

Text: Set Theory, Muenen

Office: T 10-11:30, R 1:30-3

The plan: Intro historical,
then formal axiomatic.

Grading: Problems, take-home final

Set: collection of objects

- i.e., ϵ is primitive

$$A = B \iff \forall x (x \in A \iff x \in B)$$

$\subseteq, \not\subseteq, \cup, \cap, \setminus, \emptyset$

$\{a\}, \{a, b\}$

$\{x : P(x)\} \quad \{x \in A : P(x)\}$

Algebra of sets

\subseteq, \supseteq arguments to prove eqns.

Abstractions & set-theoretic models.

Possible ur-elements?

Ordered pair concept

Set-Theoretic model: $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$

What's an ordered triple?

$A \times B$

Function concept

$$f(x) = x^2 \text{ \& } g(y) = y^2 \Rightarrow f = g$$

Motivated by expressions,
but not fully satisfactory

Set Theoretic model:

$f =$ set ordered pairs with property

$$\langle x, y_1 \rangle, \langle x, y_2 \rangle \in f \Rightarrow y_1 = y_2$$

$f(x), \text{dom}(f), \text{ran}(f), \text{fld}(f)$

$f \upharpoonright A, f \circ A \quad f: A \rightarrow B, \quad A^B$

1-1, onto, surjection, injection, $f^{-1}, f \circ g$

Relations: dom, ran, ...

Properties: symm, trans, ...

Do line $\cup, \cap, \cup \dots f(x)$, etc.

A collection is a set of objects ---

P is a "property"

$$\{a, b\} = \{x : x = a \text{ or } x = b\}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

* PROBLEM: Make a list of basic properties of $\cup, \cap, \setminus, \emptyset$.
How do you know when you're done?

- Need for Cartesian coords.

* Problem: Finish proof
 $\langle a, b \rangle = \langle a', b' \rangle \iff a = a' \text{ \& } b = b'$
 $\langle a, b, c \rangle = \langle a, \langle b, c \rangle \rangle$ isn't fully satisfactory because it's also an ordered pair.

* Propose another model for the ordered pair.

Model for $\langle, \subseteq, \dots$

1st successes of Set Theory

\mathbb{N}, \mathbb{R}

$A \approx B \quad \exists f: A \xrightarrow{\text{onto}} B$

$A \preccurlyeq B \quad \exists f: A \xrightarrow{\text{into}} B$

SCHRÖDER - BERNSTEIN

Strange fact: $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

Worse (or better)

$\{2n : n \in \mathbb{N}\} \approx \mathbb{N}$

So "the whole is not greater than its part"

And yet: $A < \mathcal{P}(A)$; esp. $\mathbb{N} < \mathcal{P}(\mathbb{N})$

- Thus, there is a hierarchy of infinities.

$\mathbb{N} < \mathbb{R}$

$\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$

Numbers & Counting

A natural number is a type

"2" is the abstract essence of twoness.

$2 = \{x : \exists y, z \ x = \{y, z\}\} ?$

Seek set-theoretic model of Natural numbers.

Possible: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots$

Better: $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots$

$S(A) = A \cup \{A\}$, successor

$\mathbb{N} = S$ -closure of $\{\emptyset\}$

$= \bigcap \{Y : \emptyset \in Y \ \& \ \forall x \in Y \ Sx \in Y\}$

$0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, 1\}, \dots$

Note: U undoes S

$U S(n) = n \quad n < m : n \in m$

n, m range over \mathbb{N}

Nat. nos., reals are abstract objects.

equipollence

Assume A, B disjoint. Give ultimate ancestor proof. Assign write-up for later, using recursion? See below

$f(\langle m, n \rangle) = 2^m \cdot (2n+1) - 1$

odd numbers $\langle 0, 0 \rangle \langle 0, 1 \rangle, \dots$
 once div. by 2 $\langle 1, 0 \rangle \langle 1, 1 \rangle, \dots$
 twice div by 2 $\langle 2, 0 \rangle \langle 2, 1 \rangle, \dots$

Let $f: A \xrightarrow{\text{into}} \mathcal{P}(A) = \{X : X \subseteq A\}$

Let $B = \{a \in A : a \notin f(a)\}$. Say $B = f(b)$
 $b \in B \iff b \notin f(b) \iff b \notin B$

Let $f: \mathbb{N} \rightarrow \mathbb{R} = [0, 1]$

$f(i) = .a_{i0} a_{i1} \dots$

Define $b_i = \begin{cases} 0 & \text{if } a_{ii} \neq 0 \\ 1 & \text{if } a_{ii} = 0 \end{cases}$ $b \notin \text{ran } f$

$f(.a_0 a_1 \dots, .b_0 b_1 \dots) =$ binary reps
 $.a_0 b_0 a_1 b_1 \dots$ ternary rep.

