* Fundamental Theorem of Calculus

ex. Let \( g(x) = \int_{0}^{x} \sin(t^2) \, dt \)

Then,

\[ g'(x) = \sin(x^2) \]

* Techniques of Integration

- Substitution rule \((u \, du)\)
- Integration by parts \((\text{IBP})\)
  \[ \int u \, dv = uv - \int v \, du \]
  indefinite integral
  \[ \int_{a}^{b} u \, dv = uv\bigg|_{a}^{b} - \int_{a}^{b} v \, du \]
  definite integral
- Trig Substitution

How to recognize one technique from another?

- Use subst. rule when \( \frac{\text{derivative of something in the integrand}}{\text{something}} \) will cancel everything else
- Use IBP when the derivative of \( u \) and the integral of \( dv \) are easier to evaluate
- Use trig. sub. to make expressions involving \( \sqrt{a^2 + x^2} \) easier to evaluate.
Examples: (4 means worked out in detail later)

1. \[ \int x^4 \ln(x) \, dx \]
2. \[ \int \frac{\ln(x)}{x} \, dx \]
3. \[ \int \frac{1 + \ln(x)}{x \ln(x)} \, dx \]
4. \[ \int \frac{x}{x-2} \, dx \]
5. \[ \int \sin(x) \cos(x) \, dx \]
6. \[ \int e^x \sqrt{1 + e^x} \, dx \]
7. \[ \int x \sin(x) \, dx \]
8. \[ \int e^x \sin(x) \, dx \]
9. \[ \int \cos(x) \ln(\sin(x)) \, dx \]
10. \[ \int \tan^2(x) \, dx \]
11. \[ \int \cot^2(x) \, dx \]
12. \[ \int \frac{1}{\sqrt{1-x^2}} \, dx \]
13. \[ \int \frac{x^2}{\sqrt{x^2-4}} \, dx \]
14. \[ \int \frac{1}{\sqrt{x^2-4}} \, dx \]
15. \[ \int \frac{x^2}{\sqrt{x^2-4}} \, dx \]

Technique

Notice the derivative of \( \ln(x) \) is easy, and the integral of \( x^4 \) isn't bad either. They combine to give \( \frac{1}{x} \left( \frac{1}{4} x^4 \right) \), which is simple to integrate... so think IBP!

This seems similar to 1 and can be solved the same way, except there is an easier way too! Notice the derivative of \( \ln(x) \) is \( \frac{1}{x} \) and dividing by \( x \) is like multiplying by \( x \) which cancels the denominator... so I think udu!

IBP would only make this uglier. Try u sub isn't applicable. Let's try udu! I will work this out later.

Wouldn't it be nice if we had \( \int \frac{x-2}{x} \, dx \) instead? We can "flip the integral over" by using udu and solve for the original variable.

What's the deriv. of \( \cos(x) \)? Right! Use udu.

What's the deriv. of \( \ln(x) \)?

Deriv of \( x = \) easy! Int of \( \sin(x) \) = easy! Use IBP!

This is IBP 2 times. We'll work it out.
Can we use udu? (Type of tricks = UFO). The integral reduces to
\[ \frac{1}{\tan x} \sec^2 x = \tan^{-1} x. \]
But how do we get it? (Type of UFO = easy, to be used wisely. Try u=\sec x already.)

Looks like (3) is much easier to solve! Notice
\[ \frac{d}{dx} \ln(\tan x) = \frac{1}{\tan x} \sec^2 x = \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin x \cos x}. \]

Look familiar? Use udu! u=\ln(\tan x)

(1) Remember the rules (pg 486) if power of tan is odd break in -p
\[ \tan^3 x = (\sec^2 x - 1) \tan x \]
l and let u = \sec x

(2) Remember the rules (pg 484) look to odd powers trig function (here \sin x)
\[ \tan^3 x = (\sec^2 x - 1) \tan x \]
and let u = the other one (here = \cos x)

(3), (4), (5)

When you see
\[ \sqrt{1-x^2} \quad \text{use } \sin^2 x \quad \text{recall } \sin^2 x = \cos^2 x \]
\[ \sqrt{1+x^2} \quad \text{use } \tan^2 x \quad \text{recall } \tan^2 x = \sec^2 x \]
\[ \sqrt{x^2-1} \quad \text{use } \sec^2 x \quad \text{recall } \sec^2 x = 1 + \tan^2 x \]

exception: \[ \int \frac{x}{\sqrt{1-x^2}} \] dx is much easier since udu works!

