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Higher Spectral Flow

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For a continuous curve of families of Dirac type operators we define a higher spectral flow as a K-group element. We show that this higher spectral flow can be computed analytically by $\hat{\eta}$-forms and is related to the family index in the same way as the spectral flow is related to the index. We introduce a notion of Toeplitz family and relate its index to the higher spectral flow. Applications to family indices for manifolds with boundary are also given. © 1998 Academic Press

1. INTRODUCTION

The spectral flow for a one parameter family of self adjoint Fredholm operators is an integer that counts the *net number* of eigenvalues that change sign. This notion is introduced by Atiyah–Patodi–Singer [APS1] in their study of index theory on manifolds with boundary and is intimately related to the η invariant, which is also introduced by Atiyah–Patodi–Singer [APS2]. It has since then found other significant applications.

In this paper, a notion of *higher spectral flow* is introduced, generalizing the usual spectral flow. The higher spectral flow is defined for a continuous one parameter family (i.e. a curve) of *families of Dirac type operators* parametrized by a compact space and is an element of the K-group of the parameter space. The (virtual) dimension of this K-group element is precisely the (usual)

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spectral flow. The definition makes use of the concept of *spectral section* introduced recently by Melrose-Piazza [MP].

Roughly speaking, a spectral section is a way of dividing the spectral data into the positive and negative parts. And two different spectral sections give rise to a difference element which lies in the K-group of the parameter space. In order to define the spectral flow for a curve of families of operators, one first needs to fix a way of dividing the spectral data at the two end points of the curve, i.e., two spectral sections. One then continuously deforms the beginning spectral section along the curve. The difference (element) of the ending spectral section with the (other) given spectral section is the (higher) spectral flow.

More precisely, let $\pi: X \to B$ be a smooth fibration with the typical fiber Z an odd dimensional closed manifold and B compact. A family of self adjoint elliptic pseudodifferential operators on Z, parametrized by B, will be called a B-family. Consider a curve of B-families, $D_u = \{D_{b,u}\}$, $u \in [0, 1]$.

Assume that the index bundle of D_0 vanishes and let Q_0, Q_1 be spectral sections of D_0, D_1 respectively. If we consider the total family $\tilde{D} = \{D_{b,u}\}$ parametrized by $B \times I$, then there is also a total spectral section $\tilde{P} = \{P_{b,u}\}$. Let P_u be the restriction of P over $B \times \{u\}$. The (higher) spectral flow sf $\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs (D_0, Q_0), (D_1, Q_1) is an element in $K(B)$ defined by

$$\mathrm{sf}\{(D_0, Q_0), (D_1, Q_1)\} = [Q_1 - P_1] - [Q_0 - P_0] \in K(B). \tag{0.1}$$

The definition is independent of the choice of the (total) spectral section $\tilde{P}$. When D_u, $u \in S^1$ is a periodic family, we choose $Q_1 = Q_0$. In this case the higher spectral flow turns out to be independent of $Q_0 = Q_1$ and therefore defines an invariant of the family, denoted by sf$\{D_u\}$.

We show that this is indeed a generalization of the notion of spectral flow (Theorem 1.3). That is, when we consider a one parameter family of self adjoint elliptic pseudodifferential operators and the standard Atiyah–Patodi–Singer spectral projections [APS2] at the end points, the higher spectral flow coincides with the usual spectral flow.

We also show that this higher version of spectral flow satisfies the basic properties of spectral flow. For example, its Chern character can be expressed analytically in terms of a generalized version of the $\hat{\eta}$ form of Bismut–Cheeger [BC1]. This gives a way of computing, analytically, the Chern character of the higher spectral flow.

THEOREM 0.1. *Let D_u be a curve of B-families of Dirac type operators and Q_0, Q_1 spectral sections of D_0, D_1 respectively. Let $\tilde{B}_t(u)$ be a curve of superconnections. Then we have the following identity in $H^*(B)$,*

$$\mathrm{ch}(\mathrm{sf}\{(D_0, Q_0), (D_1, Q_1)\}) = \hat{\eta}(D_1, Q_1) - \hat{\eta}(D_0, Q_0) - \frac{1}{\sqrt{\pi}} \int_0^1 a_0(u)\, du, \tag{0.2}$$

where $\hat{\eta}(D_i, Q_i)$ are suitable $\hat{\eta}$-forms and $a_0(u)$ is a local invariant determined by the asymptotic expansion (2.22).

In the case of periodic family, the higher spectral flow can be related to the family index, in the same way as the spectral flow is related to the index. Namely, let $D = \{D_u\}_{u \in S^1}$ be a periodic family of B-families of (self-adjoint) Dirac type operators $\{D_{b,u}\}_{b \in B, u \in S^1}$. Then for any $b \in B$, $\{D_{b,u}\}_{u \in S^1}$ determines a natural Dirac type operator D'_b on $S^1 \times Z_b$:

$$D'_b\colon \Gamma(\mathbb{C}^2 \otimes S(TZ_b) \otimes E_b) \to \Gamma(\mathbb{C}^2 \otimes S(TZ_b) \otimes E_b).$$

$D' = \{D'_b\}_{b \in B}$ then forms a family of Dirac type operators over B.

THEOREM 0.2. *The following identity holds in $H^*(B)$,*

$$\mathrm{ch}(\mathrm{ind}\, D') = \mathrm{ch}(\mathrm{sf}\{D\}).$$

This theorem generalizes the well-known result of Atiyah–Patodi–Singer [APS1] to the family case. As a consequence we obtain the following multiplicativity formula for the spectral flow.

COROLLARY 0.3. *Assume that B is also spin and for each $u \in S^1$, denote D^X_u the total Dirac operator on X. Then the following identity holds,*

$$\mathrm{sf}\{D^X_{u \in S^1}\} = \int_B \hat{A}(TB)\, \mathrm{ch}(\mathrm{sf}\{D^Z_{u \in S^1}\}).$$

Using spectral sections, we also introduce a notion of Toeplitz family, extending the usual concept of Toeplitz operator to the fibration case. Namely let $g\colon X \to GL(N, \mathbb{C})$ be a smooth map. Then g can be viewed as an automorphism of the trivial complex vector bundle $\mathbb{C}^N \to X$ over X. Thus for any $b \in B$, g induces a bounded map g_b from $L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N)$ to itself by acting as an identity on $L^2(S(TZ_b) \otimes E_b)$.

Also, for the spectral section P, P_b induces a map on $L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N)$ by acting as identity on $\mathbb{C}^N$, denoted again by P_b. Then the Toeplitz family $T_g = \{T_{g,b}\}_{b \in B}$ is a family of Toeplitz operators defined by

$$T_{g,b} = P_b g_b : P_b L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N) \to P_b L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N).$$

The Chern character of its index bundle is expressed in terms of topological data as follows.

THEOREM 0.4. *The following identity holds,*

$$\mathrm{ch}(\mathrm{ind}\, T_g) = -\int_Z \hat{A}(TZ)\,\mathrm{ch}(E)\,\mathrm{ch}(g) \qquad \text{in } H^*(B),$$

where $\mathrm{ch}(g)$ *is the odd Chern character associated to* g (*cf.* [*AH*]).

Just as the index of Toeplitz operators can be expressed via spectral flow, we interpret the index bundle of a Toeplitz family via higher spectral flow. This generalizes a result of Booss–Wojciechowski [BW, Theorem 17.17].

THEOREM 0.5. *We have the following identity in* $K(B)$,

$$\mathrm{ind}\, T_g = \mathrm{sf}\{D^E, gD^E g^{-1}\},$$

A particularly nice application of this result is a local version of Theorem 0.4, which can be viewed as an odd analogue of the Bismut local index theorem for a family of Dirac operators on even dimensional manifolds.

We will also use higher spectral flow to prove a generalization of the family index theorem for manifolds with boundary [BC2], [BC3], [MP].

Let us mention that Douglas–Kaminker [DK] has also proposed a kind of higher spectral flow concept independently. On the other hand, Wu [W] has extended our concept of higher spectral flow to noncommutative geometry. In particular the non-commutative spectral sections have been used in the recent work of Leichtnam–Piazza [LP1], [LP2], [LP3] on the Higher Atiyah–Patodi–Singer Γ-index theorem, an extension of the Connes–Moscovici higher index theorem to Galois coverings with boundary. In their work, Leichtnam–Piazza also established an analogue of Theorem 0.1 in the nocommutative geometry.

This paper is organized as follows. In Section 1, we introduce the concept of higher spectral flow, and show that it is a generalization of the classical concept of spectral flow introduced in [APS1]. In Section 2 we explore the relationship between the Chern character of the higher spectral flow and the $\hat{\eta}$-forms. In Section 3, we consider periodic families and prove Theorem 0.2. In Section 4, we introduce the concept of Toeplitz family, give a proof of Theorem 0.5, as well as a heat kernel proof of Theorem 0.4. Finally in Section 5 we prove two results for family indices of manifolds with boundary, which generalize the results in [DZ1], [MP] respectively.

The results of this paper have been announced in [DZ3].

1. SPECTRAL SECTIONS AND HIGHER SPECTRAL FLOWS

In this section we will first recall the notion of spectral flow and its basic properties in (a). Then we will give, in (b), a new interpretation using the notion of spectral sections. This new interpretation is the basis for the higher spectral flow. At the end of the section, in (c), we extend this interpretation using a generalized notion of spectral section. This gives us the flexibility we need in proving Theorem 4.4.

(a) *Spectral Flow*

We take the definition of spectral flow as in [APS1]. Thus, if D_s, $0 \leqslant s \leqslant 1$, is a curve of self-adjoint Fredholm operators, the spectral flow $\mathrm{sf}\{D_s\}$ counts the net number of eigenvalues of D_s which change sign when s varies from 0 to 1. (Throughout the paper a family always means a *continuous* (and sometimes *smooth*) family, and a curve always means a one parameter family.) The following proposition collects some of its most basic properties from [APS1] and [APS2].

PROPOSITION 1.1. (1) *If D_s, $0 \leqslant s \leqslant 1$, is a curve of self-adjoint Fredholm operators, and $\tau \in [0, 1]$, then*

$$\mathrm{sf}\{D_s, [0, 1]\} = \mathrm{sf}\{D_s, [0, \tau]\} + \mathrm{sf}\{D_s, [\tau, 1]\}. \tag{1.1}$$

(2) *If D_s, $0 \leqslant s \leqslant 1$, is a smooth curve of self-adjoint elliptic pseudodifferential operators on a closed manifold, and $\bar{\eta}(D_s) = 1/2(\eta(D_s) + \dim \ker D_s)$ is the reduced η invariant of D_s in the sense of Atiyah–Patodi–Singer* [APS2], *then $\bar{\eta}$ is smooth mod $\mathbb{Z}$ and*

$$\mathrm{sf}\{D_s\} = -\int_0^1 \frac{d\bar{\eta}(D_s)}{ds}\, ds + \bar{\eta}(D_1) - \bar{\eta}(D_0). \tag{1.2}$$

(3) *If D_s, $0 \leqslant s \leqslant 1$, is a periodic one parameter family of self-adjoint Dirac type operators on a closed manifold, and $\tilde{D}$ is the corresponding Dirac type operator on the mapping torus, then*

$$\mathrm{sf}\{D_s\} = \mathrm{ind}\, \tilde{D}. \tag{1.3}$$

(b) *Spectral Sections*

The notion of spectral section is quite new and is introduced by Melrose–Piazza [MP] in their study of the family index of Dirac operators on manifolds with boundary. It can be viewed as a generalized Atiyah–Patodi–Singer boundary condition and can be defined for a family of self adjoint first order elliptic pseudodifferential operators.

