HW # 3  Solutions
\{ 11.4, 11.6, 11.7, 11.10, 11.11 \( 6)(c) \} \\

11.4  We prove by contradiction.
Suppose \( x > 0 \), \( x < \varepsilon \) \( \forall \varepsilon > 0 \) and \( x \neq 0 \).
Then \( x > 0 \) and \( x < \varepsilon \) \( \Rightarrow x > 0 \).
Choose \( \varepsilon = \frac{x}{2} \).
Then \( \varepsilon > 0 \), but \( x \notin \varepsilon \). Contradiction.
\[ \therefore x = 0. \]

11.6  Prove a) \( |x-1| - |y-1| \leq |x-y| \) \( \left[ \text{or} \right. \]
\[ -|x-y| \leq |x-1-1| \leq |x-y| \ (\text{eq}) \]
\[ \text{Useful Trick:} \quad |x| = |x-y+y| \leq |x-y| + |y| \]
Here \( |x-1-1| \leq |x-y| \), which is the RHS of (\#).
Similarly \( |y| \leq |y-x| + |x| \).
So \( |y| - |x| \leq |y-x| \Rightarrow |x-1-y| > |x-y| \), which is LHS of (\#).
This proves part a).

b) If \( |x-y| < \varepsilon \), then \( |x| < |y| + \varepsilon \).
\[ |x| \leq |x-y| + |y| < \varepsilon + |y| \]

c) If \( |x-y| < \varepsilon \), \( \forall \varepsilon > 0 \), then \( x = y \).
\[ \text{Proof by contradiction.} \]
Suppose \( |x-y| < \varepsilon \) \( \forall \varepsilon > 0 \) and \( x \neq y \).
Since \( x \neq y \), \( |x-y| > 0 \). Say \( k = |x-y| > 0 \).
Then take \( \varepsilon = \frac{k}{2} > 0 \) making \( |x-y| > \varepsilon \). Contradiction.
\[ \therefore x = y. \]
We prove this by induction. 

Clearly \( |x_1| \leq |x_1| \) and by the triangle inequality \( |x_1-x_2| \leq |x_1|+|x_2| \) 

This takes care of cases \( n=1 \) and \( 2 \). Suppose we have the inequality for \( n<k \). Then for \( n=k \)

\[
| x_1, \ldots , x_{k-1}, x_k | \leq | x_1, \ldots , x_{k-1}, + x_k | \quad \text{by triangle inequality} \\
\leq | x_1, \ldots , | x_{k-1}, + |x_k | \quad \text{by induction hypothesis}
\]

This completes the induction.

Case 1: \( a = 0 \)

Then \( a^2 + 1 = 1 \)

Claim: \( 1 > 0 \)

\( \iff \) Suppose \( 1 < 0 \), then

\[
| 1 | > 0 \quad \text{[since negative \( \leq \) is reverse inequality when multiplied]}
\]

Hence \( 1 > 0 \) contradiction!

\[
\therefore 1 > 0 \\
\therefore a^2 + 1 > 0
\]

Case 2: \( a > 0 \)

Then \( a^2 > 0 \) by same reasoning as before

Hence \( a^2 + 1 > 0 \) so \( a^2 + 1 > 0 \)

Case 3: \( a < 0 \)

Then \( a^2 > 0 \) so \( a^2 + 1 > 1 \).

b) \( \{ -\frac{1}{2}, 3, \frac{5}{2}, -x, x^2 \} \) (smallest \( \rightarrow \) biggest)

since for instance \( -x^2 > 3-x \) since \( -x^2 - 3 + x = -1 < 0 \)

c) \( \{ \frac{x+1}{x-2}, \frac{x+2}{x-1}, \frac{x^2-2}{x-1}, \frac{x^2+2}{x-1} \} \) (smallest \( \rightarrow \) biggest)

since for instance

\[
\frac{x+1}{x-2} - \frac{x+2}{x-1} = \frac{(x^2-1)(x+1) - (x+2)(x^2-2)}{(x^2-2)(x^2-1)} = \frac{-x^2 + x + 5}{x^4 - 3x^2 + 2} = -1 < 0
\]