---

Let's get our hands dirty now!

Doing these in detail...
1. \[
\int x^4 \ln(x) \, dx = \int \ln(x) \, \frac{\partial}{\partial x} (x^5) \, dx
\]
   let \( u = \ln(x) \) \quad \Rightarrow \quad du = \frac{1}{x} \, dx \quad \Rightarrow \quad v = \frac{1}{5} x^5

\[
= \frac{1}{5} x^5 \ln(x) - \frac{1}{5} \int x^4 \, dx
\]

\[
= \frac{1}{5} x^5 \ln(x) - \frac{1}{25} x^5 + C
\]

2. \[
\int \frac{1}{x \ln(x)} \, dx
\]
   let \( u = \ln(x) \) \quad \Rightarrow \quad du = \frac{1}{x} \, dx

\[
= \int \frac{1}{x \ln(x)} \, dx = \frac{1}{5} (\ln(x))^{-1/5} \, dx
\]

\[
= \int \frac{x}{x \ln(x)} \, du = \int \frac{u^2}{u-1} \, du 
\]

\[
= \int \left( u^2 - 1 \right) \, du \quad \text{but} \quad \ln(x) = u - 1
\]

\[
= \int \left( 1 + \frac{1}{u^2-1} \right) \, du
\]

\[
= \ln|\ln(x)| + \ln(|u-1|) + C
\]

3. \[
\int \frac{x}{x^2 - 1} \, dx
\]
   Common trick when exponent powers are same top \& bottom

\[
= 2 \int \left( \frac{1}{u^2-1} \right) \, du
\]

\[
= \ln|\ln(x)| + \ln(|u-1|) + C
\]

4. \[
\int \frac{x}{x-2} \, dx
\]
   \( u = x - 2 \) \quad \Rightarrow \quad x = u + 2

\[
= \int \frac{u+2}{u} \, du
\]

\[
= \int \left( 1 + \frac{2}{u} \right) \, du
\]

\[
= \ln(x) + \ln(|u-1|) + C
\]

This is a common "flip the fraction over" trick.
\[ \int e^x \sin(x) \, dx \]

Let 
\[ u = \sin(x) \quad \text{and} \quad du = \cos(x) \, dx \]
\[ dv = e^x \, dx \quad \text{and} \quad v = e^x \]

Then,
\[ \int e^x \sin(x) \, dx = e^x \sin(x) - \int e^x \cos(x) \, dx \]
\[ = e^x \sin(x) - \int e^x \cos(x) \, dx \quad \text{since} \quad \int e^x \cos(x) \, dx = e^x \sin(x) + C \]

Putting it together,
\[ \int e^x \sin(x) \, dx = e^x \sin(x) - e^x \cos(x) \]
\[ = \int e^x \sin(x) \, dx - \int e^x \cos(x) \, dx \]
\[ = 2 \int e^x \sin(x) \, dx = \frac{1}{2} e^x (\sin(x) - \cos(x)) + C \]

\[ \int \cos(x) \ln(\sin(x)) \, dx \]

Let 
\[ u = \sin(x) \quad \text{and} \quad du = \cos(x) \, dx \]
\[ dv = \ln(u) \, du \quad \text{and} \quad v = \ln(u) \]

Then,
\[ \int \cos(x) \ln(\sin(x)) \, dx = \int \ln(u) \, du \]
\[ = u \ln(u) - u + C \]
\[ = \sin(x) \ln(\sin(x)) - \sin(x) + C \]

\[ \int \tan^2(x) \, dx \]

Let 
\[ u = \tan(x) \quad \text{and} \quad du = \sec^2(x) \, dx \]
\[ dv = \tan(x) \, dx \quad \text{and} \quad v = \tan(x) \]