DEFINITION 1.2. A spectral section for a family of first order elliptic pseudodifferential operators $D = \{D_z\}_{z \in B}$ is a family of self adjoint pseudodifferential projections P_z on the L^2-completion of the domain of D_z such that for some smooth function $R: B \to \mathbb{R}$ and every $z \in B$

$$D_z u = \lambda u \Rightarrow \begin{cases} P_z u = u & \text{if} \quad \lambda > R(z) \\ P_z u = 0 & \text{if} \quad \lambda < -R(z). \end{cases} \tag{1.4}$$

We list in the following the basic properties of spectral section. The reader is referred to [MP], [MP2] for detail.

PROPOSITION 1.3. *Let $D = \{D_s\}_{z \in B}$ be a family of first order elliptic pseudodifferential operators, and assume that the parameter space B is compact. Then*

(A) *There exists a spectral section for D if and only if the (analytic) index* [AS2] *of the family vanishes*: ind $D = 0$ *in* $K^1(B)$. *In particular, spectral sections for a one parameter family always exist.*

(B) *Given spectral sections P, Q, there exists a spectral section R such that $PR = R$ and $QR = R$. Such a spectral section will be called a majorizing spectral section.*

(C) *If R majorizes P*: $PR = R$, *then* $\ker\{P_b R_b : \mathrm{Im}(R_b) \to \mathrm{Im}(P_b)\}_{b \in B}$ *forms a vector bundle on B, denoted by $[R - P]$. Hence for any two spectral sections P, Q, the difference element $[P - Q]$ can be defined as an element in $K(B)$, as follows*:

$$[P - Q] = [R - Q] - [R - P], \tag{1.5}$$

for any majorizing spectral section R.

(D) *If P_1, P_2, P_3 are spectral sections, then*

$$[P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1]. \tag{1.6}$$

(E) *The K-group $K(B)$ is generated by all these difference elements.*

Now let D_s, $s \in [0, 1]$, be a curve of self adjoint elliptic pseudodifferential operators. Let Q_s be the spectral projection onto the direct sum of eigenspaces of D_s with nonnegative eigenvalues (the APS projection). The following theorem provides a link between the above two notions.

THEOREM 1.4. *Let P_s be a spectral section of D_s, $s \in [0, 1]$. Then $[Q_1 - P_1]$ defines an element of $K^0(pt) \cong \mathbb{Z}$ and so does $[Q_0 - P_0]$. Moreover the difference*

$[Q_1 - P_1] - [Q_0 - P_0]$ *is independent of the choice of the spectral section* P_s, *and it computes the spectral flow of* D_s:

$$\mathrm{sf}\{D_s\} = [Q_1 - P_1] - [Q_0 - P_0]. \tag{1.7}$$

Proof. We first show the independence of the choice of spectral section. Thus let P, Q' be two spectral sections and R be a majorizing spectral section. Since $[R - P]$ defines a vector bundle on $[0, 1]$, we have in particular $\dim[R_1 - P_1] = \dim[R_0 - P_0]$, and similarly, $\dim[R_1 - Q'_1] = \dim[R_0 - Q'_0]$. This, together with the basic property (1.6), gives us the independence. For (1.7), we note that both sides are additive with respect to the subdivision of the interval. Thus, by dividing into sufficiently small intervals if necessary, we can assume that there exists a positive real number a with $\pm a \notin \mathrm{spec}(D_s)$ for all s. Since we are free to choose any spectral section to compute the right hand side of (1.7), we choose P_s to be the orthogonal projection onto the direct sum of eigenspaces of D_s with eigenvalues greater than a. This reduces the problem to the finite dimensional case, where it can be easily verified. ∎

Theorem 1.4 leads us to the notion of higher spectral flow, when we consider higher dimensional families.

Let $\pi: X \to B$ be a smooth fibration with the typical fiber Z an odd dimensional closed manifold and B compact. As we mentioned before, a family of self adjoint elliptic pseudodifferential operators on Z, parametrized by B, will be called a B-family. Consider a curve of B-families, $D_u = \{D_{b,u}\}$, $u \in [0, 1]$.

Assuming that the index bundle of D_0 vanishes, the homotopy invariance of the index bundle then implies that the index bundle of each D_u vanishes. Let Q_0, Q_1 be spectral sections of D_0, D_1 respectively. If we consider the total family $\tilde{D} = \{D_{b,u}\}$ parametrized by $B \times I$, then there is a total spectral section $\tilde{P} = \{P_{b,u}\}$. Let P_u be the restriction of $\tilde{P}$ over $B \times \{u\}$.

DEFINITION 1.5. The (higher) spectral flow $\mathrm{sf}\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs (D_0, Q_0), (D_1, Q_1) is an element in $K(B)$ defined by

$$\mathrm{sf}\{(D_0, Q_0), (D_1, Q_1)\} = [Q_1 - P_1] - [Q_0 - P_0] \in K(B). \tag{1.8}$$

The definition is independent of the choice of the (total) spectral section $\tilde{P}$, as it follows again from the basic properties (1.5), (1.6). When D_u, $u \in S^1$ is a periodic family, we choose $Q_1 = Q_0$. In this case the spectral flow turns out to be independent of $Q_0 = Q_1$ and therefore defines an intrinsic invariant of the family, which we denote by $\mathrm{sf}\{D_u\}$.

Although this notion is defined generally, for the most part of this paper we are going to restrict our attention to B-families of Dirac type operators, defined as follows.

For simplicity we make the assumption that the vertical bundle $TZ \to X$ is spin and we fix a spin structure.[1]

Let g^{TZ} be a metric on TZ. We use $S(TZ)$ to denote the spinor bundle of TZ. Let (E, g^E) be a Hermitian vector bundle over X, and ∇^E a Hermitian connection on E.

For any $b \in B$, one has a canonically defined self-adjoint (twisted) Dirac operator

$$D_b^E\colon \Gamma((S(TZ) \otimes E)|_{Z_b}) \to \Gamma((S(TZ) \otimes E)|_{Z_b}). \tag{1.9}$$

This defines a smooth family of (standard, twisted) Dirac operators D^E over B.

DEFINITION 1.6. By a B-family of Dirac *type* operators on (X, E), we mean a smooth family of first order self adjoint differential operators $D = \{D_b\}_{b \in B}$ whose principal symbol is given by that of $\{D_b^E\}_{b \in B}$.

Thus, for any B-family of Dirac type operators $\{D_b\}_{b \in B}$, there is a natural homotopy between $\{D_b\}_{b \in B}$ and $\{D_b^E\}_{b \in B}$ through B-families of Dirac type operators given by

$$\tilde{D}_b(t) = (1 - t)\, D_b + t D_b^E. \tag{1.10}$$

Since the definition of higher spectral flow requires the existence of a spectral section, we now make the following

Basic Assumption 1.7. We assume that the (canonical) family D^E has vanishing index bundle.

A typical example satisfying our basic assumption is the family of signature operators on odd dimensional manifolds. More generally a B-family whose kernels have constant dimension will always satisfy the basic assumption. Another class of examples comes from the boundary family of a family of Dirac type operators on even dimensional manifolds with boundary.

Now we can speak of the higher spectral flow of a curve of B-families of Dirac type operators, given the basic assumption.

(c) *Generalized Spectral Sections*

The rest of this section is devoted to an extension of the notion of spectral section. This is what we call generalized spectral section. The higher spectral flow can also be defined in terms of generalized spectral sections. This will come into play later when we discuss Toeplitz families.

[1] Our discussion extends without difficulty to the more general case when there are smoothly varying $\mathbb{Z}_2$-graded Hermitian Clifford modules over the fibers, with graded unitary connections.

DEFINITION 1.8 [DZ2]. A generalized spectral section Q of a family of self adjoint first order elliptic pseudodifferential operators $D=\{D_z\}_{z\in B}$ is a continuous family of self-adjoint zeroth order pseudodifferential projections $Q=\{Q_z\}_{z\in B}$ whose principal symbol is the same as that of a (in fact any) spectral section P.

Remark 1.9. There are actually many equivalent definitions of generalized spectral sections. For example one can use the language of infinite dimensional Grassmannian discussed in [BW]. Also one can use the infinite dimensional Lagrangian subspaces as used by Nicolaescu in [N]. One of the most important examples of generalized spectral sections is the Caldéron projection (cf. [BW]). It has been used in an essential way in our proof of the splitting formula for family indices [DZ2].

As we have seen, the crucial ingredient in the definition of higher spectral flow is the difference element of two spectral sections. This notion generalizes to generalized spectral sections.

Thus let Q_1, Q_2 be two generalized spectral sections of D. For any $z\in B$, set

$$T_z(Q_1, Q_2)=Q_{2,z}Q_{1,z}: \mathrm{Im}(Q_1)\to \mathrm{Im}(Q_2). \tag{1.11}$$

Then $T(Q_1, Q_2)=\{T_z(Q_1, Q_2)\}_{z\in B}$ defines a continuous family of Fredholm operators over B. Thus according to Atiyah-Singer [AS1], it determines an element

$$[Q_1-Q_2]=\mathrm{ind}\, T(Q_1, Q_2)\in K(B). \tag{1.12}$$

In the special case where Q_1 and Q_2 are two spectral sections, one verifies easily that $[Q_1-Q_2]$ is the same as the difference element defined by Melrose–Piazza [MP].

Two generalized spectral sections Q_1, Q_2 of D are said to be homotopic to each other if there is a continuous curve of generalized spectral sections P_u, $0\leqslant u\leqslant 1$, of D such that $Q_1=P_0$, $Q_2=P_1$.

PROPOSITION 1.10. (1) *If Q_i, $i=1, 2, 3$, are three generalized spectral sections of D such that Q_1 and Q_2 are homotopic to each other, then one has*

$$[Q_1-Q_3]=[Q_2-Q_3] \qquad \text{in } K(B). \tag{1.13}$$

(2) *If Q_i, $i=1, 2, 3$, are three generalized spectral sections of D, then the following identity holds in $K(B)$,*

$$[Q_1-Q_2]+[Q_2-Q_3]=[Q_1-Q_3]. \tag{1.14}$$

Proof. These follow easily from some elementary arguments concerning Fredholm families. ∎

The generalized spectral sections are more flexible than the spectral sections since they only have to have the right symbol. On the other hand, we still have

THEOREM 1.11. *The higher spectral flow* (1.8) *can be computed using generalized spectral sections. That is, we can choose* $\tilde{P}$ *to be a generalized spectral section of* $\tilde{D}$.

Proof. The point here is that the right hand side of (1.8) is also independent of the choice of generalized spectral sections. To see this, let $\tilde{P}$ and $\tilde{P}'$ be two generalized spectral sections of the total family $\tilde{D}$ parametrized by $B \times I$. Then $T(P_u, P'_u)$ defines a curve of Fredholm families. By the homotopy invariance of the family index, we have

$$[P_1 - P'_1] = \operatorname{ind} T(P_1, P'_1) = \operatorname{ind} T(P_0, P'_0) = [P_0 - P'_0]. \quad ∎$$

Remark 1.12. Given generalized spectral sections P, Q, there exists a generalized spectral section R such that

$$(1-Q)\, L^2(Z, S(TZ) \otimes E|_Z) \subset (1-R)\, L^2(Z, S(TZ) \otimes E|_Z)$$

and that the family of Fredholm operators $T(1-R, 1-P)$ has vanishing cokernels, which implies that the family $T(P, R)$ has vanishing cokernels. This can be obtained by applying the procedure in [AS1] to the family of Fredholm operators

$$T(1-Q, 1-P)\colon (1-Q)\, L^2(Z, S(TZ) \otimes E|_z) \to (1-P)\, L^2(Z, S(TZ) \otimes E|_z).$$

The generalized spectral section R is an analogue of a majorizing spectral section.

