18.2
a) False.

b) False.

c) False. A sequence of rationals approaching an irrational will not converge in $\mathbb{Q}$, however the points will be getting close to each other.

18.3
a) $s_n = 1$, $s_m = \frac{1}{4}(2s_n + 5)$ for $n$

$b_n = \frac{1}{4}(7) = 7/4 > 1$ so sequence seems to be increasing

Assume $s_k > s_{k-1}$ for some $k \in \mathbb{N}$, then

$s_{k+1} = \frac{1}{4}(2s_k + 5) > \frac{1}{4}(2s_{k-1} + 5) = s_k$

Hence by induction $\{s_k\}$ is monotonically inc.

Suppose $s_k < 100$ for some $k \in \mathbb{N}$, then

$s_m = \frac{1}{4}(2\cdot 100 + 5) = \frac{205}{4} < 100$

Hence $s_k$ is bounded by induction.

To find the limit, we solve

$s = \frac{1}{4}(2s + 5) \Rightarrow s = \frac{1}{4}s + \frac{5}{4} \Rightarrow \frac{1}{4}s = \frac{5}{4} \Rightarrow s = 10/4$

so $\lim_{n \to \infty} s_n = 5/4$

18.5
a) Counterexample: $a_n = x^2$, $b_n = -x$. Then $a_n + b_n = x^2 - x = x(x-1)$, see below.

b) Counterexample: $a_n = x$, $b_n = x - 1$, see above.
(18.7) \( s_1 = \sqrt{6}, \ s_{n+1} = \sqrt{6 + s_n} \)

To show \( \{s_n^2\} \) converges we show it is monotone and bounded.

For \( s_1 = \sqrt{6}, \ s_2 = \sqrt{6 + \sqrt{6}} > s_1 \),

Assume \( s_k > s_{k-1} \), for some \( k \in \mathbb{N} \), then

\[
s_{k+1} = \sqrt{6 + s_k} > \sqrt{6 + s_{k-1}} = s_k
\]

So by induction \( \{s_k\} \) is monotone.

Assume \( s_k < 100 \) for some \( k \), then

\[
s_{k+1} = \sqrt{6 + s_k} < \sqrt{106} < 11 < 100
\]

Hence by induction \( \{s_n\} \) is bounded.

1. \( s_n \) converges.

To find its limit, solve

\[
s = \sqrt{6 + s} \quad \Rightarrow \quad s^2 = 6 + s \quad \Rightarrow \quad s^2 - s - 6 = 0
\]

So \( s \in \{-3, 2\} \)

Since \( s_n > 0 \) for \( k \), \( s = 2 \).

(18.8) \( s_1 = k, \ s_{n+1} = \sqrt{4s_n - 1} \)

Note the sequence is monotone, since for some \( n \)

If \( s_n > s_{n-1} \), then \( s_{n+1} = \sqrt{4s_n - 1} > \sqrt{4s_{n-1} - 1} = s_n \), so \( \{s_n\} \) is increasing.

If \( s_n < s_{n-1} \), then similarly \( \{s_n\} \) is decreasing.

Hence to know which values of \( k \) \( s_n \) will be increasing we need to find out what value makes \( s_2 > s_1 \), i.e. we need to solve for \( k \):

\[
\sqrt{4k - 1} > k
\]

So \( 4k - 1 > k^2 \Rightarrow k^2 - 4k + 1 < 0 \Rightarrow (k - (2 + \sqrt{3}))(k - (2 - \sqrt{3})) < 0 \)

To make sure the LHS is negative we need

\[
2 - \sqrt{3} < k < 2 + \sqrt{3}
\]

Similarly \( \{s_n\} \) will be monotone decreasing for

\[
k < 2 - \sqrt{3} \quad \text{and} \quad k > 2 + \sqrt{3}
\]
a) \( S = \{0\} \), \( \limsup = \liminf = 0 \)

b) \( S = \mathbb{N} \), \( \limsup = \infty \), \( \liminf = 0 \)

19.7

a) True, by Thm 19.7 every bounded sequence has a convergent sequence and every convergent sequence is Cauchy (19.10)

b) False, \( S_n = n \), \( t_n = \frac{n}{n+1} \), \( S_n > t_n > 0 \)

c) True, take \( r_n = \frac{1}{n} + c \), \( t_n = \frac{1}{n} - c \), where \( c > 0 \)

14.9

Suppose \( \lim s_n = r \) and \( r \neq s \)

If \( r > s \) then, since \( r \) is a subsequential limit

\[ S = \limsup (s_n) > r > s \] , contradiction.

Similarly, if \( r < s \)

Hence \( r \) must be equal to \( s \)

19.15

a) Let \( s = \liminf (s_n) \), \( t = \liminf (t_n) \)

Then for \( \forall \varepsilon > 0 \), \( \exists N \) s.t. \( \forall n > N_n \), \( s_n > s - \varepsilon \)

and \( s_n < s + \varepsilon \)

Similarly, \( \exists N_2 \) s.t. \( \forall n > N_2 \), \( t_n > t - \varepsilon \)

and \( t_n < t + \varepsilon \)

Let \( N = \max \{N_1, N_2\} \), then for \( n > N \) \[ \text{note that } N \text{ depends on } \varepsilon \]

\[ s + 2\varepsilon < s_n + t_n < s + t + 2\varepsilon \]

Now let \( \varepsilon = \frac{1}{2k} \), then for each \( k \), \( \exists N_k > N \) s.t.

\[ |(s_n + t_n) - (s + t)| > \frac{1}{2} / k \]

as \( k \to \infty \),

\[ s_n + t_n \to s + t \]

Hence \( s + t \in \{ \text{set of subsequential limits of } (s_n + t_n) \} \)

Since \( \liminf (s_n + t_n) \) is the infimum of this set (ie greatest lower bound)

\[ \liminf (s_n + t_n) \geq s + t = \liminf (s_n) + \liminf (t_n) \]

b) Take \( s_n = (-1)^n \), \( t_n = (-1/n) \), note \( s_n + t_n = 0 \) \( \forall \ n \)

while \( s = t = -1 \)