ALEXIS BÈS. *Decidability and definability results related to the elementary theory of ordinal multiplication*. ***Fundamenta Mathematica***, vol. 171 (2002), pp. 197-211.

The metamathematics of ordinal arithmetic has not received much attention of late, so it was a pleasure to read this fine paper on definability and decidability for the theory of ordinal multiplication $\times$ by itself. Of course, much is already known about these matters for $+$ and $\times$ together, and for $+$ alone. Bès shows that an ordinal $\alpha > 0$ is definable in the theory of ordinal multiplication if and only if it is $< \omega^{\omega^{\omega^\omega}}$ and is of form $\omega^{\beta_1} + \cdots + \omega^{\beta_k}$ with $\omega^\omega > \beta_1 > \cdots > \beta_k$, and that $\mathrm{Th}\langle\alpha; \times\rangle$ is decidable if and only if $\alpha < \omega^\omega$ ($\times$ is the three-place multiplication relation, suitably restricted).

Most of the definitions needed in the proofs are formulated in terms of the notions of left- and right-hand divisibility of ordinals, denoted by $|_l$ and $|_r$, respectively (e.g., $\alpha \mid_r \beta$ means that $\beta = \gamma\alpha$ for some γ). Bès considers the theory based just on these notions and shows that $\mathrm{Th}\langle\alpha; |_r, |_l\rangle$ is decidable if and only if $\alpha < \omega^\omega$. On the other hand, for any ξ, $\mathrm{Th}\langle\omega^{\omega^\xi}; |_r\rangle$ and $\mathrm{Th}\langle\omega^\xi; |_l\rangle$ are each decidable. The proofs are interesting applications of Feferman-Vaught generalized weak direct powers.

In the interests of brevity I will limit my outline of Bès' proofs to the definability and decidability results about the theory of ordinal multiplication. Let's begin by reviewing the prior situation: $\mathrm{Th}\langle\omega; +\rangle$ is decidable, and every natural number is definable in this theory (Presburger); all $\mathrm{Th}\langle 2^\alpha; +\rangle$ are decidable (Büchi); an ordinal α is definable in the theory of addition iff $\alpha < \omega^{\omega^\omega}$, and in the theory of both addition and multiplication, iff $\alpha < \omega^{\omega^{\omega^\omega}}$ (Ehrenfeucht). $\mathrm{Th}\langle\omega; \times\rangle$ is decidable (Skolem, Mostowski), but only 0 and 1 are definable in this theory: Given $m > 1$, let p be a prime factor of m, q be some other prime not a factor of m, and let f be the mapping on ω that simply interchanges p and q in the prime decomposition of its argument. Then f is an automorphism of $\langle\omega; \times\rangle$ and $f(m) \neq m$, so m is not definable. Bès has filled in the main blank in the picture: what happens with multiplication alone in the transfinite domain.

Key to the development is an obscure but beautiful 1909 theorem of Jacobsthal, included in Sierpinski's book *Cardinal and Ordinal Numbers*, that just as with finite numbers, every transfinite ordinal has a prime decomposition. A prime is an ordinal $\geqslant 2$ which is not a product of two smaller ordinals. Transfinite primes can be shown to be ordinals of one of the forms ω^{ω^ξ} or $\omega^\gamma + 1$ with $\gamma > 0$. Thus, there are limit primes and infinite successor primes. The prime factorization of an infinite ordinal is not unique in the same way as for finite ordinals, as the equation $\omega\omega = (\omega + 1)\omega$ illustrates. However, the factorization is unique if we simply add the requirement that all limit prime factors must appear first in non-increasing order. If an ordinal α has the Cantor Normal Form (CNF) $\omega^{\alpha_1}a_1 + \omega^{\alpha_2}a_2 + \cdots + \omega^{\alpha_k}a_k$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ and the a_i are finite and > 0, and if the CNF of α_k is $\omega^{\mu_1}m_1 + \omega^{\mu_2}m_2 + \cdots + \omega^{\mu_n}m_n$, then the Prime Factorization (PF) of α is obtained from

$$(\omega^{\omega^{\mu_1}})^{m_1}(\omega^{\omega^{\mu_2}})^{m_2}\cdots(\omega^{\omega^{\mu_n}})^{m_n}a_k(\omega^{\alpha_{k-1}-\alpha_k}+1)a_{k-1}\cdots(\omega^{\alpha_1-\alpha_2}+1)a_1$$

by replacing the finite numbers $a_1, \ldots, a_k$ by their prime factorizations.