2. $\hat{\eta}$-FORM AND THE CHERN CHARACTER OF HIGHER SPECTRAL FLOWS

In this section we show that the Chern character of the higher spectral flow can be calculated by the heat kernel methods through $\hat{\eta}$-forms. For simplicity we will restrict ourselves to Dirac type operators while at the same time we allow the freedom to use the superconnections generalizing the Bismut superconnection [B]. In this sense the $\hat{\eta}$-form of this section can be seen as certain generalizations of those of Bismut–Cheeger [BC1] and Melrose–Piazza [MP].

This section is organized as follows. In (a), we recall some basic results of Melrose–Piazza [MP]. In (b), we introduce a generalized superconnection. In (c), we define the $\hat{\eta}$-forms and prove certain basic properties. In (d), we prove a relative formula generalizing a result of [MP]. Finally in (e), we prove the main result of this section which expresses the Chern character of higher spectral flow via $\hat{\eta}$-forms.

(a) *Melrose–Piazza Operator of a Spectral Section*

Now, let $D=\{D_b\}_{b\in B}$ be a B-family of self adjoint Dirac type operators as defined in Section 1, and let $P=\{P_b\}_{b\in B}$ be a spectral section of $\{D_b\}_{b\in B}$. Then by [MP] there exists a family of zeroth order finite rank pseudodifferential operators $\{A_b\}_{b\in B}$,

$$A_b\colon \Gamma((S(TZ)\otimes E)|_{Z_b})\to \Gamma((S(TZ)\otimes E)|_{Z_b}),$$

such that, for any $b\in B$,

(i) $\tilde{D}_b=D_b+A_b$ is invertible;

(ii) P_b is precisely the Atiyah–Patodi–Singer projection [APS2] of $\tilde{D}_b$.

DEFINITION 2.1. We call this $A=\{A_b\}_{b\in B}$ a Melrose–Piazza operator associated to the spectral section P.

(b) *Superconnections Associated to Dirac Type Families*

We now choose a connection for the fibration which amounts to a splitting

$$TX=TZ\otimes T^HX. \tag{2.1}$$

We also have the identification $T^HX=\pi^*TB$.

Endow B with a metric g^{TB} and let g^{TX} be the metric defined by

$$g^{TX}=g^{TZ}\otimes \pi^*g^{TB}. \tag{2.2}$$

Let P, $P^\perp$ be the orthogonal projections of TX onto TZ, T^HX respectively and denote by ∇^{TX}, ∇^{TB} the Levi–Civita connections of g^{TX}, g^{TB} respectively. Following Bismut [B], let ∇^{TZ} be the connection on the vertical bundle defined by $\nabla^{TZ}=P\,\nabla^{TX}P$. This is a connection compatible with the metric g^{TZ} and is independent of the choice of the metric g^{TB}.

Then the connection lifts to a connection on the spinor bundle. Also following Bismut we view $\Gamma(S(TZ)\otimes E)$ as the space of sections of an infinite dimensional vector bundle H_∞ over B, with fiber

$$H_{\infty,b}=\Gamma(S(TZ_b)\otimes E_b). \tag{2.3}$$

Then $\nabla^{S(TZ)\otimes E}$ determines a connection on H_∞ by the prescription:

$$\tilde{\nabla}_X h = \nabla_{X^H} h, \tag{2.4}$$

where $Y^H \in \Gamma(T^H X)$ is the horizontal lift of $Y \in \Gamma(TB)$.

Now for any $1 \leqslant i \leqslant \dim B$, let

$$B_i \in \Omega^i(T^*B) \mathbin{\hat{\otimes}} \Gamma(\mathrm{End}(S(TZ) \otimes E)) = \Omega^i(T^*B) \mathbin{\hat{\otimes}} \mathrm{cl}(TZ) \otimes \mathrm{End}(E)$$

be an odd element (with respect to the natural $\mathbb{Z}_2$-grading). And let A be a Melrose–Piazza operator associated to the spectral section P. Furthermore, choose a cut-off function ρ in t: $\rho(t) = 1$ when $t > 8$ and $\rho(t) = 0$ when $t < 2$.

Definition 2.2. For any $t > 0$, the (rescaled) superconnection B_t is a superconnection on H_∞ given by

$$B_t = \tilde{\nabla} + \sqrt{t}\,(D + \rho(t)\,A) + \sum_{i=1}^{\dim B} t^{(1-i)/2} B_i, \tag{2.5}$$

We also set $B = B_1 = \tilde{\nabla} + D + \sum_{i=1}^{\dim B} B_i$. (This should not be confused with the base manifold.)

(c) *$\hat{\eta}$-Functions and Their Residues*

We define, for $\Re(s) \gg 0$, the $\hat{\eta}$-function for the superconnection B_t as follows

$$\hat{\eta}(B, A, s) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{s/2} \mathrm{Tr}^{\mathrm{even}} \left[\frac{dB_t}{dt}\, e^{-B_t^2} \right] dt. \tag{2.6}$$

This defines an even differential form on B which depends holomorphically on s for $\Re(s) \gg 0$. By standard arguments it extends to a meromorphic function of s in the whole complex plane with only simple poles.

Remark 2.3. Note that our definition has an extra factor of $1/2$, coming from dB_t/dt, in comparing with those of [BC1], [MP].

Proposition 2.4. *The residue of $\hat{\eta}(B, A, s)$ at $s = 0$ is an exact form.*

Proof. Our proof consists of two steps. We first show that, modulo exact forms, $\mathrm{Res}_{s=0}\{\hat{\eta}(B, A, s)\}$ is invariant under the smooth deformation of Dirac type families. The main ingredient here is the transgression formula (2.8).

But note first of all, since $\rho(t) = 0$ for $t < 1$, $\mathrm{Res}_{s=0}\{\hat{\eta}(B, A, s)\}$ does not depend on either A or P.

Now let $\{D_u\}_{u\in[0,1]}$ be a smooth curve of B-families $D_u=\{D_{b,u}\}_{b\in B}$. We view $\widetilde{D}=\{D_{b,u}\}_{b\in B,\,u\in[0,1]}$ as a $B\times I$-family of Dirac type operators. Note that this family also verifies the basic assumption.

Let $\widetilde{P}$ be a spectral section of $\widetilde{D}$ and $\widetilde{A}$ a Melrose–Piazza operator for $\widetilde{P}$. By pulling back B_i to an odd element $\widetilde{B}_i$ on $B\times I$, we obtain a superconnection $\widetilde{B}_t$ on $B\times I$:

$$\widetilde{B}_t=\frac{\partial}{\partial u}\,du+\widetilde{\nabla}+\sqrt{t}\,(\widetilde{D}+\rho(t)\,\widetilde{A})+\sum_{i=1}^{\dim B} t^{(1-i)/2}\widetilde{B}_i. \tag{2.7}$$

We now proceed as in [Q], [B]. Namely, from the fact that $e^{-((\partial/\partial t)\,dt+\widetilde{B}_t)^2}$ is $(\partial/\partial t)\,dt+(\partial/\partial u)\,du+d_B$ closed, and equating the dt component, we deduce

$$\frac{\partial}{\partial t}\operatorname{Tr}^{\mathrm{odd}}[\exp(-\widetilde{B}_t^2)]=\left(\frac{\partial}{\partial u}\,du+d_B\right)\operatorname{Tr}^{\mathrm{even}}\left[\frac{\partial\widetilde{B}_t}{\partial t}\exp(-\widetilde{B}_t^2)\right]. \tag{2.8}$$

Therefore, for $\Re(s)\gg 0$,

$$\begin{aligned}\left(\frac{\partial}{\partial u}\,du+d_B\right)&\int_0^\infty t^{s/2}\operatorname{Tr}^{\mathrm{even}}\left[\frac{\partial\widetilde{B}_t}{\partial t}\exp(-\widetilde{B}_t^2)\right]dt\\&=-\frac{s}{2}\int_0^\infty t^{s/2-1}\operatorname{Tr}^{\mathrm{odd}}[\exp(-\widetilde{B}_t^2)]\,dt,\end{aligned} \tag{2.9}$$

where we have used the fact that $\widetilde{D}+\widetilde{A}$ is invertible.

Now when $\dim Z$ is odd, by proceeding as in [B], [BF], one has the following small time asymptotics

$$\operatorname{Tr}^{\mathrm{odd}}[\exp(-\widetilde{B}_t^2)]=\frac{a_{-k}}{t^k}+\frac{a_{-k+1}}{t^{k-1}}+\cdots+a_0+O(t) \tag{2.10}$$

for some $k\in\mathbb{Z}$ as $t\to 0$. This gives

$$\operatorname{Res}_{s=0}\left\{-\frac{s}{2}\int_0^\infty t^{s/2-1}\operatorname{Tr}^{\mathrm{odd}}[\exp(-\widetilde{B}_t^2)]\,dt\right\}=0. \tag{2.11}$$

For any $u\in[0,1]$, set

$$B(t,u)=\widetilde{\nabla}+\sqrt{t}\,(D_{b,u}+\rho(t)\,A_{t,u})+\sum_{i=1}^{\dim B} t^{(1-i)/2}B_{i,u}. \tag{2.12}$$

Then

$$\widetilde{B}_t^2=\left(\frac{\partial}{\partial u}\,du+B(t,u)\right)^2=B(t,u)^2+du\,\frac{\partial B(t,u)}{\partial u}. \tag{2.13}$$

Thus

$$
\begin{aligned}
\mathrm{Tr}^{\mathrm{even}}&\left[\frac{\partial \widetilde{B}_t}{\partial t}\exp(-\widetilde{B}_t^2)\right]\\
&=\mathrm{Tr}^{\mathrm{even}}\left[\frac{\partial B(t,u)}{\partial t}\exp\Big(-B(t,u)^2-du\,\frac{\partial B(t,u)}{\partial u}\Big)\right]\\
&=\mathrm{Tr}^{\mathrm{even}}\left[\frac{\partial B(t,u)}{\partial t}\exp(-B(t,u)^2)\right]\\
&\quad+\frac{\partial}{\partial \tau}\left\{\mathrm{Tr}^{\mathrm{even}}\left[\frac{\partial B(t,u)}{\partial t}\exp\left(-B(t,u)^2-\tau\,\frac{\partial B(t,u)}{\partial u}\,du\right)\right]\right\}\Big|_{\tau=0}.
\end{aligned}
\tag{2.14}
$$

From (2.14), (2.11) (2.9) and (2.6), one deduces that

$$
\begin{aligned}
&\mathrm{Res}_{s=0}\left\{\frac{\partial\hat{\eta}(B_u, A_u, s)}{\partial u}\right\}du\\
&\quad+d_B\left\{\frac{1}{\sqrt{\pi}}\,\mathrm{Res}_{s=0}\left\{\int_0^\infty t^{s/2}\,\frac{\partial}{\partial\tau}\left\{\mathrm{Tr}^{\mathrm{even}}\left[\frac{\partial B(t,u)}{\partial t}\exp\left(-B(t,u)^2-\tau\right.\right.\right.\right.\right.\\
&\quad\left.\left.\left.\left.\left.\times\frac{\partial B(t,u)}{\partial u}\,du\right)\right]\right\}\Big|_{\tau=0}\right\}\right\}=0.
\end{aligned}
\tag{2.15}
$$

Therefore

$$
\mathrm{Res}_{s=0}\left\{\frac{\partial\hat{\eta}(B_u, A_u, s)}{\partial u}\right\}\in d\Omega(B). \tag{2.16}
$$

With this invariance property at hand, we can now easily finish our proof. Let D^E be the B-family of (standard) Dirac operators, and B^E the Bismut superconnection. Then by [BC1], [MP],

$$
\mathrm{Res}_{s=0}\{\hat{\eta}(B^E, A_1, s)\}=0, \tag{2.17}
$$

for any Melrose–Piazza operator A_1 of D^E. Applying the invariance property to the curve of B-families $D_u=(1-u)\,D+uD^E$ yields the desired result. ▮

(d) $\hat{\eta}$-*Forms*

With the help of Proposition 2.4 we can now define the $\hat{\eta}$-forms.

