot\frac{\mu}{(\sqrt{\mu}-1)^2}\right]^{1/p_k}\mu^{k/p_k}H(p_k,\tau_k,R_k)^{1/p_k} .$$

Hence

$$H(p_{m+1},\tau_{m+1},R_{m+1})^{1/p_{m+1}}\le C_S^{\sum_{k=0}^m\frac{2}{p_{k+1}}}\left[C_1+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}+\frac{4e^{lT}}{R^2}\cdot\frac{\mu}{(\sqrt{\mu}-1)^2}\right]^{\sum_{k=0}^m\frac{1}{p_k}}\mu^{\sum_{k=0}^m\frac{k}{p_k}}H(p_0,\tau_0,R_0)^{\frac{1}{p_0}}$$

Letting $m\to\infty$ we conclude

$$|f(x,t)| \le (1+\frac{2}{n})^{\frac{n^2+2n}{4p_0}} C_S^{\frac{n}{p_0}} \left(C_1 + \frac{1}{t} + \frac{e^{lT}}{R^2}\right)^{\frac{n+2}{2p_0}} \left(\int_0^T \int_{B_R} f^{p_0}\right)^{1/p_0}.$$

□

3. Bounding curvature tensor

Let $(M, g_0) \in \mathcal{M}(n, r_0)$ for some n, r_0. Consider the Ricci flow on M with initial metric g_0. It is well-known that the Riemann curvature tensor Rm satisfies the following evolution equation along the Ricci flow [14]:

$$\frac{\partial Rm}{\partial t} = \Delta Rm + Q(Rm), \tag{3.1}$$

where $Q(Rm)$ is a tensor that is quadratic in Rm. From this it follows that

$$\frac{\partial}{\partial t}|Rm| \le \Delta|Rm| + c(n)|Rm|^2. \tag{3.2}$$

In order to apply Moser's weak maximum principle we need bounds on

$$\begin{array}{l} \max_{0\le t\le T} \|Rm\|_{p_0}, \\ \max_{0\le t\le T} C_S, \\ \max_{0\le t\le T} |Ric|. \end{array}$$

We are going to employ suitable evolution equations associated with the Ricci flow such as (3.1) and (3.2) to derive these bounds from the initial bounds (i.e. bounds at $t = 0$). The basic logic of the argument goes as follows. By smoothness of the Ricci flow, the initial bounds can at least be extended to a very short time interval. We then derive a uniform estimate for the length of the maximal interval of extension.

To get initial bounds we actually have to pull the metric up to the tangent spaces. For any fixed $x \in M$, we lift the metric g_0 by the exponential map to $\tilde{g}_x(0)$ on $\tilde{B}_{r_0}(x) \subset T_xM$. Then

$$|\mathrm{Ric}_{\tilde{B}_{r_0}}| \le 1, \quad inj_{\tilde{B}_{r_0/2}} \ge r_0/2.$$

By [3], the metric $\tilde{g}_x(0)$ also comes with the following uniform bounds:

$$\begin{aligned} \|Rm\|_{p_0,\tilde{B}_{r_0/2}(x)} &\le K(n, r_0, p_0), \text{ for all } 1 \le p_0 < \infty, \\ C_S(\tilde{B}_{r_0/2}(x)) &\le \chi(n, r_0). \end{aligned}$$

This furnishes us with the required initial bounds.

Let $\tilde{g}_x(t)$ be the lifted Ricci flow on $\tilde{B}_{r_0/2}(x)$. Choose $p_0 > n/2 + 1$, e.g. $p_0 = n + 2$. Then we have

Proposition 3.1. Let $[0, T_{\max})$ be a maximal time interval on which the Ricci flow $g(t)$ has a smooth solution such that each lifted Ricci flow $\tilde{g}_x(t)$ on $\tilde{B}_{r_0/2}(x)$ satisfies the following estimates:

$$\|Rm(\tilde{g}_x(t))\|_{p_0, \tilde{B}_{r_0/4}(x)} \leq 2K(n, r_0, p_0), \tag{3.3}$$

$$|\mathrm{Ric}(\tilde{g}_x(t))| \leq 2, \tag{3.4}$$

$$C_S(\tilde{g}_x(t)) \leq 2\chi(n, r_0). \tag{3.5}$$

Then $T_{\max} \geq T(n, r_0, p_0)$ for some $T(n, r_0, p_0) > 0$.

The proof of this proposition requires four lemmas. Set $K = K(n, r_0, p_0)$.