Shows $\mathbb{R} \times \mathbb{R} < \mathbb{R}$

"Ultimate Ancestor" Proof of Schröder - Bernstein:

A, B disjoint, $f: A \xrightarrow{\text{into}} B, g: B \xrightarrow{\text{into}} A$.

Trace back: $x, g^{-1}x, f^{-1}g^{-1}x \dots$

3 groups:
 ultimate ancestor in A
 " " in B
 no ultimate ancestor.

Given $x \in A$

$h(x) = \begin{cases} f(x) & \text{if ult. ancest. } x \text{ in } A \\ & \text{or no ult. ancest.} \\ g^{-1}(x) & \text{if ult. ancest. in } B \end{cases}$

\bar{x} = closure of x

$= \bigcap \{Y \subseteq A \cup B : x \in Y \ \& \ z \in Y \cap A \Rightarrow f(z) \in Y \ \& \ z \in Y \cap B \Rightarrow g(z) \in Y\}$

u is ult. ancestor of x if $x \in \bar{u}$, & u has no ancestor.

Proof by induction

$$\frac{Q(0) \quad \forall n(Q(n) \Rightarrow Q(Sn))}{\forall n Q(n)}$$

prove $Q(0)$
 Assume $Q(n)$, prove $Q(Sn)$
 Assuming these proofs, let
 $X = \{n : Q(n)\}$; then $N \subseteq X$.

- (1) $0 \leq n$
- (2) $m < n \Rightarrow Sm < n$
- (3) $m < n$ or $m = n$ or $n < m$
- (4) $m \in n \Rightarrow m \subseteq n$ (n is transitive)
- (5) $m \cap n \in N$ (with ⁽³⁾ $h(4)$, $m, n \in \{m, n\}$)
- (6) $m \cup n \in N$ (with ⁽³⁾ $h(4)$, $m, n \in \{m, n\}$)

* PROBLEM: PROVE THESE
 (Give in random order
 2 1 5 4 6 3
 - Prove (4) by induct. on n
 as example.

Least element principle:
 $0 \neq X \subseteq N \Rightarrow \cap X \in X$

Every non-empty set of
 Nat'l nos has a least element.
 $\mu X = \text{least in } X$
 $\mu n Q(n) = \cap \{n : Q(n)\}$, if not \emptyset

Sequences:
 $\langle b_0, \dots, b_{n-1} \rangle$ is a function
 with domain n .
 $b : n \rightarrow \text{ran}(b)$
 b_i is alt. notation for $b(i)$

So $f(n) = \underbrace{h(h(\dots h(a)\dots))}_{n \text{ times}}$

Recursive Definitions

$$\begin{cases} f(0) = a \\ f(Sn) = h(f(n)) \end{cases}$$

- (1): $n=0, g = \{\langle 0, a \rangle\}$
 $n = Sm$: Have g' with $\text{dom } g' = S(m)$
 $g = g' \cup \{\langle n, h(g'(m)) \rangle\}$

$$Y = \{ \langle g_0, \dots, g_n \rangle : g_0 = a \ \& \ \forall i < n \ g_{i+1} = h(g_i) \}$$

$$= \{ g : \exists n \ \text{dom } g = Sn \ \& \ \forall i < n \ i=0 \Rightarrow g(i) = a \ \& \ i = S(u_i) \Rightarrow g(i) = h(g(u_i)) \}$$

- (2) say $\text{dom } g_1 = n_1, \text{dom } g_2 = n_2$.
 say $n_1 \subseteq n_2$. S'pose $g_1 \neq g_2$
 Let m least in $\text{dom } g_1, \exists$
 $g_1(m) \neq g_2(m)$.
 $m \neq 0; m = S(k)$.
 $g_1(m) = h(g_1(k)) = h(g_2(k)) = g_2(m)$

Application: defn of +
 Show:

- (1) $\forall n \exists g \in Y \ \text{dom } g = Sn$
- (2) $g_1, g_2 \in Y \Rightarrow g_1 \subseteq g_2 \vee g_2 \subseteq g_1$
- (3) $f = \cup Y$ has req'd properties

(3) f is fcn. etc.

General Theorem:
 Given h There is unique f
 $f(n) = h(f \upharpoonright n)$

* Problem: give proof

Peano Structure $\langle A, \omega, \sigma \rangle$
 $\neg \omega(x) = \sigma$
 $\omega x = \omega y \Rightarrow x = y$
 $\sigma \in X \ \& \ \forall x [x \in X \Rightarrow \omega x \in X] \Rightarrow X = A$

THM If $\langle A, \omega, \sigma \rangle$ is Peano, then
 $\langle \mathbb{N}, S, 0 \rangle \cong \langle A, \omega, \sigma \rangle$

ORDER TYPES

R totally orders A :
 $R \subseteq A \times A$
 Trichotomy: $aRb \vee a = b \vee bRa$
 Transitivity
 Irreflexive

$\langle A, R \rangle$ is ordering.