DEFINITION 2.5. The $\hat{\eta}$-form (associated to B and A) is defined as

$$\hat{\eta}(B, A) = \left\{ \hat{\eta}(B, A, s) - \frac{\mathrm{Res}_{s=0}\{\hat{\eta}(B, A, s)\}}{s} \right\}_{s=0}. \tag{2.18}$$

Remark 2.6. The $\hat{\eta}$-form for a general superconnection is defined in [BC1] in the finite dimensional case.

A variational argument shows

PROPOSITION 2.7. *The value of $\hat{\eta}(B, A)$ in $\Omega^*(B)/d\Omega^*(B)$ is independent of the choice of the cut-off function ρ and the Melrose–Piazza operator A.*

Proof. If $\gamma(t)$ is another cut-off function, we will set $\rho_u(t) = (1-u)\,\rho(t) + u\gamma(t)$.

Also, if A' is another Melrose–Piazza operators, by [MP], there is a smooth family $A(u)$ of Melrose–Piazza operators such that $A(0) = A$, $A(1) = A'$.

Then the rest of the proof follows along the same line as that in the proof of Proposition 2.4. Namely, the variation of the $\hat{\eta}$-forms is, up to an exact form, given by a local term (Cf. formula (2.24) below). This local term is the constant term in the asymptotic expansion (2.22). Since $\rho(t) = 0$ for $t \leqslant 2$ (and all u), and D is independent of u, this term is zero. ∎

This enables us to give an intrinsic definition.

DEFINITION 2.8. Given a spectral section P and a superconnection B, the $\hat{\eta}$-form $\hat{\eta}(D, P)$ is defined to be an element of $\Omega^*(B)/d\Omega^*(B)$ determined by $\hat{\eta}(B, A)$ for any choice of A.

(e) *Difference Element and $\hat{\eta}$-Forms*

The dependence of $\hat{\eta}$-form on the spectral section is also well understood. The following is a slight generalization of a result of [MP].

THEOREM 2.9. *If P_0 and P_1 are spectral sections of the family D, then*

$$\mathrm{ch}(P_1 - P_0) = \hat{\eta}(D, P_1) - \hat{\eta}(D, P_0) \qquad \text{in } H^*(B). \tag{2.19}$$

Proof. We note first that if we fix the family D, the spectral section P, and vary the other terms in the superconnection, the variation of the $\hat{\eta}$-form will be given by a local term, which is independent of the spectral section. This shows that the right hand side of (2.19) is independent of such variations. Now we can connect any superconnection to the Bismut superconnection through such a deformation. Since (2.19) is true for the Bismut superconnection by [MP], the proof is complete. ∎

(f) $\hat{\eta}$-*Forms and the Chern Character of Higher Spectral Flow*

Now consider a smooth curve of B-families of Dirac type operators $\tilde{D} = \{\tilde{D}_u\}_{u \in [0,1]}$. We view it as a total family over $B \times I$. Also, let $\tilde{P}$ be a total spectral section for $\tilde{D}$ and $\tilde{A}$ a corresponding Melrose–Piazza operator. As before, set

$$B(t, u) = \tilde{\nabla} + \sqrt{t}\,(D_u + \rho(t) A_u) + \sum_{i=1}^{\dim B} t^{(1-i)/2} B_{i,u}. \tag{2.20}$$

Proceeding as in [B], [BF], one verifies the following asymptotic expansion

$$\begin{aligned} &\frac{\partial}{\partial \tau} \left\{ \operatorname{Tr}^{\text{even}} \left[\exp\left(-B(t,u)^2) - \tau \frac{\partial B(t,u)}{\partial u}\right)\right] \right\}_{\tau=0} \\ &\quad = \frac{a_{-k}}{t^k} + \frac{a_{-k+1}}{t^{k-1}} + \cdots + a_0 + O(t) \end{aligned} \tag{2.21}$$

for some $k \in \mathbb{Z}$ as $t \to 0$. Here $a_i = a_i(u)$ are smooth forms on B.

THEOREM 2.10. *Let P_0, P_1 be the restrictions of $\tilde{P}$ at $u = 0, 1$ respectively. Then*

$$\hat{\eta}(B_1, P_1) - \hat{\eta}(B_0, P_0) = \frac{1}{\sqrt{\pi}} \int_0^1 a_0(u)\, du, \qquad \text{in } \Omega^*(B)/d\Omega^*(B). \tag{2.22}$$

Proof. From (2.13), (2.9), (2.14), one deduces

$$\begin{aligned} &\frac{\partial \hat{\eta}(B_u, A_u, s)}{\partial u}\, du + d_B \frac{1}{\sqrt{\pi}} \int_0^\infty t^{s/2} \\ &\quad \times \frac{\partial}{\partial \tau} \left\{ \operatorname{Tr}^{\text{even}} \left[\frac{\partial B_u(t)}{\partial t} \exp\left(-B_u(t)^2 + \tau \frac{\partial B_u(t)}{\partial t}\, du\right)\right] \right\}_{\tau=0} dt \\ &= -\frac{s}{2\sqrt{\pi}} \frac{\partial}{\partial \tau} \left\{ \int_0^\infty t^{s/2-1} \operatorname{Tr}^{\text{odd}} \left[\exp\left(-B_u(t)^2 - \tau \frac{\partial B_u(t)}{\partial u}\, du\right)\right] dt \right\}_{\tau=0}. \end{aligned} \tag{2.23}$$

We now let $s \to 0$ and equate the constant terms in the corresponding asymptotic expansions by standard methods, and then integrate from 0 to 1. ∎

Remark 2.11. The main point here is that the $\hat{\eta}$-forms $\hat{\eta}(B_u, P_u)$ is continuous in u. Thus, there is no "jump phenomena".

The following theorem generalizes the well known relationship between the spectral flow and the η invariant ([APS1]).

THEOREM 2.12. *Let D_u be a curve of B-families of Dirac type operators and Q_0, Q_1 spectral sections of D_0, D_1 respectively. Let $\tilde{B}_t(u) = \tilde{\nabla} + \sqrt{t}\,(D_u + \rho(t)\,\tilde{A}) + \sum_{i \geqslant 1} t^{(1-i)/2} B_i(u)$ be a curve of superconnections. Then we have the following identity in $H^*(B)$:*

$$\mathrm{ch}(\mathrm{sf}\{(D_0, Q_0), (D_1, Q_1)\}) = \hat{\eta}(D_1, Q_1) - \hat{\eta}(D_0, Q_0) - \frac{1}{\sqrt{\pi}} \int_0^1 a_0(u)\, du, \tag{2.24}$$

where $a_0(u)$ is determined by (2.21).

Proof. This follows from the definition of higher spectral flow, Theorems 2.9 and 2.10. ∎

Theorem 2.12 provides a way to compute, analytically, the Chern character of higher spectral flow. We will use formula (2.24) to compute the higher spectral flow of both a periodic family and a Toeplitz family.

3. PERIODIC FAMILY INDEX AND THE HIGHER SPECTRAL FLOW

In this section, we use the results of Section 2 to calculate the Chern character of the higher spectral flow of periodic families and show that it equals the Chern character of the family index of the associated families of Dirac operators on *even* dimensional manifolds.

This section is organized as follows. In (a), we introduce the periodic family and the associated higher spectral flow. In (b), we examine the relation between the Chern character of the higher spectral flow and of the family index.

(a) *Periodic Families*

By definition, a periodic B-family of self-adjoint Dirac type operators is just a closed curve of B-families of operators. Equivalently, one can also view it as a $u \in [0, 1]$ family of B-families of Dirac type operators, such that $D_0 = D_1$.

In this special case, if we give D_0, D_1 the same spectral section Q and let $\tilde{P}$ be a total spectral section over $B \times I$, then one has

$$\mathrm{sf}\{(D_0, Q), (D_1, Q)\} = [P_1 - Q] - [P_0 - Q] = [P_1 - P_0]. \tag{3.1}$$

Thus one has

PROPOSITION 3.1. *The higher spectral flow* $\mathrm{sf}\{(D_0, Q), (D_1, Q)\}$ *for the periodic family does not depend on* Q.

Therefore $\mathrm{sf}\{(D_0, Q), (D_1, Q)\}$ defines an intrinsic invariant of the periodic family. We will denote it by $\mathrm{sf}\{D\}$.

(b) *Relations of Higher Spectral Flow with Index Bundles for Periodic Families*

Now let $D = \{D_u\}_{u \in S^1}$ be a periodic family of B-families of (self-adjoint) Dirac type operators $\{D_{b,u}\}_{b \in B, u \in S^1}$. Then for any $b \in B$, $\{D_{b,u}\}_{u \in S^1}$ defines a natural Dirac operator D'_b on $S^1 \times Z_b$:

$$D'_b = \begin{pmatrix} 0 & \dfrac{\partial}{\partial u} + D_{b,u} \\ -\dfrac{\partial}{\partial u} + D_{b,u} & 0 \end{pmatrix}:$$

$$\Gamma(\mathbb{C}^2 \otimes S(TZ_b) \otimes E_b) \to \Gamma(\mathbb{C}^2 \otimes S(TZ_b) \otimes E_b), \tag{3.2}$$

where we have lift TZ_b, E_b canonically to $S^1 \times Z_b$ and made the identification that $S(T(S^1 \times Z_b)) = \mathbb{C}^2 \otimes S(TZ_b)$. $D' = \{D'_b\}_{b \in B}$ then forms a family of Dirac operators over B.

The connection $\tilde{\nabla}$ in (2.6) lifts naturally to $\Gamma(S(T(S^1 \times Z)) \otimes E)$, which we still denote by $\tilde{\nabla}$.

Then by the fundamental result of Bismut [B], for any $t > 0$, the following differential form is a representative of $\mathrm{ch}(D')$,

$$\alpha_t(b) = \mathrm{Tr}_s^{S^1 \times Z}[\exp(-(\tilde{\nabla} + \sqrt{t}\, D'_{b,u})^2)]. \tag{3.3}$$

Now for $\varepsilon > 0$, set

$$D'_{b,\varepsilon} = \begin{pmatrix} 0 & \sqrt{\varepsilon}\, \dfrac{\partial}{\partial u} + D_{b,u} \\ -\sqrt{\varepsilon}\, \dfrac{\partial}{\partial u} + Db, u & 0 \end{pmatrix} \tag{3.4}$$

Then for any $t > 0$, $\varepsilon > 0$, one clearly has that

$$\alpha_{t,\varepsilon}(b) = \mathrm{Tr}_s[\exp(-(\tilde{\nabla} + \sqrt{t}\, D'_{b,u,\varepsilon})^2)] \tag{3.5}$$

is also a representative of $\mathrm{ch}(D')$.

Now we take the adiabatic limit $\varepsilon \to 0$. By proceeding as in [BF, Section 2f] and [B] in a very simple situation, one gets

$$\beta_t(b) = \lim_{\varepsilon \to 0} \alpha_{t,\varepsilon}(b) = \frac{1}{\sqrt{\pi}} \int_{S^1} \mathrm{Tr}^{Z,\,\mathrm{odd}} \left[\exp\left(-\left(du \frac{\partial}{\partial u} + \tilde{\nabla} + \sqrt{t}\, D_{b,u} \right)^2 \right) \right], \tag{3.6}$$

which is also a representative of $\mathrm{ch}(D')$ for any $t > 0$.