Lemma 3.2. For $0 \leq t < T_{\max}$ there holds

$$\|Rm(\tilde{g}_x(t))\|_\infty \leq C(n, p_0, \chi, K)t^{-\frac{n+2}{2p_0}}. \tag{3.6}$$

This follows immediately from Moser's weak maximum principle Theorem 2.1.

Lemma 3.3. For $0 \leq t < T_{\max}$ there holds

$$\max_{x \in M} \|Rm(\tilde{g}_x(t))\|_{p_0, \tilde{B}_{r_0/4}(x)} \leq \frac{K}{1 - C(n, p_0, \chi, K)t}. \tag{3.7}$$

Lemma 3.4. For $0 \leq t < T_{\max}$ there holds

$$|\mathrm{Ric}(g(t))| \leq e^{C(n, p_0, K, \chi)t^{\frac{2p_0 - n - 2}{2p_0}}}. \tag{3.8}$$

Lemma 3.5. For $0 \leq t < T_{\max}$ there holds

$$\max_{x \in M} C_S(\tilde{g}_x(t)) \leq \chi e^{C(n, p_0, K, \chi)t^{\frac{2p_0 - n - 2}{2p_0}}}. \tag{3.9}$$

The proof of Lemmas 3.3-3.5 is presented in the next section. Granting the lemmas the proof of Proposition 3.1 can be seen as follows. First of all we quote the following theorem of Hamilton [14]:

Theorem 3.6. Let $g(t)$, $0 \leq t < T$ be a (smooth) solution to the Ricci flow (1.1) on a compact manifold M. If the sectional curvatures of $g(t)$ remain bounded as $t \to T$, then the solution extends smoothly beyond T.

By (3.7)-(3.9) there exists $T(n, r_0, p_0) > 0$ such that if $t < \min(T(n, r_0, p_0), T_{\max})$ then strict inequalities hold in (3.3), (3.4), (3.5). Consequently, if $T_{\max} < T(n, r_0, p_0)$, by (3.6) and Hamilton's theorem, the solution to the Ricci flow can be extended beyond $T_{\max}$ with (3.3), (3.4), (3.5) still holding, contradicting the maximality of $T_{\max}$.

Proof of Theorem 1.1. Theorem 1.1 follows immediately from Proposition 3.1 and (3.6) if we let $p_0 = n + 2$. □

4. Proof of the three basic lemmas

Proof of Lemma 3.3. By (3.1) and the bounds implied by the definition of T_{max}, we can apply (2.4) with $p = p_0$ to deduce

$$\frac{\partial}{\partial t}\int_{\tilde{B}_{r_0/2}(x)}|Rm|^p\eta^2 \le 2\int_{\tilde{B}_{r_0/2}(x)}|\nabla\eta|^2|Rm|^p + C_1(n,p_0,K,\chi)\int_{\tilde{B}_{r_0/2}(x)}|Rm|^p\eta^2. \tag{4.1}$$

We choose η so that the following is true.

$$\frac{\partial}{\partial t}\int_{\tilde{B}_{r_0/4}(x)}|Rm|^p \le (2+C_1)\int_{\tilde{B}_{r_0/2}(x)}|Rm|^p. \tag{4.2}$$

Now cover the ball $\tilde{B}_{r_0/2}(x)$ by balls $\tilde{B}_{r_0/4}(\tilde{y}_i)$ with $\tilde{y}_i \in \tilde{B}_{r_0/2}(x)$. By the following proposition of Gromov the number of covering is uniformly bounded.

Proposition 4.1 (Gromov). (see [6, Proposition 3.11]) Let the Ricci curvature of M^n satisfy $\mathrm{Ric}_{M^n} \ge (n-1)H$. Then given $r, \epsilon > 0$ and $p \in M^n$, there exists a covering, $B_p(r) \subset \cup_1^N B_{p_i}(\epsilon)$, ($p_i$ in $B_p(r)$) with $N \le N_1(n, Hr^2, r/\epsilon)$. Moreover, the multiplicity of this covering is at most $N_2(n, Hr^2)$.

