PA FIRST-ORDER PEANO'S ARITHMETIC

$$Sx \approx Sy \rightarrow x \approx y \qquad \mathbf{S+1}$$

$$\neg(\bar{0} \approx Sx)$$

$$\begin{cases} x + \bar{0} \approx x \\ x + Sy \approx S(x+y) \end{cases}$$

$$\begin{cases} x \cdot \bar{0} \approx \bar{0} \\ x \cdot Sy \approx (x \cdot y) + x \end{cases}$$

$$\begin{cases} x \uparrow \bar{0} \approx S\bar{0} \\ x \uparrow Sy \approx (x \uparrow y) \cdot x \end{cases}$$

and, for each formula $\varphi(x, \vec{y})$ with a free variable x and possibly some others $\vec{y} = y_1, \ldots, y_n$

$$\varphi(\bar{0}, \vec{y}) \wedge \forall x \left(\varphi(x, \vec{y}) \rightarrow \varphi(Sx, \vec{y})\right) \rightarrow \forall x \, \varphi(x, \vec{y})$$

Logical axioms:

$$x \approx y \wedge y \approx z \rightarrow x \approx z$$

$$(\varphi(x) \rightarrow \psi) \rightarrow (\exists x\, \varphi(x) \rightarrow \psi)$$

provided x not free in ψ,

$$\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x \varphi \rightarrow \forall x \psi)$$

etc.

Rule of inference:

Infer ψ from φ and $\varphi \rightarrow \psi$

Proof of φ:

A sequence $\varphi_1, \ldots, \varphi_n = \varphi$ where for each i, either

φ_i is an axiom

or

for some $j, k < i$

$\varphi_k = \varphi_j \rightarrow \varphi_i$

$\vdash \varphi$ means "φ has a proof"

IMPORTANT: The formulas are strings of symbols built up from

$$\approx \quad S \quad (\quad) \quad \rightarrow \quad \wedge \quad \vee \quad \leftrightarrow \quad \neg$$
$$\forall \quad \exists \quad \bar{0} \quad v \quad + \quad \cdot \quad \uparrow$$

By <u>interpreting</u> these symbols, you can attach meanings to the formulas. But the formulas are not mere abbreviations for these meanings, they are objects themselves.

Formal variables: Need infinitely many.

$$v \;,\; vv \;,\; vvv \;,\; vvvv \;,\; \ldots$$
$$\| \qquad \| \qquad \| \qquad \|$$
$$v_0 \qquad v_1 \qquad v_2 \qquad v_3$$

$x, y, z, \ldots$ — formal variables

τ, π — terms

e.g. x, $(x + y) \uparrow (S\bar{0} + z)$

φ, ψ, ϑ — formulas

σ — sentences (formulas with no free variables)

$\varphi[x / \tau]$ — replace the free occurrences of x in φ by τ

$\varphi(z, \tau)$ is conventional:

You have some understanding that φ has two free variables x, y; then

$$\varphi(z, \tau) = \varphi[x/z][y/\tau]$$

Numerals

$$\bar{0}$$

$$\bar{1} = S\,\bar{0}$$

$$\bar{2} = S\,\bar{1} = S\,S\,\bar{0}$$

$$\bar{3} = S\,\bar{2} = S\,S\,S\,\bar{0}$$

$$\vdots$$

$$Prime(x) = \neg\, x \approx \bar{0} \wedge \forall u \forall v \,(x \approx u \cdot v \rightarrow u \approx \bar{1} \vee u \approx x)$$

$$even(x) = \exists y\,(x \approx \bar{2} \cdot y)$$

$$x < y = \exists z\,(\neg\, z \approx \bar{0} \wedge y \approx x + z)$$

$$\forall x \,(even(x) \wedge \bar{0} < x \rightarrow \exists y\, \exists z\, (Prime(y) \wedge Prime(z) \wedge x \approx y + z))$$

PA expresses and proves essentially all of Hardy & Wright

HISTORICAL OVERVIEW

Paradise : Cantor's Set Theory.

The blossoming of infinitistic and non-constructive mathematics. Set theory provides a coherent framework for unifying mathematics, dealing with infinite sets. It is very elegant and *obviously correct.*

Paradise Lost: The paradoxes.

Russell's paradox:

Let S be the set of all those sets which are not members of themselves. Is S a member of S, or not?

Ricard's paradox:

List the possible properties of natural numbers; $P_0(n)$, $P_1(n)$, ... etc. Some numbers n possess property P_n, and others do not. That n does not have property P_n is itself a property of natural numbers, and must appear in the list. Say it's P_k. Do we have $P_k(k)$, or not?

Hilbert's Program:

"We shall not be driven from the paradise to which Cantor has led us." Using only finitary methods, prove that non-finitary procedures can't lead to error.

