Review of Surface Integrals

What is a parametrization?

It is a mapping from a flat domain to a curved space (e.g., curve, surface, solid).

This mapping lets us associate functions defined on a curved space to functions defined on a flat space.

e.g., Let $S$ be a cylinder of radius 1 along the $y$-axis, i.e., the set of points satisfying $x^2 + z^2 = 1$.

We can parameterize $S$ by the mapping

$$T(u, v) = (\cos(u), v, \sin(u)) \quad 0 \leq u < 2\pi \quad a \leq v \leq b$$

Now say we have a function $f(x, y, z)$ defined on $S$.

Essentially, every point of $S$ has a number associated to it.

To calculate

$$\iint f(x, y, z) \, dS = \sum \sum f \, dS$$

means to "sum up" all these numbers (and multiply by an infinitesimal surface area).

But what is an infinitesimal surface area?

What if indeed, we transferred all these numbers to the $u,v$ plane?

$$\iint_D f(T(u, v)) \, dD = \iint_S f(x, y, z) \, dS$$

and now note that an infinitesimal surface element has area $= dA \parallel t_x \times t_y$

be a number on $S$ has its value smeared out over a different area than on $D$. 

So for example, say \( f(x,y,z) = x^2 + z^3 \)

Then
\[
\iint_S f \, dS = \iint_S f(T(u,v)) \| T_u \times T_v \| \, dA = \int_{-\pi/2}^{\pi/2} \int_0^1 f(T(u,v)) \| T_u \times T_v \| \, du \, dv
\]

\[
= \iint_D f(x,y) \, dA
\]

Note that \( T(u,v) \) hasn't stretched out any area (since \( dA \) and \( ds \))

what if \( T(u,v) = (\cos(2\pi u), v, \sin(2\pi v)) \) \( u,v \in [0,1] \)

was used?

The area could be stretched out by \( 2\pi \)!

Let's try this one:

Let \( S \) be the surface defined by the plane \( x+y+z = 1 \)

lying in the 1st quadrant. Let \( f(x,y,z) = z \)

What is \( \iint_S f \, dS ? \)

\[
\iint_S f \, dS = \iint_S (1-x-y) \sqrt{3} \, dy \, dx = \int_0^1 \int_0^{1-z} \sqrt{3} \, dy \, dz = \int_0^1 \frac{2}{3} \, dz = \frac{2}{3}
\]

\[
= \sqrt{3}/6
\]

These correspond to the different parameterizations

\((x,y,1-x-y)\), \((1-y-z,y,z)\), \((x,1-x-y,z)\)

respectively.
2) Let \( F = (\frac{\partial}{\partial y}, 2x) \). Then
\[
\int_C F \cdot d\mathbf{r} = \iint_D \frac{\partial}{\partial y} \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \frac{\partial}{\partial y} \, dA
\]
\[
= \iint_D 2 - 5 \, dA
\]
\[
= -3 (\pi \, \text{area})
\]
\[
= -12\pi
\]

If you wanted to evaluate these path integrals directly, you would need to parameterize the path like so:

2) Let \( c(t) = (\cos(t), \sin(t)) \), \( 0 \leq t \leq 2\pi \)

Then
\[
\int_C y \, dx + 15x \, dy = \int_0^{2\pi} \sin(t) \, d((\cos(t)) + 15(\cos(t)) \, d((\sin(t)))
\]
\[
= \int_0^{2\pi} -\sin(t) \, dt + 15(\cos(t)) \, dt
\]
\[
= -\sin^2(t) + 15\cos^2(t) \, dt
\]
\[
= 14\pi
\]
Quiz

Let \( S \) be defined by the plane \( 2x + y + z = 2 \) lying in the 1st octant.

Let \( f(x, y, z) = xy + z \).

Compute

\[ \int_S f \, ds \]

\( \int_S f \, ds = \int (2-x) \sqrt{6} \, dA \)

\[ = \sqrt{6} \int_0^1 \int_0^{2-x} (2-x) \, dy \, dx \]

\[ = \sqrt{6} \int_0^1 2y - xy \, |_{y=0}^{y=2-x} \, dx \]

\[ = \sqrt{6} \int_0^1 2(2-x) - x(2-x) \, dx \]

\[ = \sqrt{6} \int_0^1 2x^2 - 6x + 4 \, dx \]

\[ = \sqrt{6} \left[ \frac{2}{3}x^3 - 3x^2 + 4x \right]_0^1 \]

\[ = \sqrt{6} \left( \frac{2}{3} - 3 + 4 \right) \]

\[ = \frac{5\sqrt{6}}{3} \]