Then,
\[ \int \tan^2(x) \, dx = \frac{1}{2} \tan^2(x) + \ln |\sec(x)| + C \]

\[ \int \sec^2(x) \tan(x) \, dx - \int \tan(x) \, dx \]

Let 
\[ u = \tan(x) \quad \text{and} \quad du = \sec^2(x) \, dx \]
\[ dv = \sec^2(x) \, dx \quad \text{and} \quad v = \sec(x) \]

Then,
\[ \int \sec^2(x) \tan(x) \, dx = \frac{1}{2} \tan^2(x) + \ln |\sec(x)| + C \]
\[
\frac{\tan^2 \theta - 1}{\sec^2 \theta - 1} = 2 \tan \theta \sec \theta
\]

\[
\frac{4 \sec^2 \theta}{2 \tan \theta} \sec \theta = 2 \int \frac{\sec^2 \theta}{\tan \theta} d\theta
\]

\[
= 2 \int \frac{1}{u} du
\]

\[
= 2 \ln |u| + C
\]

\[
= 2 \ln |\tan \theta| + C
\]

\[
= 2 \ln \left( \frac{\sqrt{x^2 - 4}}{2} \right) + C
\]

---

**Rational Functions**

- **First Step**: if \( \text{deg}(\text{numerator}) < \text{deg}(\text{denominator}) \) divide!

\[
\frac{x^3 + 1}{x^2 - 10x - 10}
\]

\[
= \frac{x^2 - x - 6}{x^2 - 10x - 10}
\]

\[
= \frac{x + 1}{x - 6}
\]

**Cases**

1. **Simplest**: denom. breaks up into non-repeating linear factors

\[
\frac{x - 9}{x^2 - 7x - 10} = \frac{x - 9}{(x + 5)(x - 2)} = \frac{A}{x + 5} + \frac{B}{x - 2}
\]

By pulling this over a common denominator we find

\[
A(x - 2) + B(x + 5) = x - 9
\]

Re-expressing (group in terms of \( x^2 \)) gives

\[
x(A + B) - 2A + 5B = x - 9
\]

\[
\Rightarrow A + B = 1 \Rightarrow A = 1 - B
\]

\[
-2A + 5B = -9 \Rightarrow -2(1-B) + 5B = -9 \Rightarrow B = -1 \Rightarrow A = 2
\]
5) **Repeated root**

**Example:** \( \frac{3x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \)

Putting the fractions under a common denominator,

\[ A(x+1) + B = 3x+3 \]

or

\[ x(A) + (A+B) = 2x+3 \]

So

\[ A = 2 \]

\[ A+B = 3 \Rightarrow B = 1 \]

\[ \int \frac{3x+3}{(x+1)^2} \, dx = 3 \ln(x+1) + \int \frac{1}{x+1} \, dx \]

\[ = 3 \ln(x+1) - \frac{1}{x+1} \]

See pg 499 for a harder example

6) **Irreducible quadratic root**

**Example:** \( \frac{2x^3-x+4}{x^3+4x} = \frac{2x^3-x+4}{x(x^3+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \)

See pg 500 ex 5 for solution.

**Summary:**

Factor

\[ \frac{x^2+1}{x(x+1)(x^2+1)(x^3+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x^2+1} + \frac{Ex+F}{x^3+1} + \frac{Gx+H}{(x^3+1)^2} + \frac{Ix+J}{(x^3+1)^3} \]
Improper Integrals

Good things to know:

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx \]
- is convergent if \( p > 1 \), divergent if \( p \leq 1 \).

\[ \int_{0}^{\infty} \frac{1}{x^p} \, dx \]
- is divergent if \( p > 1 \), convergent if \( p \geq 1 \).