Finitary methods: No reference to infinities as completed; only as 'potential.' Use only simplest rules of logic. Apply induction only to prove equations of constructive functions.

Prove the consistency of normal mathematics (set theory?) by using only that part of it with which no one can quarrel.

Consistency: You can't prove both a statement and its negation.

Gödel: Hilbert's program can't be done.

If a theory is strong enough to do even a minimum part of finitary mathematics, then it can't prove its own consistency.

Gödel Numbers

Each symbol gets a 5-bit code, e.g.

$\approx$	00100	S	10001
v	01000	+	01001

Consider the equation

$$v_0 \approx S v_2 + v_1$$

Officially, it is

$$v \approx S v v v + v v$$

The Gödel number is, in binary,

01000 00100 10001 01000 01000 01000 01001 01000 0100

$v \approx S v v v + v v$

$\ulcorner v_0 \approx S v_2 + v_1 \urcorner =$

$\ulcorner \alpha \urcorner$ = Gödel Number of α

By adding symbol , we can have Gödel numbers for lists of strings:

$\ulcorner \alpha_1, \ldots, \alpha_n \urcorner$

Property $P(n)$ is represented by φ:

$P(n)$ holds $\Rightarrow$ $\vdash \varphi(\bar{n})$

$P(n)$ fails $\Rightarrow$ $\vdash \neg\varphi(\bar{n})$

Function $f(n)$ is represented by φ:

$f(n) = m \Rightarrow \vdash \varphi(\bar{n}, \bar{m})$

$f(n) \neq m \Rightarrow \vdash \neg\varphi(\bar{n}, \bar{m})$

All primitive recursive functions and predicates are represented.

Usually, more simple properties are formally provable, e.g., p.r. functions are represented by formulas φ for which

$\vdash \varphi(x,y) \wedge \varphi(x,z) \rightarrow y \approx z$

There is a primitive recursive function $sbn(m, n)$ which does this:

> Substitute $\bar{n}$ for the first free variable in the formula with Gödel number m, and return the Gödel number of the result.

$$sbn(\ulcorner \varphi(x) \urcorner, n) = \ulcorner \varphi(\bar{n}) \urcorner$$

Let $Sbn(x, y, z)$ represent sbn; then

$$\vdash Sbn(\overline{\ulcorner \varphi(x) \urcorner}, \bar{n}, \overline{\ulcorner \varphi(\bar{n}) \urcorner})$$

$\varphi(x), n \qquad \varphi(\bar{n})$

$\ulcorner \varphi(x) \urcorner, n \qquad \ulcorner \varphi(\bar{n}) \urcorner$

This is primitive recursive:

$prf(m,n) \overset{\Delta}{\iff}$ m is the Gödel number of a proof of the formula with Gödel number n

Let $Prf(x,y)$ represent prf. Then if $\varphi_1, \ldots, \varphi_n$ is a proof of φ, we have

$$prf(\ulcorner \varphi_1, \ldots, \varphi_n \urcorner, \ulcorner \varphi \urcorner).$$

and hence

$$\vdash Prf(\overline{\ulcorner \varphi_1, \ldots, \varphi_n \urcorner}, \overline{\ulcorner \varphi \urcorner})$$

DIAGONAL LEMMA:

Given a formula $\varphi(x)$ there is a sentence σ such that

$$\vdash \sigma \leftrightarrow \varphi(\overline{\ulcorner \sigma \urcorner})$$

Proof: Let

$$\vartheta(y) = \forall x (sbn(y, y, x) \rightarrow \varphi(x))$$

$$m = \ulcorner \vartheta(y) \urcorner$$

$$\sigma = \vartheta(\bar{m})$$

Now

$$sbn(m, m) = \ulcorner \vartheta(\bar{m}) \urcorner = \ulcorner \sigma \urcorner$$

so

$$\vdash sbn(\bar{m}, \bar{m}, x) \leftrightarrow x \approx \overline{\ulcorner \sigma \urcorner}$$

and hence

$$\vdash \underbrace{\forall x (sbn(\bar{m}, \bar{m}, x) \rightarrow \varphi(x))}_{\sigma} \leftrightarrow \varphi(\overline{\ulcorner \sigma \urcorner})$$

RICARD'S PARADOX, FORMALIZED

Suppose there is a truth definition; a formula $Tr(x)$ such that for each sentence σ,

$Tr(\overline{\ulcorner\sigma\urcorner})$ is true iff σ is true.

Apply the Diagonal Lemma:

$$\vdash \sigma \leftrightarrow \neg Tr(\overline{\ulcorner\sigma\urcorner})$$

Anything provable is true, so

σ is true iff $\neg Tr(\overline{\ulcorner\sigma\urcorner})$ is true.