One verifies

$$\left(du \frac{\partial}{\partial u} + \tilde{\nabla} + \sqrt{t}\, D_{b,u} \right)^2 = (\tilde{\nabla} + \sqrt{t}\, D_{b,u})^2 + \sqrt{t}\, du \frac{\partial D_{b,u}}{\partial u}. \tag{3.7}$$

From (3.6) and (3.7), one deduces that

$$\begin{aligned} \beta_t(b) &= \frac{1}{\sqrt{\pi}} \int_{S^1} \mathrm{Tr}^{\mathrm{odd}} \left[\exp\left(-(\tilde{\nabla} + \sqrt{t}\, D_{b,u})^2 - \sqrt{t}\, du \frac{\partial D_{b,u}}{\partial u} \right) \right] \\ &= -\frac{1}{\sqrt{\pi}} \int_{S^1} \frac{\partial}{\partial s} \left\{ \mathrm{Tr}^{\mathrm{even}} \left[\exp\left(-(\tilde{\nabla} + \sqrt{t}\, D_{b,u})^2 - s\sqrt{t} \frac{\partial D_{b,u}}{\partial u} \right) \right] \right\}_{s=0} du. \end{aligned} \tag{3.8}$$

Now since we have a periodic family, and also that we have taken all $B_i = 0$ $(i \geqslant 1)$ in the superconnection formalism, we see that the $\hat{\eta}$-forms at the end points (or the same point) are the same, so they cancel in (2.24).

Comparing (3.8) with (2.21), (2.24), we get finally the following result.

THEOREM 3.2. *The following identity holds in $H^*(B)$,*

$$\mathrm{ch}(\mathrm{ind}\, D') = \mathrm{ch}(\mathrm{sf}\{D\}). \tag{3.9}$$

Remark 3.3. Theorem 3.2 generalizes the well-known result of Atiyah–Patodi–Singer [APS1] to higher dimensions.

Remark 3.4. It was pointed out in ([DZ3]) that (3.9) should still hold on the K-theoretic level. That is, one should have

$$\mathrm{ind}\, D' = \mathrm{sf}\{D\} \qquad \text{in } K(B). \tag{3.10}$$

Fangbing Wu [W] has since then proven this true.

Now if B is also spin, for any $u \in S^1$, one can define the total Dirac type operator D_u^X on X. Then one has also a spectral flow $\mathrm{sf}\{D^X\}$. The following result is a relation between this spectral flow and the higher spectral flow of the fibered Dirac operators.

COROLLARY 3.5. *The following identity holds,*

$$\mathrm{sf}\{D^X_{u\in S^1}\} = \int_B \hat{A}(TB)\,\mathrm{ch}(\mathrm{sf}\{D^Z_{u\in S^1}\}), \tag{3.11}$$

where now the Chern character is the usual Chern character, not the normalized one (*i.e. here one has the factor* $2\pi\sqrt{-1}$).

Remark 3.6. The proof of Theorem 3.2 above is not the same as the one indicated in [DZ3]. To pursue the proof in [DZ3], one first reduces the problem to the case of (standard) Dirac operators family and then uses the Bismut superconnection formalism to evaluate the Chern character. Since we will encounter the Bismut superconnection in the next section, we think it is better to avoid it in this section.

4. TOEPLITZ FAMILIES AND HIGHER SPECTRAL FLOWS: A HEAT KERNEL COMPUTATION FOR THE INDEX BUNDLE OF TOEPLITZ FAMILIES

In this section we introduce what we call Toeplitz families which extend the usual concept of Toeplitz operators to fibration case. Just as the index of Toeplitz operators can be expressed via spectral flow, we will interpret the index bundle of a Toeplitz family via higher spectral flow. This in turn allows us to give an evaluation of the Chern character of the index bundle of Toeplitz families via $\hat{\eta}$-forms. In particular, when the operators are of Dirac type, we get an explicit local index computation of the Chern character, which can be viewed as an odd analogue of the Bismut local index theorem [B] for families of Dirac operators on even dimensional manifolds. Besides the Bismut superconnection, which should be involved naturally here in our odd analogue, we will also use a conjugation of it. See (d) and (e) for details.

This section is organized as follows. In (a), we introduce what we call a Toeplitz family and give the formula for the Chern character of its index bundle. In (b), we establish a relationship between the index bundle of Toeplitz families and a natually associated higher spectral flow. In (c), we give a heat kernel formula computing the Chern character of the index bundle of Toeplitz families. In (d), we introduce a conjugate form of the Bismut superconnection [B] and finally in (e), we give the local index evaluation of the Chern character of the index bundle of Toeplitz families.

(a) *Toeplitz Families*

We consider the same geometric objects as in Sections 1 and 2. That is, we have a fibration $Z \to M \xrightarrow{\pi} B$ with compact closed odd dimensional fibers and compact base B. We assume the vertical bundle $TZ \to M$ is spin and carries a fixed spin structure. $E \to M$ a complex vector bundle with a Hermitian metric g^E and a Hermitian connection ∇^E. Also TZ has a metric g^{TZ}. So we have a canonically defined B-family of self-adjoint (twisted) Dirac operators $D^E = \{D_b^E\}_{b \in B}$.

Recall that we have also made the basic assumption that the associated Atiyah–Singer index bundle

$$\operatorname{ind} D^E = 0 \qquad \text{in } K^1(B). \tag{4.1}$$

Let $D = \{D_b\}_{b \in B}$ be a smooth B-family of self-adjoint Dirac type operators (that is, for any $b \in B$, the symbol of D_b is the same as that of D_b^E). Recall that by Remark 1.6, the K^1-index of D also vanishes.

By the fundamental result of Melrose–Piazza [MP], which we already recalled in Section 1, there is then a spectral section $P = \{P_b\}_{b \in B}$ of the family D.

Now let g: $M \to GL(N, \mathbb{C})$ be a smooth map. Then g can be viewed as an automorphism of the trivial complex vector bundle $\mathbb{C}^N \to M$ over M. For any $b \in B$, g induces an automorphism of $\mathbb{C}^N \to Z_b$. Thus for any $b \in B$, g induces a bounded map g_b from $L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N)$ to itself by acting as an identity on $L^2(S(TZ_b) \otimes E_b)$.

Also it is clear that P_b induces a map on $L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N)$ by acting as identity on sections of $\mathbb{C}^N$. We still note this map by P_b.

Definition 4.1. The Toeplitz family $T_g = \{T_{g,b}\}_{b \in B}$ is a family of Toeplitz operators defined by

$$T_{g,b} = P_b g_b \colon P_b L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N) \to P_b L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N). \tag{4.2}$$

Clearly, T_g is a continuous family. In fact one can even make it a smooth family (cf. [MP]). Furthermore, since g is invertible, for any $b \in B$, $T_{g,b}$ is a Fredholm operator. Thus we get a family of Fredholm operators. By Atiyah–Singer [AS1] it then defines an index bundle

$$\operatorname{ind} T_g \in K(B). \tag{4.3}$$

This index bundle is independent of the choice of the spectral section. In fact, using a trick due to Baum and Douglas [BD], we set

$$\tilde{T}_{g,b} = I - P_b + P_b g_b P_b : L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N) \to L^2(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N). \tag{4.4}$$

Then by a simple calculation of symbol, we see that each $\tilde{T}_{g,b}$ is a zeroth order elliptic pseudodifferential operator. Thus $\tilde{T}_g = \{\tilde{T}_{g,b}\}_{b \in B}$ is a continuous family of elliptic pseudodifferential operators. Furthermore, we have obviously

$$\operatorname{ind} \tilde{T}_g = \operatorname{ind} T_g \qquad \text{in } K(B). \tag{4.5}$$

Now it is clear that a different choice of spectral section results in a finite rank perturbation of $\tilde{T}_g$. Hence the index bundle is independent of the choice of spectral section. Further, by the Atiyah–Singer family index theorem, we see that ind T_g is equal to a topological index bundle whose Chern character has the following very natural form.

THEOREM 4.2. *The following identity holds,*

$$\operatorname{ch}(\operatorname{ind} T_g) = -\int_Z \hat{A}(TZ)\operatorname{ch}(E)\operatorname{ch}(g) \qquad \text{in } H^*(B), \tag{4.6}$$

where ch(g) *is the odd Chern character associated to g (cf.* [AH]).

This formula can be proved in the same way as in [BD] where the case of single operator is discussed. In the rest of this section we will give a purely analytic proof of (4.6) via the heat kernel method.

(b) *Toeplitz Families and Higher Spectral Flows*

We make the same assumption and use the same notation as in (a). It is clear that each D_b can also be extended as an operator from $\Gamma(S(TZ_b) \otimes E_b \otimes \mathbb{C}^N \mid Z_b)$ to itself by acting as identity on $\mathbb{C}^N$.

Also, as $GL(N, \mathbb{C})$ is homotopically equivalent to $U(N)$, we assume, without loss of generality, $g \in U(N)$.

Set for any $b \in B$,

$$D_{g,b} = g_b D_b g_b^{-1}. \tag{4.7}$$

Since $g \in U(N)$, one verifies easily that $D_g = gD_g^{-1} = \{D_{g,b}\}_{b \in B}$ is also a B-family of self-adjoint Dirac type operators.

For any $0 \leqslant t \leqslant 1$, set

$$D_b(t) = (1-t)\, D_b + tD_{g,b}. \tag{4.8}$$

Then we get a natural curve $D(t)=\{D_b(t)\}_{b\in B}$, $t\in[0,1]$ of B-families of self-adjoint Dirac type operators.

Let P be a spectral section of D. Then clearly $gPg^{-1}=\{g_bP_bg_b^{-1}\}_{b\in B}$ is a spectral section of gDg^{-1}. And we have a naturally defined higher spectral flow

$$\mathrm{sf}\{(D,P),(gDg^{-1},gPg^{-1})\}\in K(B) \tag{4.9}$$

associated to the curve (4.8) (See Section 1).

Proposition 4.3. (i), $\mathrm{sf}\{(D,P),(gDg^{-1},gPg^{-1})\}$ *does not depend on* P; (ii), $\mathrm{sf}\{(D,P),(gDg^{-1},gPg^{-1})\}$ *depends only on the symbol of* D.

Proof. (i) Let Q be another spectral section of D. Then one verifies easily that

$$\begin{aligned}\mathrm{sf}\{(D,P),(gDg^{-1},gPg^{-1})\}-\mathrm{sf}\{(D,Q),(gDg^{-1},gQg^{-1})\}\\ =[gPg^{-1}-gQg^{-1}]-[P-Q]\in K(B).\end{aligned} \tag{4.10}$$

Now by Kuiper's theorem [K], as a map into the unitary group of a Hilbert space, g is homotopic to Id via unitary operators. Thus one has, by the homotopy invariance of the K-group elements, that

$$[gPg^{-1}-gQg^{-1}]=g[P-Q]\,g^{-1}=[P-Q] \qquad \text{in } K(B). \tag{4.11}$$

The conclusion of (i) follows from (4.10) and (4.11).

(ii) Let D' be another B-family of self-adjoint Dirac type operators with the same symbol as D. Take any spectral section R of D'. Since D, D' has the same symbol, a spectral section of D is a generalized spectral section of D' and vice versa. Furthermore, since for each $t\in[0,1]$, $D(t)$ defined in (4.8) have the same symbol as D, one deduces that

$$\begin{aligned}\mathrm{sf}\{(D,P),(gDg^{-1},gPg^{-1})\}-\mathrm{sf}\{(D,R),(gD'g^{-1},gRg^{-1})\}\\ =[gPg^{-1}-gRg^{-1}]-[P-R] \qquad \text{in } K(B),\end{aligned} \tag{4.12}$$

which vanishes by the same reason as in the proof of (i). ∎

Thus, what we get is an intrinsic invariant depending only on g and the symbol of D^E. We will denote it by $\mathrm{sf}\{D^E,gD^Eg^{-1}\}$.

We now state the main result of this subsection, which can be seen as an extension to the fibration case of the classical result of Booss–Wojciechowski (cf. [BW, Theorem 17.17]).

