The May model. This model has been proposed by the contemporary ecologist, R. M. May, to incorporate more realistic assumptions about the encounters between predators (foxes) and their prey (rabbits). So that you can work with quantities that are about the same size (and therefore plot them on the same graph), let \( y \) be the number of foxes and let \( x \) be the number of rabbits divided by 100. We are thus measuring rabbits in units of what we could call “centirabbits.”

In his model, May makes the following assumptions.

- In the absence of foxes, the rabbits grow logistically.
- The number of rabbits a single fox eats in a given time period is a function \( D(x) \) of the number of rabbits available. \( D(x) \) varies from \( a \) if there are no rabbits available to some value \( c \) (the saturation value) if there is an unlimited supply of rabbits. The total number of rabbits consumed in the given time period will thus be \( D(x)y \).
- The fox population is governed by the logistic equation, and the carrying capacity is proportional to the number of rabbits.

a) Explain why

\[
D(x) = \frac{cx}{x + d}
\]

(where \( d \) is some constant) might be a reasonable model for the function \( D(x) \). Include a sketch of the graph of \( D \) in your discussion. What is the role of the parameter \( d \)? That is, what feature of rabbit-fox interactions is reflected by making \( d \) smaller or larger?

b) Explain how the following system of equations incorporates May’s assumptions.

\[
x' = ax \left( 1 - \frac{x}{b} \right) - \frac{cxy}{x + d}
\]

\[
y' = ey \left( 1 - \frac{y}{fx} \right)
\]

The parameters \( a, b, c, d, e, \) and \( f \) are all positive.

c) Assume you begin with 2000 rabbits and 10 foxes. (Be careful: \( x(0) \neq 2000 \).) What does May’s model predict will happen to the rabbits and foxes over time if the values of the parameters are \( a = .6, b = 10, c = .5, d = 1, e = .1, \) and \( f = 2 \)?

d) Make a phase portrait (i.e., vector field with \( y \) vs. \( x \)) using the same parameters. Are there any stable equilibria?

e) Using the same parameters, describe what happens if you begin with 2000 rabbits and 20 foxes; with 1000 rabbits and 10 foxes; with 1000 rabbits and 20 foxes. Does the eventual long-term behavior depend on the initial condition?

f) Using 2000 rabbits and 20 foxes as the initial values, let’s see how the behavior of the solutions is affected by changing the values of the parameter \( c \), the saturation value for the number of rabbits (measured as centirabbits; remember) a single fox can eat in a month. Keeping all the other parameters \( (a, b, d, \ldots) \) fixed at the values given above, get solution curves for \( c = .3, c = .45, c = .4, c = .15, \) and \( c = .1 \). The solutions undergo qualitative change somewhere between \( c = .3 \) and \( c = .25 \). Describe this change. Can you pinpoint the crucial value of \( C \) more closely? This phenomenon is an example of Hopf bifurcation. The May model undergoes a Hopf bifurcation as you vary each of the other parameters as well. Choose a couple of them and locate approximately the associated bifurcation values.

g) Make a phase portrait for one of the interesting cases. What types of equilibria exist? Can they always be reached?

\[1\] While a term like “centirabbits” is deliberately whimsical, it echoes the common and sensible practice of choosing units that allow us to measure things with numbers that are neither too small nor too large. For example, we wouldn’t describe the distance from the earth to the moon in millimeters, and we wouldn’t describe the mass of a raindrop in kilograms.