" iff $Tr(\overline{\ulcorner\sigma\urcorner})$ is false

" iff σ is false.

So there is no truth definition

(Tarski)

$$Pr(x) \triangleq \exists y\, Prf(y, x)$$

The first derivability condition:

(D1) $\quad \vdash \varphi \implies \vdash Pr(\overline{\ulcorner \varphi \urcorner})$

For if $\vdash \varphi$, then φ has a proof $\varphi_1, \ldots, \varphi_n$, so

$$Prf(\ulcorner \varphi_1, \ldots, \varphi_n \urcorner, \ulcorner \varphi \urcorner) \quad \text{holds},$$

so

$$\vdash Prf(\overline{\ulcorner \varphi_1, \ldots, \varphi_n \urcorner}, \overline{\ulcorner \varphi \urcorner})$$

so

$$\vdash Pr(\overline{\ulcorner \varphi \urcorner})$$

Gödel's 1st Incompleteness Theorem follows the pattern of Ricard, but using provability instead of truth.

1ST INCOMPLETENESS THEOREM

THERE IS A TRUE SENTENCE σ SUCH THAT, IF PA IS CONSISTENT, THEN $\not\vdash \sigma$.

Proof. Using the Diagonal Lemma, let

$$\vdash \sigma \leftrightarrow \neg Pr(\overline{\ulcorner \sigma \urcorner})$$

Then

$$\vdash \sigma \overset{(D1)}{\Rightarrow} \vdash Pr(\overline{\ulcorner \sigma \urcorner}) \Rightarrow \vdash \neg \sigma$$

So if PA is consistent, then $\not\vdash \sigma$. Further, for each n, n isn't the G.N. of a proof of σ, so

$$prf(n, \ulcorner \sigma \urcorner)$$

is false for each n. Thus,

$$\exists y\, Prf(y, \overline{\ulcorner \sigma \urcorner})$$

i.e., $Pr(\overline{\ulcorner \sigma \urcorner})$ is false. So σ is true.

Two more derivability conditions:

(D2) $\vdash Pr(\overline{\ulcorner\varphi\urcorner}) \wedge Pr(\overline{\ulcorner\varphi \to \psi\urcorner}) \to Pr(\overline{\ulcorner\psi\urcorner})$
for any φ, ψ.

(D3) $\vdash Pr(\overline{\ulcorner\varphi\urcorner}) \to Pr\left(\overline{\ulcorner Pr(\overline{\ulcorner\varphi\urcorner})\urcorner}\right)$

(D2) is easy.

(D3) is straight forward, but very lengthy. It is a formalization of (D1)

$$Con_{PA} \triangleq \neg Pr(\overline{\ulcorner \bar{0} \approx \bar{1} \urcorner})$$

Note that, with (D1), (D2) one easily shows

$$\vdash Pr(\overline{\ulcorner\varphi\urcorner}) \to \left(Pr(\overline{\ulcorner\neg\varphi\urcorner}) \to \neg Con_{PA}\right)$$

$$\vdash Con_{PA} \leftrightarrow \neg\left(Pr(\overline{\ulcorner\varphi\urcorner}) \wedge Pr(\overline{\ulcorner\neg\varphi\urcorner})\right)$$

2nd Incompleteness Theorem

PA consistent $\Rightarrow \nvdash Con_{PA}$

Proof. We have $\vdash \sigma \leftrightarrow \neg Pr(\overline{\ulcorner \sigma \urcorner})$

By (D3),

$$\vdash Pr(\overline{\ulcorner \sigma \urcorner}) \rightarrow Pr(\overline{\ulcorner Pr(\overline{\ulcorner \sigma \urcorner}) \urcorner})$$

With $\vartheta = Pr(\overline{\ulcorner \sigma \urcorner})$, this is

(*) $$\vdash \vartheta \rightarrow Pr(\overline{\ulcorner \vartheta \urcorner})$$

But also,

$$\vdash \sigma \rightarrow \neg \underbrace{Pr(\overline{\ulcorner \sigma \urcorner})}_{\vartheta}$$

so by (D1), (D2)

$$\vdash Pr(\overline{\ulcorner \sigma \urcorner}) \rightarrow Pr(\overline{\ulcorner \neg \vartheta \urcorner})$$

(**) i.e. $$\vdash \vartheta \rightarrow Pr(\overline{\ulcorner \neg \vartheta \urcorner})$$

Lines (*), (**) show

$$\vdash Con_{PA} \rightarrow \neg \vartheta$$

But $\vdash \sigma \leftrightarrow \neg \vartheta$, so

$$\vdash Con_{PA} \rightarrow \sigma$$

$\nvdash \sigma$, so $\nvdash Con_{PA}$.