THEOREM 4.4. *The following identity holds in $K(B)$,*

$$\operatorname{ind} T_g = \operatorname{sf}\{D^E, gD^E g^{-1}\}. \tag{4.13}$$

Proof. The proof of (4.13) is already implicitly contained in the proof of (ii) of Proposition 4.3. The main observation is that for any $t \in [0, 1]$, the symbol of $D(t)$ is the same as that of D^E. Thus P is a generalized spectral section in the sense of Definition 1.8 for all $D(t)$, $0 \leqslant t \leqslant 1$. By the (modified) definition of higher spectral flow, one then gets

$$\operatorname{sf}\{D^E, gD^E g^{-1}\} = [gPg^{-1} - P] = \operatorname{ind} T_g \in K(B). \tag{4.14}$$

The proof of Theorem 4.4 is completed. ∎

(c) *A Heat Kernel Formula for he Chern Character of the Index Bundle of Toeplitz Families*

We combine Theorem 4.4 and Theorem 2.12 to give a heat kernel formula for $\operatorname{ind} T_g$. This amounts first to introduce as in Section 2 an orthogonal splitting

$$\begin{aligned} TM &= TZ \oplus T^H M, \\ g^{TM} &= g^{TZ} \oplus \pi^* g^{TB}. \end{aligned} \tag{4.15}$$

We now follow the same strategy as in Section 2 by taking $D_1 = gDg^{-1}$, and consider the higher spectral flow $\operatorname{sf}\{(D, P), (gDg^{-1}, gPg^{-1})\}$.

For our special situation, we make the assumption that in the notation of (2.21), one has

$$B(t, 1) = gB(t, 0)\, g^{-1}. \tag{4.16}$$

This clearly can always be achieved.

By (4.16), one gets immediately

$$\hat{\eta}(B_1, gPg^{-1}) = \hat{\eta}(B_0, P). \tag{4.17}$$

Thus by Theorem 2.12, one gets

PROPOSITION 4.5. *The following identity holds in $\Omega(B)/d\Omega(B)$,*

$$\operatorname{ch}(\operatorname{ind} T_g) = -\frac{1}{\sqrt{\pi}} \int_0^1 a_0(u)\, du, \tag{4.18}$$

where $a_0(u)$ is the asymptotic term in (2.22) *subject to the condition* (4.16) *of the superconnection $B(t, u)$.*

Proof. By (4.17) and Theorem 2.12, one has

$$\mathrm{ch}(\mathrm{sf}\{D^E, gD^E g^{-1}\}) = -\frac{1}{\sqrt{\pi}} \int_0^1 a_0(u)\, du. \tag{4.19}$$

(4.18) follows from (4.19) and Theorem 4.4. ∎

(d) *A Conjugate Form of the Bismut Superconnection*

We use the same notation as in Section (2c). Also, following Bismut [B], let S be the tensor defined by

$$\nabla^{TM} = \nabla^{TZ} + \pi^* \nabla^{TB} + S. \tag{4.20}$$

Clearly, the connection $\tilde{\nabla}$ defined in (2.6) extends canonically as a connection on $H_\infty \otimes \mathbb{C}^N$.

Let $e_1, ..., e_n$ be an orthonormal base of TZ. Set

$$k = -\tfrac{1}{2} \sum_1^n S(e_i)\, e_i, \tag{4.21}$$

and let $\tilde{\nabla}^u$ be the connection on H_∞ defined by

$$\tilde{\nabla}^u_Y = \tilde{\nabla}_Y + \langle k, Y^H \rangle, \qquad Y \in TB. \tag{4.22}$$

Then $\tilde{\nabla}^u$ is a unitary connection on $H_\infty \otimes \mathbb{C}^N$, and does not depend on g^{TB}.

Also let T be the torsion of $\nabla^{TZ} + \pi^* \nabla^{TB}$. Set

$$c(T) = -\sum_{\alpha < \beta} \langle T(f_\alpha, f_\beta), e_i \rangle\, c(e_i)\, dy^\alpha\, dy^\beta, \tag{4.23}$$

where $\{f_\alpha\}$ is an orthonormal base of TB and $\{dy^\alpha\}$ the dual base of T^*B (We identify TB with $\pi^* TB$).

Definition 4.6 [B]. The Bismut superconnection $B(0)$ is defined by

$$B(0) = \tilde{\nabla}^u + D^E - \frac{c(T)}{4}. \tag{4.24}$$

The rescaled Bismut superconnection is then given by

$$B_t(0) = \tilde{\nabla}^u + \sqrt{t}\, D^E - \frac{c(T)}{4\sqrt{t}}. \tag{4.25}$$

Recall that g is unitary as in (b).

DEFINITION 4.7. The g-conjugate Bismut superconnection $B(1)$ is defined by

$$B(1) = gB(0)\, g^{-1}. \tag{4.26}$$

The rescaled g-conjugate Bismut superconnection $B_t(1)$ is then given by

$$B_t(1) = gB_t(0)\, g^{-1}. \tag{4.27}$$

For any $u \in [0, 1]$, set

$$B_t(u) = (1-u)\, B_t(0) + uB_t(1). \tag{4.28}$$

We will use $B_t(u)$ to calculate the asymptotic term $a_0(u)$ in (4.18). The result will be stated in Theorem 4.8, which implies an explicit evaluation of $\mathrm{ch}(\mathrm{ind}\, T_g)$.

(e) *A Local Formula for the Chern Character of the Index Bundle of Toeplitz Families*

We will use the superconnections in (d) to calculate the asymptotic term $a_0(u)$ in (4.18).

Recall that $a_0(u)$ is defined in (2.22) by an asymptotic formula. In using notation in (d),

$$\begin{aligned} &\frac{\partial}{\partial b} \left\{ \mathrm{Tr}^{\mathrm{even}} \left[\exp\left(-B_t^2(u) - b\, \frac{\partial B_t(u)}{\partial u} \right) \right] \right\}_{b=0} \\ &\quad = \frac{a_{-k}}{t^k} + \cdots + \frac{a_{-1}}{t} + a_0 + o(1), \qquad t \to 0. \end{aligned} \tag{4.29}$$

The following is the main result of this section.

THEOREM 4.8. *The following identity holds,*

$$\begin{aligned} &\lim_{t \to 0} \mathrm{Tr}^{\mathrm{even}} \left[\frac{\partial B_t(u)}{\partial u} \exp(-B_t^2(u)) \right] \\ &\quad = -\left(\frac{1}{2\pi\sqrt{-1}} \right)^{(\dim Z + 1)/2} \sqrt{\pi} \int_Z \hat{A}(R^{TZ})\, \mathrm{Tr}[\exp(-R^E)] \\ &\qquad \times \mathrm{Tr}[g^{-1} dg \exp((u-u^2)(g^{-1}dg)^2))], \end{aligned} \tag{4.30}$$

where R^{TZ} is the curvature of ∇^{TZ} and R^E is the curvature of ∇^E.

Proof. By (4.27), one has

$$B_t(1) = gB_t(0)\, g^{-1} = B_t(0) + g[B_t(0),\, g^{-1}]. \tag{4.31}$$

Thus for any $u > 0$,

$$B_t(u) = B_t(0) + ug[B_t(0),\, g^{-1}] \tag{4.32}$$

and

$$\frac{\partial B_t(u)}{\partial u} = g[B_t(0),\, g^{-1}]. \tag{4.33}$$

Also one verifies easily that

$$\begin{aligned} g[B_t(0),\, g^{-1}] &= g\left[\tilde{\nabla}^u + \sqrt{t}\, D - \frac{c(T)}{4\sqrt{t}},\, g^{-1}\right] \\ &= g[\tilde{\nabla}^u + \sqrt{t}\, D,\, g^{-1}] \\ &= g \sum_\alpha dy^\alpha \left(\frac{\partial}{\partial y^\alpha}\, g^{-1}\right) + g \sum_i \sqrt{t}\, c(e_i) \left(\frac{\partial}{\partial e_i}\, g^{-1}\right) \\ &= -\sum_\alpha dy^\alpha \left(\frac{\partial}{\partial y^\alpha}\, g\right) g^{-1} - \sum_i \sqrt{t}\, c(e_i) \left(\frac{\partial}{\partial e_i}\, g\right) g^{-1}. \end{aligned} \tag{4.34}$$

Also from (4.32), one deduces that

$$\begin{aligned} B_t^2(u) &= B_t^2(0) + u[B_t(0),\, g[B_t(0),\, g^{-1}]] + u^2 g[B_t(0),\, g^{-1}]\, g[B_t(0),\, g^{-1}] \\ &= B_t^2(0) + u[B_t(0),\, g][B_t(0),\, g^{-1}] + ug[B_t(0),\, [B_t(0),\, g^{-1}]] \\ &\quad + u^2(g[B_t(0),\, g^{-1}])^2. \end{aligned} \tag{4.35}$$

Now one verifies by (4.34) that

$$\begin{aligned} [B_t(0),\, g] &= \sum_\alpha dy^\alpha \frac{\partial g}{\partial y^\alpha} + \sqrt{t} \sum_i c(e_i) \frac{\partial g}{\partial e_i} \\ &= -g[B_t(0),\, g^{-1}]\, g. \end{aligned} \tag{4.36}$$

If we make the formal change that $\sqrt{t}\, c(e_i) \to e_i \in T^*Z$, then one gets

$$[B_t(0),\, g^{-1}] \to dg^{-1} \tag{4.37}$$

and

$$[B_t(0),\, [B_t(0),\, g^{-1}]] \to d^2 g^{-1} = 0. \tag{4.38}$$

From (4.33), (4.35) to (4.38), and by proceeding as in [B], [BF], and [BGV, Chap. 10] the by now standard local index techniques, which can be adapted easily here in odd dimensions, one gets that as $t \to 0$, one has

$$\lim_{t\to 0} \operatorname{Tr}^{\text{even}}\left[\frac{\partial B_t(u)}{\partial u}\exp(-B_t^2(u))\right]$$
$$= \left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim Z+1)/2} \sqrt{\pi} \int_Z \hat{A}(R^{TZ}) \operatorname{Tr}[\exp(-R^E)]$$
$$\times \operatorname{Tr}[g\, dg^{-1} \exp((u-u^2)(g\, dg^{-1})^2)]. \tag{4.39}$$

Now clearly

$$g\, dg^{-1} = -(dg)\, g^{-1} = -g(g^{-1}\, dg)\, g^{-1}. \tag{4.40}$$

(4.30) follows from (4.39) and (4.40). ∎

From (4.30), one finds

$$\frac{1}{\sqrt{\pi}}\int_0^1 \lim_{t\to 0} \operatorname{Tr}^{\text{even}}\left[\frac{\partial B_t(u)}{\partial u}\exp(-B_t^2(u))\right] du$$
$$= -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim Z+1)/2} \int_Z \hat{A}(R^{TZ}) \operatorname{Tr}[\exp(-R^E)]$$
$$\times \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \operatorname{Tr}[(g^{-1}\, dg)^{2n+1}]. \tag{4.41}$$

We can now state the main result of this section, which is a local form of Theorem 4.2.

THEOREM 4.9. *The following differential form on* B,

$$-\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim Z+1)/2} \int_Z \hat{A}(R^{TZ}) \operatorname{Tr}[\exp(-R^E)]$$
$$\times \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \operatorname{Tr}[(g^{-1}dg)^{2n+1}], \tag{4.42}$$

is closed and represents $\operatorname{ch}(\operatorname{ind} T_g)$ *in* $H^*(B)$.

Remark 4.10. In the definition of Toeplitz families, one can in fact replace g by any automorphism of an arbitary vector bundle over M. From the topological point of view, however, the trivial bundle is sufficiently general, for $H^{\text{odd}}(M)$ is generated by the odd Chern characters of automorphisms of trivial vector bundles (cf. [AH]).

