1. Find the prime factorization of 111111.
   \textit{Solution:} 111111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37

2. (a) Which positive integers have exactly three positive divisors?
   \textit{Solution:} \( n = p^2 \), where \( p \) is prime.

   (b) Which positive integers have exactly four positive divisors?
   \textit{Solution:} \( n = p_1p_2 \), where \( p_1 \) and \( p_2 \) are distinct primes, and \( n = q^3 \), where \( q \) is prime.

   (c) Suppose \( n \geq 2 \) is an integer with the property that whenever a prime \( p \) divides \( n \), \( p^2 \) also divides \( n \) (i.e. all primes in the prime factorization of \( n \) appear at least to the power 2). Prove that \( n \) can be written as the product of a square and a cube.

   \textit{Proof.} Let \( n = p_1^{a_1}p_2^{a_2} \cdots p_m^{a_m} \) be the prime factorization of \( n \), where each \( p_i \) is a distinct prime and \( a_i \geq 2 \) for all \( i \). It suffices to prove that we can find a factorization of \( n \) in which the exponent of each factor is either a multiple of 2 or a multiple of 3. So, if every exponent \( a_i \) is already either a multiple of 2 or a multiple of 3, then we are happy and done! Therefore, we suppose there is some exponent \( a_k \) that is neither a multiple of 2 nor a multiple of 3 (5 is an example of such a positive integer). Note that \( a_k \) is an odd integer greater than 3. Hence \( a_k - 3 \) is even. Thus, if there is any prime power \( p_k^{a_k} \) in the factorization above, where \( a_k \) is neither a multiple of 2 nor 3, we write \( p_k^{a_k} = p_k^{a_k - 3}p_k^3 \). Therefore, the prime factorization of \( n \) can be written in such a way that each exponent is either a multiple of 2 or a multiple of 3 (and note that now this factorization may not have each prime distinct). \( \Box \)

4. Prove that \( \text{lcm}(a, b) = \frac{ab}{\gcd(a, b)} \) for any positive integers \( a, b \) without using prime factorization.

   \textit{Proof.} This is a sketch of the proof. You are left to fill in the details.

   Let’s start with basic notation. Let \( m = \text{lcm}(a, b) \) and \( d = \gcd(a, b) \). We want to show that \( ab = dm \).

   (a) First show that since \( d \) divides \( a \) and \( d \) divides \( b \), then \( d \) must also divide the product \( ab \).

   (b) Once you’ve shown the above, this means (by definition) that we can write \( ab = dn \) for some integer \( n \). Now the goal of the problem is to show that \( n \) must actually be equal to \( m \).

   (c) Next, show that \( n \) is a common multiple of \( a \) and \( b \). That is, show \( a \) divides \( n \) and \( b \) divides \( n \).

   (d) Finally, show that \( n \) divides \( m \).
(e) Note that the previous two steps yield $n = m$. From item (c), we can conclude that $m \leq n$ (since $m$ is the LEAST common multiple of $a$ and $b$ it must be less than or equal to every common multiple of $a$ and $b$). From item (d) we can conclude that $n \leq m$. Thus, these two inequalities yield $n = m$. \qed

6. On your own or discuss in section.

8. Find all solutions $x, y \in \mathbb{Z}$ to the following Diophantine equations:

(a) $x^2 = y^3$
   Solution: Any integer that is both a square and a cube is a 6th power, and conversely, every integer that is a 6th power is both a square and a cube. So the solutions are $x = a^3$ and $y = a^2$ for every integer $a$.

(b) $x^2 - x = y^3$
   Solution: Factor the left hand side as $x(x - 1)$. The two integers $x$ and $x - 1$ are coprime, and their product is a cube. Thus, by Proposition 12.4, both $x$ and $x - 1$ are cubes, and in particular, their difference is 1. The only integers $x$ that make this true are $x = 0, 1$. Hence the solutions are $x = 0, y = 0$ and $x = 1, y = 0$.

(c) $x^2 = y^4 - 77$
   Solution: $x = 4, y = 3$ is one solution. Are there any others?

(d) $x^3 = 4y^2 + 4y - 3$
   Solution: Factor the right hand side to obtain $x^3 = (2y - 1)(2y + 3)$ now mimic Example 12.1.