Midterm 1 solutions

You should be aware that there are frequently multiple solutions to each problem. I am including only one solution here. Feel free to ask me about other valid solutions.

Problem 1:

Each person shakes hands with n-2 people (not self, not spouse).
This gives a total of \( n(n-2) \) shakes. However, this counts person A shaking hands with person B, as well as person B shaking hands with person A. Thus, we have overcounted by a factor of 2. Hence, there are \( \frac{n(n-2)}{2} \) total shakes.
Problem 2: (use subtraction principle)

Let $P$ be the set of all subsets of $S = \mathbb{Z}_{1, 2, \ldots, 10}$.

Let $A$ be the subsets of $S$ that do not contain 1, 3, and 7.

We want to count $\overline{A}$, the complement of $A$ in $P$.

Notice that $A$ is the set of all subsets of $S \setminus \{1, 3, 7\}$.

We have proven in class that an $n$-element set has $2^n$ subsets.

Thus,

$$|\overline{A}| = |P| - |A| = 2^{10} - 2^7$$
Problem 3:

Suppose the parent sits at the table. Now we seat the children relative to the parent.

There are two cases: either a boy is in the seat immediately to the right of the parent or a girl is in the seat immediately to the right of the parent.

Case 1:

\[
\begin{align*}
\text{G} & \quad \text{P} & \quad \text{B} & \quad 5 \text{ chairs} \\
\text{G} & \quad \text{B} & \quad \text{G} & \quad 5 \text{ chairs} \\
\text{B} & \quad \text{G} & \quad \text{B} & \quad 4 \text{ chairs} \\
\text{B} & \quad \text{G} & \quad \text{B} & \quad 4 \text{ chairs} \\
\end{align*}
\]

Thus there are \((5!)^2\) possibilities in case 1.

Case 2 is similar and hence has \((5!)^2\) possibilities. So by the addition principle there are \(2 \cdot (5!)^2\) ways to seat these people.
Problem 4: (similar to "sticks" problem in Homework 4)

The number of ways to choose 3 integers between 1 and 20, no two of which are consecutive, is in 1-1 correspondence with non-negative integral solutions of the equation \( x_1 + x_2 + x_3 + x_4 = 17 \) when \( x_1 \geq 0, \ x_2 \geq 1, \ x_3 \geq 1, \ x_4 \geq 0 \).

(Recall: \( x_1 \) represents number of integers less than 1st choice,
\( x_2 \) represents number of integers between 1st & 2nd choice,
\( x_3 \) represents number of integers between 2nd & 3rd choice,
\( x_4 \) represents number of integers greater than 3rd choice.)

Change variables so that each variable represents a non-negative integer.
So

\[ x_1 = y_1 \]
\[ x_2 - 1 = y_2 \]
\[ x_3 - 1 = y_3 \]
\[ x_4 = y_4 \]

Now substitute: \( y_1 + y_2 + y_3 + y_4 = 15 \)

Now by Thm 2.5.1 using \( k = 4 \) and \( r = 15 \) we have that the number of non-negative integral solutions to the equation above is \( \binom{r + k - 1}{r} = \binom{18}{15} \).
Problem 5:

First consider the case that $n$ is odd. The number of even subsets of $S$ is given by

\[ \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-3} + \binom{n}{n-1} \]

and the number of odd subsets of $S$ is given by

\[ \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-2} + \binom{n}{n} \]

Now using the symmetry property, $\binom{n}{k} = \binom{n}{n-k}$

we see that the number of even subsets of $S$ equals the number of odd subsets of $S$.

(Aside: why does this argument not hold for even $n$?)
Now suppose that \( n \) is even.

Choose an element \( x_0 \in S \) and consider \( S' = S - \{x_0\} \). By previous argument, we know that the number of even subsets of \( S' \) equals the number of odd subsets of \( S'' \) (\( S' \) has odd order).

Notice that subsets of \( S \) can be partitioned into subsets containing \( x_0 \) and those that do not. That is, each subset of \( S \) either belongs to \( S' \) — in which case, we have \( A \subseteq S' \), or is of the form \( A \cup \{x_0\} \) for some \( A \subseteq S' \).

If \( A \) is odd, then \( A \cup \{x_0\} \) is even, and if \( A \) is even, then \( A \cup \{x_0\} \) is odd.

Thus every odd subset of \( S \) is either...
an odd subset of $S'$ or $A \cup \{x_0\}$ for some even subset $A$ of $S'$.

Similarly, every even subset of $S$ is either an even subset of $S'$ or $A \cup \{x_0\}$ for some odd subset $A$ of $S'$.

So the odd subsets of $S$ is exactly twice the number of odd subsets of $S'$.

and the number of even subsets of $S$ is exactly twice the number of even subsets of $S'$.

So since the property holds for $S'$, it also holds for $S$. 