Therefore $\tilde{B}_{r_0/2}(x) \subset \bigcup_1^N \tilde{B}_{r_0/4}(\tilde{y}_i)$ with $N \le N(n, r_0)$.
Let $y_i = \exp_x \tilde{y}_i$. We now construct a map

$$\exp_{y_i}^{-1}\circ\exp_x : \tilde{B}_{r_0/4}(\tilde{y}_i) \subset T_xM \to \tilde{B}_{r_0/4}(y_i) \subset T_{y_i}M. \tag{4.3}$$

For any $\tilde{y} \in \tilde{B}_{r_0/4}(\tilde{y}_i)$ connect it with the center $\tilde{y}_i$ by the unique minimal geodesic. This projects down to a geodesic on M which can then be lifted to a geodesic in $\tilde{B}_{r_0/4}(y_i)$. The end point of this geodesic gives the image of $\tilde{y}$. Thus defined, $\exp_{y_i}^{-1}\circ\exp_x$ is one to one and an isometry. It follows then

$$\int_{\tilde{B}_{r_0/4}(\tilde{y}_i)}|Rm|^p = \int_{\tilde{B}_{r_0/4}(y_i)}|Rm|^p.$$

This combines with (4.2) implies

$$\frac{\partial}{\partial t}\int_{\tilde{B}_{r_0/4}(x)}|Rm|^p \le (2+C_1)N\max_{y\in M}\int_{\tilde{B}_{r_0/4}(y)}|Rm|^p.$$

Integrating this, we obtain

$$\int_t |Rm(\tilde{g}_x)|^p \le \int_{t=0}|Rm(\tilde{g}_x)|^p + (2+C_1)Nt\max_{y\in M}\int|Rm(\tilde{g}_y)|^p.$$

Therefore

$$(1-(2+C_1)Nt)\max_{y\in M}\int|Rm(\tilde{g}_y)|^p \le \int_{t=0}|Rm(\tilde{g}_x)|^p \le K,$$

and this implies (3.7). □

Proof of Lemma 3.4. Note that for Ricci curvature we do the estimate directly, without passing to the tangent space.

Recall that the Ricci curvature tensor satisfies the following evolution equation [14].

$$\frac{\partial}{\partial t}\text{Ric} = \Delta\text{Ric} + 2Rm(\text{Ric}) - 2\text{Ric}\cdot\text{Ric}.$$

Let $\varphi(t) = \max_{x\in M}|\text{Ric}(g(t))|$, then

$$\frac{\partial}{\partial t}\varphi(t) \le c(n)|Rm(g(t))|\varphi(t).$$

Using (3.6), together with $\varphi(0)\le 1$ and integrating with respect to t gives

$$\varphi(t) \le e^{\int_0^t C(n,p_0,K,\chi)\tau^{-\frac{n+2}{2p_0}}\,d\tau} = e^{\frac{2p_0}{2p_0-n-2}C(n,p_0,K,\chi)t^{\frac{2p_0-n-2}{2p_0}}}.$$

This implies (3.8). □

Proof of Lemma 3.5. For $u \in C^\infty(\tilde{B}_{r_0/2}(x))$ and vanishing on the boundary, define

$$E_t[u] = \left(\frac{||u||_{\frac{2n}{n-2},\tilde{B}_{r_0/2}}}{||\nabla u||_2}\right)^2.$$

Then a straightforward computation shows that

$$\frac{\partial}{\partial t}E_t[u] \le c(n)||\text{Ric}(\tilde{g}_x(t))||_\infty E_t[u].$$

Integrating this gives

$$C_S(\tilde{g}_x(t)) \le \chi e^{C(n,p_0,K,\chi)t^{\frac{2p_0-n-2}{2p_0}}},$$

which is (3.9). □

5. Applications

By Theorem 1.1, the deformed metric $g(t)$ has uniform sectional curvature bound (away from $t=0$) and $g(t)$ is close to $g(0)$. To apply the results with sectional curvature bounds to $g(t)$, we need to show that other geometric quantities like diameter and volume are also under control. We first prove the following lemma.

Lemma 5.1. Let $g(t)$ be the Ricci flow in Theorem 1.1. Then for $0\le t\le T(n,r_0)$,

$$e^{-2t}\text{diam}(0) \le \text{diam}(t) \le e^{2t}\text{diam}(0), \tag{5.1}$$
$$e^{-4nt}\text{vol}(0) \le \text{vol}(t) \le e^{4nt}\text{vol}(0), \tag{5.2}$$

where e.g. diam(t) means diam(g(t)).

Proof. For the estimate on the diameter we consider the evolution of the length functional. Thus fix a curve c and let $l_c(t)$ denote its length in the metric $g(t)$. Then

$$-2l_c(t) \le \frac{\partial l_c(t)}{\partial t} \le 2l_c(t), \tag{5.3}$$

from which we obtain $e^{-2t}l_c(0) \le l_c(t) \le e^{2t}l_c(0)$. This gives

$$e^{-2t}d_{p,g}(0) \le d_{p,q}(t) \le e^{2t}d_{p,g}(0), \tag{5.4}$$

where $d_{p,q}(t)$ denote the distance between p and q in the metric $g(t)$ for p, q in M, in particularly (5.1).