5. HIGHER SPECTRAL FLOWS AND FAMILY INDICES FOR MANIFOLDS WITH BOUNDARY

In this section we use the idea of higher spectral flow to study the index theory for families of manifolds with boundary. Among the two main results of this section, one generalizes an earlier result in [DZ1, Theorem 1.1] to family case. It characterizes the changes of family indices under continuous deformations of Dirac type operators. The other one extends the Melrose–Piazza index theorem [MP] to the general superconnection case, not necessary restricted to the Bismut superconnections. Of course then, the local index density can not be directly identified with the characteristic forms.

This section is organized as follows. In (a), we define what we call a B-family of Dirac type operators on (even) dimensional manifolds with boundary. And following [MP], for any spectral section on the boundary family, we define an index bundle of the B-family of Dirac type operators. In (b), we prove our result on the change of index bundles under various deformations. In (c), we prove the index formula mentioned above.

(a) *A Family of Dirac Type Operators on Even Dimensional Manifolds with Boundary*

Let $Y \to M \xrightarrow{\pi_Y} B$ be a smooth fibration with the typical fibre an even dimensional compact manifold Y with boundary Z. We assume B is compact. Again, for simplicity we make the assumption that the vertical bundle TY is spin and carries a fixed spin structure. Then the boundary fibration $Z \to \partial M \xrightarrow{\pi_Z} B$ verifies the properties of previous sections.

Let g^{TY} be a metric on TY. Let g^{TZ} be the restriction of g^{TY} on TZ. We make the assumption that there is a neighborhood $[-1, 0] \times \partial M$ of ∂M in M such that for any $b \in B$, on $\pi_Y^{-1}(b) \cap ([-1, 0] \times \partial M) = [-1, 0] \times Z_b$, g^{TY} takes the form

$$g^{TY}|_{[-1,0] \times Z_b} = dt^2 \oplus g^{TZ_b}. \tag{5.1}$$

We denote by $\pi_b : [-1, 0] \times Z_b \to Z_b$ the projection onto the second factor.

Let $S(TY)$ be the spinor bundle of (TY, g^{TY}). Since $\dim Y$ is even, we have a canonical splitting of $S(TY)$ to the positive and negative spinor bundles: $S(TY) = S_+(TY) \oplus S_-(TY)$.

Let E be a complex vector bundle over M. Let g^E be a metric on E such that

$$g^E|_{[-1,0] \times Z} = \pi^* g^E|_Z. \tag{5.2}$$

Let ∇^E be a Hermitian connection on E such that

$$\nabla^E|_{[-1,0]\times Z} = \pi^*\nabla^E|_Z. \tag{5.3}$$

The existence of such a pair (g^E, ∇^E) is clear.

Let $S(TZ)$ be the spinor bundle of (TZ, g^{TZ}). We will still use the notation in previous sections for Z here. Thus there are canonically defined (twisted) Dirac opeartors $D_b^{Z,E}$ associated to g^{TZ_b}, $g^{E|_{Z_b}}$ and $\nabla^{E|_{Z_b}}$.

Since $Z \to \partial M \to B$ is a boundary family, it is known ([Sh], [MP], see also [N] for a new proof) that the $K^1(B)$ index of $D^{Z,E} = \{D_b^{Z,E}\}_{b\in B}$ vanishes.

Let P be a spectral section of $D^{Z,E}$, the existence of which is guaranteed by [MP].

Now for any $b \in B$, there is also a (twisted) Dirac operator $D_b^{Y,E}: \Gamma(S_+(TY_b) \otimes E|_{Y_b}) \to \Gamma(S_-(TY_b) \otimes E|_{Y_b})$ canonically associated to g^{TY_b}, $g^{E|_{Y_b}}$ and $\nabla^{E|_{Y_b}}$. Clearly on $U_b = [-1, 0] \times Z_b$, $D_b^{Y,E}$ takes the form

$$D_b^{Y,E} = c\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t} + D_b^{Z,E}\right). \tag{5.4}$$

Definition 5.1. By a B-family of Dirac type operators on $Y \to M \to B$, we will mean a B-family of first order differential operators $D^Y = \{D_b^Y\}_{b\in B}$ with each $D_b^Y: \Gamma(S_+(TY_b) \otimes E|_{Y_b}) \to \Gamma(S_-(TY_b) \otimes E|_{Y_b})$ has the same symbol as that of $D_b^{Y,E}$ and such that on $U_b = [-1, 0] \times Z_b$, D_b^Y can be written in the form

$$D_b^Y = c\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t} + D_b^Z\right), \tag{5.5}$$

where $D^Z = \{D_b^Z\}_{b\in B}$ is a B-family of (self-adjoint) Dirac type operators of $Z \to \partial M \to B$.

Recall that if P is a spectral section of $D^{Z,E}$, it is then a generalized spectral section of D^Z. For any $b \in B$, we then have a generalized Atiyah–Patodi–Singer elliptic boundary problem (D_b^Y, P_b). And the family $(D^Y, P) = \{(D_b^Y, P_b)\}_{b\in B}$ gives a B-family of Fredholm operators. According to Atiyah and Singer [AS1], it then determines an index bundle

$$\operatorname{ind}(D^Y, P) \in K(B). \tag{5.6}$$

(b) *A Variation Formula for Index Bundles*

Now consider a curve of B-families of metrics and connections $(g^{TY,u}, g^{E,u}, \nabla^{E,u})$ verifying the conditions (5.1) to (5.3) for each $u \in [0, 1]$.

Let $\tilde{D} = \{D^Y(U)\}_{u \in [0,1]}$ be a smooth curve of B-family of Dirac type operators such that for any u, $D^Y(u)$ has the same symbol as that of $D^{Y,E}(u)$ corresponding to $(g^{TY,u}, g^{E,u}, \nabla^{E,u})$.

Let P_0, P_1 be two generalized spectral sections of $D^Z(0)$, $D^Z(1)$ respectively. Then as in (a), one has two naturally defined index bundles

$$\begin{aligned} &\operatorname{ind}(D^Y(0), P_0) \in K(B), \\ &\operatorname{ind}(D^Y(1), P_1) \in K(B). \end{aligned} \tag{5.7}$$

On the other hand, $(D^Z(0), P_0)$, $(D^Z(1), P_1)$ and the curve $\{D^Z(u)\}_{u \in [0,1]}$ determine a higher spectral flow by Section 1:

$$\operatorname{sf}\{(D^Z(0), P_0), (D^Z(1), P_1)\} \in K(B). \tag{5.8}$$

The first main result of this section, which generalizes [DZ1, Theorem 1.1], can be stated as follows.

THEOREM 5.2. *The following identity holds in* $K(B)$,

$$\operatorname{ind}(D^Y(0), P_0) - \operatorname{ind}(D^Y(1), P_1) = \operatorname{sf}\{(D^Z(0), P_0), (D^Z(1), P_1)\}. \tag{5.9}$$

Proof. Let $\tilde{Q}$ be a generalized spectral section of the total family $\tilde{D}^Z = \{D_b^Z(u)\}$. Then we get a continuous family of elliptic boundary problems $(D^Y(u), \tilde{Q}(u))_{u \in [0,1]}$.

Thus by the homotopy invariance of the index bundles, one gets

$$\operatorname{ind}(D^Y(0), \tilde{Q}(0)) = \operatorname{ind}(D^Y(1), \tilde{Q}(1)) \qquad \text{in } K(B). \tag{5.10}$$

Now by the relative index theorem in [DZ2], which extends the relative index theorem of Melrose–Piazza [MP] to generalized spectral sections, one has

$$\operatorname{ind}(D^Y(0), \tilde{Q}(0)) = \operatorname{ind}(D^Y(0), P_0) + [P_0 - \tilde{Q}(0)] \qquad \text{in } K(B) \tag{5.11}$$

and

$$\operatorname{ind}(D^Y(1), \tilde{Q}(1)) = \operatorname{ind}(D^Y(1), P_1) + [P_1 - \tilde{Q}(1)] \qquad \text{in } K(B). \tag{5.12}$$

From (5.10) to (5.12), one gets

$$\begin{aligned}\operatorname{ind}(D^Y(0), P_0) - \operatorname{ind}(D^Y(1), P_1) &= [P_1 - \tilde{Q}(1)] - [P_0 - \tilde{Q}(0)] \\ &= \operatorname{sf}\{(D^Z(0), P_0), (D^Z(1), P_1)\} \qquad \text{in } K(B).\end{aligned} \tag{5.13}$$

The proof of Theorem 5.2 is completed. ▮

(c) *A Heat Kernel Asymptotic Formula for Index Bundles*

In this subsection, we give a heat kernel asymptotic formula for the Chern character of the index bundle (5.6) in terms of a local index density in the interior and an $\hat{\eta}$-form on the boundary. Since the $\hat{\eta}$-form is only well-defined with repect to spectral sections, not the generalized ones, we will restrict in this subsection to spectral sections.

We first introduce the superconnection formalism.

For the boundary fibration $Z \to \partial M \to B$, we choose a splitting as in Section 2:

$$\begin{aligned} T\,\partial M &= TZ \oplus \pi_Z^* TB, \\ g^{T\,\partial M} &= g^{TZ} \oplus \pi_Z^* g^{TB}. \end{aligned} \tag{5.14}$$

This splitting induces a trivial product splitting on $U = [-2, 0] \times \partial M$:

$$\begin{aligned} TU &= T([-2, 0]) \oplus T\,\partial M, \\ g^{TU} &= g^{T([-2,0])} \oplus g^{T\,\partial M}. \end{aligned} \tag{5.15}$$

We assume (TM, g^{TM}) also carries a splitting

$$\begin{aligned} TM &= TY \oplus \pi_Y^* TB, \\ g^{TM} &= g^{TY} \oplus \pi_Y^* g^{TB} \end{aligned} \tag{5.16}$$

with

$$(TM, g^{TM})|_U = (TU, g^{TU}). \tag{5.17}$$

Let $\tilde{M}$ be the double of M along ∂M via the product structure near ∂M. Then $\tilde{M}$ is a fibration $\tilde{Y} \to \tilde{M} \xrightarrow{\tilde{\pi}} B$ such that for any $b \in B$, $\tilde{\pi}^{-1}(b) = \tilde{Y}_b$ is the double of Y_b via the product structure near $Z_b = \partial Y_b$. The splittings (5.16), (5.17) determine a splitting of $(T\tilde{M}, g^{T\tilde{M}})$.

Let $D^Y = \{D_b^Y\}_{b \in B}$ be a B-family of Dirac type operators for π_Y. Then for any $b \in B$, D_b^Y extends canonically to an invertible Dirac type operator on $\tilde{Y}_b$ (cf. [BW]). Thus we get a B-family of invertible Dirac type operators $D^{\tilde{Y}} = \{D^{\tilde{Y}_b}\}_{b \in B}$.

Now the splitting (5.16) determines a connection $\tilde{\nabla}$ of the infinite dimensional bundle

$$H_\infty(\tilde{Y}, E) = \Gamma(S(T\tilde{Y}) \otimes E) \tag{5.18}$$

over B by (2.6), which preserves the natural splitting

$$\begin{aligned} \Gamma(S(T\tilde{Y}) \otimes E) &= H_{\infty,+} \oplus H_{\infty,-} \\ &= \Gamma(S_+(T\tilde{Y}) \otimes E) \oplus \Gamma(S_-(T\tilde{Y}) \otimes E). \end{aligned} \tag{5.19}$$

Set $\tilde{U} = [-2, 2] \times \partial M$, the product neighborhood of ∂M in $\tilde{M}$ induced canonically by $U = [-2, 0] \times \partial M \subset M$. The metric splitting and connection splitting clearly holds on $\tilde{U}$. In particular, $D^{\tilde{Y}}$ is of product structure near $\tilde{U}$.