Example:

\[ \int_{1}^{\infty} \frac{1}{x^a} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^a} \, dx = \lim_{t \to \infty} \left( \frac{t^a}{a} - \frac{1}{a} \right) = \frac{1}{a} (t^a - 1) \]

Comparison Test:

If \( f(x) \geq g(x) \geq 0 \) for all \( x \geq a \),

\[ \int_{a}^{\infty} f(x) \, dx \geq \int_{a}^{\infty} g(x) \, dx \]

Example:

\[ \int_{0}^{\infty} \frac{\sin(x)}{x^2} \, dx \]

\( 0 \leq \int_{0}^{\infty} \frac{\sin(x)}{x^2} \, dx \leq \int_{0}^{\infty} \frac{1}{x^2} \, dx \) which is convergent since \( p = 1 \).
Arclength + Surface Area

\[ \text{Arclength} = \int_{a}^{b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

ex. Find length of curve \( y = \ln(\sec x) \) from \( 0 \leq x \leq \frac{\pi}{4} \)

\[ \frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x \]

\[ L = \int_{0}^{\pi/4} \sqrt{1 + \tan^2 x} \, dx = \int_{0}^{\pi/4} \sec x \, dx = \ln \left| \sec x + \tan x \right| \bigg|_{0}^{\pi/4} = \ln \left( 1 + \frac{\sqrt{2}}{\sqrt{2}} \right) - \ln(1) \]

\[ \cos \left( \frac{\pi}{4} \right) = \frac{1}{\cos \left( \frac{\pi}{4} \right)} = \frac{\sqrt{2}}{\sqrt{2}} \quad \sec \left( \frac{\pi}{4} \right) = 1 \]

\[ \tan \left( \frac{\pi}{4} \right) = \frac{\sin \left( \frac{\pi}{4} \right)}{\cos \left( \frac{\pi}{4} \right)} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}} = 1 \quad \tan \left( \frac{\pi}{4} \right) = 0 \]

Surface Area

\[ \text{Surface Area} = \int_{a}^{b} 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \quad \text{(about x-axis)} \]

\[ \int_{0}^{\pi/4} 2\pi \sqrt{1 + \tan^2 x} \, dx \quad \text{(about y-axis)} \]

ex. Find the surface area of \( y = \cos(2x) \) rotated about the x-axis \( 0 \leq x \leq \frac{\pi}{6} \)

\[ S = \int_{0}^{\pi/6} \pi \cos(2x) \sqrt{1 + \sin^2(2x)} \, dx \quad \text{let } u = \sin(2x) \]

\[ du = 4 \cos(2x) \, dx \]

\[ = \frac{\pi}{2} \int_{0}^{1} \sqrt{1 + u^2} \, du \quad \text{when } x = 0 \quad u = \sin(0) = 0 \]

\[ x = \frac{\pi}{6} \quad u = \sin \left( \frac{\pi}{6} \right) = 1 \]

two ways to proceed, interpret \( \int \) as area

of \( \bigcirc \) = \( \frac{\pi}{4} \) or let \( u = \tan(x) \) and solve

\[ = \frac{\pi^2}{8} \]
In conclusion let's try a WEIRD problem! 

Let \( u = \frac{1}{x} \) \( \ 1 \leq x \leq \infty \)

\[
\begin{align*}
V &= \lim_{t \to \infty} \pi \int_{t}^{\infty} x^{-2} \, dx = \lim_{t \to \infty} \pi \left( \frac{-1}{x} \right) \bigg|_{t}^{\infty} \\
&= \lim_{t \to \infty} \left( -\frac{\pi}{t} + \frac{\pi}{t} \right) \\
&= \pi \quad \text{since} \quad \frac{\pi}{t} \to 0
\end{align*}
\]

The surface area

\[
S = \int_{1}^{\infty} 2\pi \left( \frac{1}{x} \right) \sqrt{1 + h^2(x)} \, dx
\]

let \( u = \ln(x) \)

\( du = \frac{1}{x} \, dx \)

\[
= 2\pi \int_{0}^{\infty} \sqrt{1 + u^2} \, du
\]

when \( x = 1 \quad u = \ln(1) = 0 \)

\( r = \infty \quad u = \infty \)

\[
\leq 2\pi \int_{0}^{\infty} u^2 \, du = 2\pi \int_{0}^{\infty} u \, du
\]

which diverges!

This surface has finite volume and infinite surface area! Wow! That means you can fill it with a finite amount of paint, but if you wanted to paint it, you'd never finish.