For the volume let $\omega(t)$ denote the volume form (or density if M is not orientable) of $g(t)$. Consider $A(t) = \omega(t)/\omega(0)$. One computes

$$\frac{\partial A(t)}{\partial t} = \mathrm{Tr}_{g(t)}\frac{\partial g(t)}{\partial t})A(t).$$

Then

$$-4nA(t) \le \frac{\partial A(t)}{\partial t} \le 4nA(t).$$

Hence $e^{-4nt} \le A(t) \le e^{4nt}$ and the volume estimate follows. □

Proof of Theorem 1.2. Let $g(t)$ be the unique solution to (1.1) with the given metric as the initial data. By (1.3), for $0 < t \le T(n, r_0)$,

$$|K(t)| \le C(n, r_0)t^{-1/2} \tag{5.5}$$

If $\epsilon_0(n)$ is the small constant in the original Gromov's almost flat manifold theorem [12], we choose $t_0 = T(n, r_0)$, $\epsilon = \mathrm{diam}(0) \le \left(\frac{\epsilon_0(n)t_0^{1/2}}{C(n,r_0)e^{4t_0}}\right)^{1/2}$. Then from (5.1) and (5.5) $|K(t_0)D^2(t_0)| \le \epsilon_0(n)$. Applying Gromov's almost flat manifold theorem to $g(t_0)$ gives Theorem 1.2. □

Proof of Theorem 1.3. Consider the Ricci flows $g_M(t)$, $g_N(t)$ on M and N respectively, starting with the given metrics. Then

$$|K_M(t)| \le C(n, r_0)t^{-1/2}, \quad |K_N(t)| \le C(m, \mu)t^{-1/2},$$

for $0 < t \le T(n, r_0)$ and $0 < t \le T(m, \mu)$ respectively. Let $T = \min(T(n, r_0), T(m, \mu))$, and $C = \max(C(n, r_0), C(m, \mu))$. Rescale the metrics

$$h_M(t) = Ct^{-1/2}g_M(t), \quad h_N(t) = Ct^{-1/2}g_N(t)$$

so that with respect to $h_M(t), h_N(t)$, where $0 < t \le T$,

$$|K_M| \le 1, \quad |K_N| \le 1. \tag{5.6}$$

Since $\mathrm{inj}_N \ge \mu$ for the initial metric on N, we have $C_S(g_N(0)) \le \chi(m, \mu)$ and therefore by (3.5)

$$C_S(h_N(t)) = C_S(g_N(t)) \le 2C_S(g_N(0)) \le 2\chi(m, \mu),$$

for $0 < t \le T$. By [10]

$$\mathrm{inj}_{h_N(t)} \ge \mu_1(m, \mu)$$

for all $0 < t \le T$.

Now for $0 < t \le T$,

$$\begin{aligned} d_H\left(M(h(t)), N(h(t))\right) &= Ct^{-1/2} d_H(M(g(t)), N(g(t))) \\ &\le Ct^{-1/2}\left[d_H(M(g(t)), M(g(0))) + d_H(M(g(0)), N(g(0))) + d_H(N(g(0)), N(g(t)))\right] \\ &\le Ct^{-1/2}\left[8t + d_H(M(g(0)), N(g(0)))\right]. \end{aligned}$$

Let $\lambda(n)$ be the small constant in the refined fibration theorem (cf. [7, Theorem 2.6]). We choose $t_0 \le \min\{T, \left(\frac{\mu_1(n,\mu)\lambda(n)}{16C}\right)^2\}$, $\epsilon = d_H(M(g(0)), N(g(0))) \le \frac{1}{2}\left(\frac{\mu_1(n,\mu)\lambda(n)t_0^{1/2}}{C}\right)$. Then $\frac{d_H(M(h(t_0)), N(h(t_0)))}{\mathrm{inj}_{h_N(t_0)}} \le \lambda(n)$. Applying Theorem 2.6 in [7] finishes the proof. □

With (5.2) the proof of Theorem 1.4 is quite similar. It follows from Theorem 1.1, and Corollary 0.4 in [16].

Theorem 1.5 and 1.6 can be proved by using Theorem 1.3 as in [17].

Acknowledgement. The first and second authors would like to thank Jeff Cheeger for interesting discussions and constant encouragement.

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