Now let $B_i^Y \in \Omega^i(B) \hat{\otimes} \Gamma(\mathrm{cl}(TY) \otimes \mathrm{End} E)$, $i \geqslant 1$ be odd with respect to the natural $\mathbf{Z}_2$ grading of $\Lambda(T^*B) \otimes \mathrm{cl}(TY)$, and that $B_i^Y \mid U$ does not depend on $u \in [-2, 0]$. Then B_i^Y extends to $\tilde{Y}$ as $B_i^{\tilde{Y}}$, the restriction of which to $\tilde{U}$ does not depend on $u \in [-2, 2]$.

For any $t > 0$, set

$$B_t^{\tilde{Y}} = \tilde{\nabla} + \sqrt{t}\, D^{\tilde{Y}} + \sum_{i \geqslant 1} t^{(1-i)/2} B_i^{\tilde{Y}}. \tag{5.20}$$

Then $B_t^{\tilde{Y}}$ is a superconnection on $H_\infty(\tilde{Y}, E)$ in the sense of Quillen [Q] and Bismut [B].

One verifies easily that as $t \to 0$, one has the uniform asymptotic expansion on $\tilde{Y}$,

$$\mathrm{Tr}_s[\exp(-B_t^{\tilde{Y}, 2})(y, y)] = \sum_{i \geqslant (\dim y)/2 - [(\dim B)/2]}^{0} a_i t^i + o(1),\ y \in \tilde{Y} \tag{5.21}$$

where $\exp(-B_t^{\tilde{Y}, 2})(x, y)$, $x, y \in \tilde{Y}$ is the C^∞ kernel of $\exp(-B_t^{\tilde{Y}, 2})$ along the fibre $\tilde{Y}$.

On the other hand, each B_i^Y also induces a natural element B_i^Z in $\Omega^i(B) \hat{\otimes} \Gamma(\mathrm{cl}(TZ) \otimes E|_{\partial M})$. Also $\tilde{\nabla}$ induces a connection $\tilde{\nabla}^Z$ on

$$H_\infty(Z, E|_Z) = \Gamma(S(TZ) \otimes E|_{\partial M}) \tag{5.22}$$

and one has the induced superconnection for any $t > 0$

$$B_t^Z = \tilde{\nabla}^Z + \sqrt{t}\, D^Z + \sum_{i \geqslant 1} t^{(1-i)/2} B_i^Z. \tag{5.23}$$

Let P be a spectral section of D^Z in the sense of [MP]. Then by Section 2, one has a well-defined $\hat{\eta}$ form (class)

$$\hat{\eta}(D^Z, P, B^Z) \in \Omega(B)/d\Omega(B). \tag{5.24}$$

We can now state the second main result of this section, which extends the family index theorems in [BC2], [BC3] and [MP].

THEOREM 5.3. *The following identity holds in* $H^*(B)$,

$$\mathrm{ch}(\mathrm{ind}(D^Y, P)) = \int_Y a_0 - \hat{\eta}(D^Z, P, B^Z), \tag{5.25}$$

where a_0 *is given by* (5.21).

Proof. Let $D^{Y,E}$ be the standard B-family of (twisted) Dirac operators associated to (g^{TY}, g^E, ∇^E). Then

$$D^Y(s) = (1-s)\, D^Y + sD^{Y,E}, \qquad 0 \leqslant s \leqslant 1, \tag{5.26}$$

forms a curve of B-families of Dirac type operators over B.

Let Q be a spectral section of $D^{Z,E}$, which is the boundary family induced by $D^{Y,E}$. Then by Theorem 5.2, one has

$$\mathrm{ind}(D^Y, P) - \mathrm{ind}(D^{Y,E}, Q) = \mathrm{sf}\{(D^Z, P), (D^{Z,E}, Q)\}. \tag{5.27}$$

Now let $B^{Y,E}$ be the Bismut superconnection [B] associated to $D^{Y,E}$ and the splitting (5.16). Let $B^{Z,E}$ be the induced Bismut superconnection on the boundary family. Then it is a result of Melrose–Piazza [MP], which generalizes the earlier results of Bismut–Cheeger [BC2], [BC3], that (5.25) holds for $(B^{Y,E}, B^{Z,E})$. That is, one has the formula

$$\mathrm{ch}(\mathrm{ind}(D^{Y,E}, Q)) = \int_Y a_0^E - \hat{\eta}(D^{Z,E}, Q, B^{Z,E}), \qquad \text{in } H^*(B). \tag{5.28}$$

Now for any $s \in [0, 1]$, $t > 0$,

$$B_t^Y(s) = (1-s)\, B_t^Y +, sB_t^{Y,E} \tag{5.29}$$

is a (rescaled) superconnection associated to the B-family of Dirac type operators $D^Y(s)$. Denote its extension to $\tilde{M}$ by $B_t^{\tilde{Y}}(S)$.

For any $t > 0$, $s \in [0, 1]$, $y, y' \in \tilde{\pi}^{-1}(b)$, $b \in B$, let $P_{t,s}(y, y') = \exp(-I_{t,s})(y, y')$ be the smooth kernel associated to $I_{t,s} = (B_t^{\tilde{Y}}(s))^2$. By using the

Duhamel principle and by proceeding as in [B, Section 21, one deduces easily that

$$\begin{aligned}\frac{\partial}{\partial s}\operatorname{Tr}_s[P_{t,s}(y,y)] &= -\operatorname{Tr}_s\left[\frac{\partial I_{t,s}}{\partial s}P_{t,s}(y,y')\right]_{y=y'}\\ &= -\operatorname{Tr}_s\left[\left[B_t^{\tilde{Y}}(s),\frac{\partial B_t^{\tilde{Y}}(s)}{\partial s}\right]\exp(-B_t^{\tilde{Y}}(s)^2)(y,y')\right]_{y=y'}\\ &= -\operatorname{Tr}_s\left[\left[B_t^{\tilde{Y}}(s),\frac{\partial B_t^{\tilde{Y}}(s)}{\partial s}\exp(-B_t^{\tilde{Y}}(s)^2)\right](y,y')\right]_{y=y'}\\ &= -d\operatorname{Tr}_s\left[\frac{\partial B_t^{\tilde{Y}}(s)}{\partial s}\exp(-B_t^{\tilde{Y}}(s)^2)(y,y')\right]_{y=y'}\\ &\quad -\sqrt{t}\operatorname{Tr}_s\left[\left[D^{\tilde{Y},E},\frac{\partial B_t^{\tilde{Y}}(s)}{\partial s}\exp(-B_t^{\tilde{Y}}(s)^2)\right](y,y')\right]_{y=y'}\end{aligned} \tag{5.30}$$

Thus one has

$$\begin{aligned}&\frac{\partial}{\partial s}\operatorname{Tr}_s[\exp(-B_t^{\tilde{Y}}(s)^2)(y,y)]\\ &\quad= -\sqrt{t}\operatorname{Tr}_s\left[\left[D^{\tilde{Y},E},\frac{\partial B_t^{\tilde{Y}}}{\partial s}\exp(-B_t^{\tilde{Y}}(s)^2)\right](y,y')\right]_{y=y'} \bmod(\tilde{\pi}d^B)\,\Omega(\tilde{Y}).\end{aligned} \tag{5.31}$$

Integrating (5.31) from $s=0$ to $s=1$, one gets

$$\begin{aligned}&\operatorname{Tr}_s[\exp(-B_t^{\tilde{Y},2})(y,y)] - \operatorname{Tr}_s[\exp(-B_t^{\tilde{Y},E,2})(y,y)]\\ &\quad= \sqrt{t}\int_0^1 \operatorname{Tr}_s\left[\left[D^{\tilde{Y},E},\frac{\partial B^{\tilde{Y}_t(s)}}{\partial s}\exp(-B_t^{\tilde{Y}}(s)^2)\right](y,y')\right]_{y=y'} ds\\ &\qquad \bmod(\tilde{\pi}d^B)\,\Omega(\tilde{Y}).\end{aligned} \tag{5.32}$$

Now we let $t\to 0$ in (5.32). Proceeding as in [B] and by comparing the coefficients of the resulting asymptotics, we get

$$a_0 - a_0^E = d^{\tilde{Y}}\int_0^1 b_0(s)\,ds \bmod(\tilde{\pi}d^B)\,\Omega(\tilde{Y}), \tag{5.33}$$

where $b_0(s)$ is the constant term in the asymptotics as $t\to 0$ of

$$\operatorname{Tr}_s\left[\frac{\partial B_t^{\tilde{Y}}(s)}{\partial s}\exp(-B_t^{\tilde{Y}}(s)^2)(y,y')\right]_{y=y'}. \tag{5.34}$$

Also, by the product structure of metrics and connections near the boundary fibration ∂M, we see that if c_0 is the constant term of the asymptotics of

$$\operatorname{Tr}^{\text{even}}\left[\frac{\partial B_t^Z(s)}{\partial s}\exp(-B_t^Z(s)^2)(z, z')\right]_{z=z'}, \qquad z, z' \in Z, \tag{5.35}$$

where $B_t^Z(s)$ is the restriction of $B_t^{\tilde{Y}}(s)$ on Z, then we have

$$b_0|_Z = \frac{1}{\sqrt{\pi}}\, c_0. \tag{5.36}$$

By (5.33), (5.36), one then gets

$$\int_Y a_0 - \int_Y a_0^E = \frac{1}{\sqrt{\pi}} \int_Z c_0 \qquad \text{in } \Omega(B)/d\Omega(B). \tag{5.37}$$

Now again using the results in Section 2, one has

$$\frac{1}{\sqrt{\pi}} \int_0^1 \left(\int_Z c_0\right) ds = -\hat{\eta}(D^{Z,E}, P_1, B^{Z,E}) + (\hat{\eta}(D^E, P_0, B^Z) \qquad \text{in } \Omega(B)/d\Omega(B), \tag{5.38}$$

where $\{P_s\}_{s\in[0,1]}$ is a (in fact any) curve of spectral sections of $\{D(s) = (1-s)\, D^Z + sD^{Z,E}\}_{s\in[0,1]}$.

Combining (5.37), (5.38), we get

$$\int_Y a_0 - \hat{\eta}(D^Z, P_0, B^Z) = \int_Y a_0^E - \hat{\eta}(D^{Z,E}, P_1, B^{Z,E}) \qquad \text{in } \Omega(B)/d\Omega(B). \tag{5.39}$$

From (5.39), one deduces that

$$\begin{aligned}
\int_Y a_0 - \hat{\eta}(D^Z, P, B^Z) &= \int_Y a_0^E - \hat{\eta}(D^{Z,E}, Q, B^{Z,E}) \\
&\quad + \hat{\eta}(D^{Z,E}, Q, B^{Z,E}) - \hat{\eta}(D^{Z,E}, P_1, B^{Z,E}) \\
&\quad + \hat{\eta}(D^Z, P_0, B^Z) - \hat{\eta}(D^Z, P, B^Z) \\
&= \operatorname{ch}(\operatorname{ind}(D^{Y,E}, Q)) + \operatorname{ch}([Q - P_1] + [P_0 - P]) \\
&= \operatorname{ch}(\operatorname{ind}(D^{Y,E}, Q)) - \operatorname{ch}(\operatorname{sf}\{(D^{Z,E}, Q), (D^Z, P)\}) \\
&= \operatorname{ch}(\operatorname{ind}(D^Y, P)),
\end{aligned} \tag{5.40}$$

where we have used results proved in this section and in Section 2.

The proof of Theorem 5.3 is completed. ∎

Remark 5.4. (5.25) may be viewed as an analogue in the family case of the general index theorem of Atiyah–Patodi–Singer [APS2] for manifolds with boundary.

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