Let T be the theory of ordinal multiplication, i.e., $T = \mathrm{Th}\langle\mathbf{On}; \times\rangle$ where $\mathbf{On}$ is the class of all ordinals. Ehrenfeucht showed $T = \mathrm{Th}\langle\omega^{\omega^{\omega^\omega}}; \times\rangle$. 0,1 are definable in T, but what else? Not all ordinals $< \omega^{\omega^{\omega^\omega}}$: given $\alpha = \omega^{\alpha_1}a_1 + \omega^{\alpha_2}a_2 + \cdots + \omega^{\alpha_k}a_k$ (CNF) with some $a_i > 1$, we use the same idea as in the finite domain, and define a function f on ordinals using the PF; f interchanges two different finite primes only one of which is a factor of α_i, and is the identity on infinite primes. Then f is an automorphism of $\langle\omega^{\omega^{\omega^\omega}}; \times\rangle$ and $f(\alpha) \neq \alpha$. Bès goes on to show that all other infinite ordinals $< \omega^{\omega^{\omega^\omega}}$

are definable in T—that is, strictly decreasing sums of powers of ω are definable, but nothing else.

The idea is to first show that every power of ω is definable, and then that sums of such powers $\omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k}$ are as well. Let's consider this last step first. If $\alpha < \omega^{\omega^{\omega^{\omega}}}$ is a strictly decreasing sum of powers of ω, we can look at both its CNF and its PF:

$$\begin{aligned}\alpha &= \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}\\ &= (\omega^{\omega^{\mu_1}})^{m_1}(\omega^{\omega^{\mu_2}})^{m_2}\cdots(\omega^{\omega^{\mu_n}})^{m_n}(\omega^{\alpha_{k-1}-\alpha_k}+1)\cdots(\omega^{\alpha_1-\alpha_2}+1)\end{aligned}$$

where the CNF of α_k is $\omega^{\mu_1}m_1 + \omega^{\mu_2}m_2 + \cdots + \omega^{\mu_n}m_n$. From this it follows that if each power of ω and each infinite successor prime (both $< \omega^{\omega^{\omega^{\omega}}}$, of course) are definable, then α is definable.

Now we turn to the definability of powers of ω. Unfortunately, there are a couple of errors in Bès' argument at this point; of the definitions given on page 201, two (for powers of ω and of ω itself) don't work. But the problem is easy to fix. The definitions of the set of primes as those ordinals with exactly two right-hand divisors and of the set of limit primes as those primes with more than two left-hand divisors are fine. The powers of ω are exactly those ordinals with a limit prime as a right-hand divisor. We note that although the full order relation is not definable in T, an important fragment of it is: $0 < \alpha \leqslant \omega^\gamma$ if and only if α is a left-hand divisor of ω^γ. ω is the least limit prime. The successor primes are those infinite primes that are not limit primes.

$\langle \omega^{\omega^{\omega}}; + \rangle$ is isomorphic to $\langle \{\omega^\gamma : \gamma < \omega^{\omega^{\omega}}\}; \times \rangle$ via the mapping $\gamma \mapsto \omega^\gamma$. Since every ordinal $< \omega^{\omega^{\omega}}$ is definable in $\mathrm{Th}\langle \omega^{\omega^{\omega}}; + \rangle$, it follows that every ω^γ with $\gamma < \omega^{\omega^{\omega}}$ is definable in T. In particular, each individual limit prime $< \omega^{\omega^{\omega^{\omega}}}$ is definable. Each successor prime $\omega^\gamma + 1$ is definable since it is the only successor prime between ω^γ and $\omega^{\gamma+1}$. Limit ordinals are the ones with ω as a left-hand divisor, successors are the others. Combining all these facts and examining the formula for the PF, we see that every infinite ordinal $< \omega^{\omega^{\omega^{\omega}}}$ which does not have a finite prime among its factors, i.e., every strictly decreasing sum of powers of ω, is definable.

Bès next considers the decidability of T, and answers this question in the negative by interpreting the two-letter word problem into T. Consider the alphabet $\Sigma = \{1, 2\}$ and the mapping

$$m_0 m_1 \ldots m_n \;\mapsto\; (\omega^{m_0}+1)(\omega^{m_1}+1)\ldots(\omega^{m_n}+1)$$

where $m_0 m_1 \ldots m_n$ is a word over Σ^*. Via this mapping, the two-letter word problem is reduced to the decision problem for T.

Up to now, I have discussed $T = \mathrm{Th}\langle \omega^{\omega^{\omega^{\omega}}}; \times \rangle$, whereas Bès' results apply to other theories $\mathrm{Th}\langle \alpha; \times \rangle$. In particular, he shows that ordinal definability in $\mathrm{Th}\langle \omega^\xi; \times \rangle$ is exactly what one might expect—the ordinals definable there are those definable in T which are also $< \omega^\xi$. Further, the undecidability result applies to any theories $\mathrm{Th}\langle \alpha; \times \rangle$ with $\alpha \geqslant \omega^\omega$. But when $\alpha < \omega^\omega$, we do get decidability, which Bès shows by interpreting the relevant theories into $\mathrm{Th}\langle \omega; \times, 0, 1, 2, \ldots \rangle$, a theory shown decidable by Mostowski.

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