Order type: $\sigma \langle A, < \rangle$

η = type of rationals
 λ = type of reals
 ω = type of $\langle \mathbb{N}, < \rangle$

Arithmetic of order types:

$\alpha + \beta, \alpha \cdot \beta$

Finite types: $\hat{n} = \sigma \langle n, \epsilon_n \rangle$

Assoc. law, not commut.

$\alpha \cdot (\beta_1 + \beta_2) = \alpha \cdot \beta_1 + \alpha \cdot \beta_2$

$1 + \omega = \omega \neq \omega + 1$

$\eta + \eta = \eta, \eta \cdot \omega = \eta$

Unclear how we can produce set-theoretic models for order types. (or infinite cardinals)

α^* = reversal of α

ω^* = type of neg. ints
 $\omega + \omega^*$ = type of ints

$|\alpha| = |A|$, where $\sigma \langle A, <_A \rangle = \alpha$

$\eta^* = \eta, \lambda^* = \lambda, \lambda + \lambda = \lambda$

ie, $\forall x (\neg \omega(x) = \sigma)$

ie, $\forall x [\sigma \in X \ \& \ \dots \Rightarrow X = A]$

A brief intro. to cardinal arith.

$m + n, m \cdot n, \aleph_0 + \aleph_0, \aleph_0 \cdot \aleph_0, \aleph = |\mathbb{R}|$

$m^m, m < 2^m$, standard arith. laws

$2^{\aleph_0} = \aleph_0^{\aleph_0}$

Countable: $\approx A \subseteq \mathbb{N}$

ctble union ctbl sets is countable.

enumeration: A map $f: \mathbb{N} \xrightarrow{1-1} \text{set}$
 why this needs a special axiom.

$\langle A, < \rangle$ or $\langle A, <_A \rangle$

✓ Sometimes, drop $R \subseteq A \times A$,
 in that case, use $R \cap A \times A$ indef.

Nat'l corresp: $<$ and \leq

$\tau \langle A, <_A \rangle = \alpha, \tau \langle B, <_B \rangle = \beta$

wlog, assume $A \cap B = \emptyset$

(else, replace A by $\{0\} \times A, B$ by $\{1\} \times B$)

$\alpha + \beta = \tau \langle A \cup B, <_{A \cup B} \cup A \times B \rangle$

$\alpha \cdot \beta = \tau \langle A \times B, R \rangle$,

$\langle a_1, b_1 \rangle R \langle a_2, b_2 \rangle \Leftrightarrow b_1 <_B b_2, \text{ or } b_1 = b_2 \ \& \ \alpha_1 <_A \alpha_2$

" β copies of α "

Cantor: If $\langle A, <_A \rangle$ is l.o., no first or last, and

$\forall x, y \ x <_A y \Rightarrow \exists z \ x <_A z <_A y$

then $\langle A, <_A \rangle \cong \langle \mathbb{Q}, < \rangle$

THM Every ctbl o.t. embeddable in η .

* Problem: $\langle A, < \rangle$ totally ordered, w. 1st & lth, each element has immed successor, each but first has immed. pred.,

$\Leftrightarrow \sigma \langle A, < \rangle = \omega + (\omega^* + \omega) \cdot \xi, \text{ some } \xi$

L

$\alpha < \beta : \alpha = \tau \langle B_a, S \rangle$ where $\beta = \tau \langle B, S \rangle, a \in B$ i.e., α is type of proper init. seg.

4) $<$ is total on ordinals.

Trichotomy: by 3), comparability of WO's.

Irreflex: $\alpha = \tau \langle A, R \rangle,$
 $\alpha < \alpha \Rightarrow \alpha = \tau \langle A_\alpha, R \rangle,$ so
 $\langle A, R \rangle \cong$ init seg itself.

$\bar{\alpha} = \{ \beta : \beta < \alpha \}$

Transitivity: $\alpha < \beta < \gamma$
 $\beta = \tau \langle C_x, T \rangle, \alpha = \tau \langle B_y, S \rangle$
 $\langle B, S \rangle \cong \langle C_x, T \rangle, \text{ so}$
 $\langle B_y, S \rangle \cong \langle (C_x)_{f(y)}, T \rangle = \langle C_{f(y)}, T \rangle$

5) $\tau \langle A, R \rangle = \alpha \Rightarrow \langle A, R \rangle \cong \langle \bar{\alpha}, < \rangle$

$\langle \bar{\alpha}, < \rangle$ is a W.O.

$\tau \langle \bar{\alpha}, < \rangle = \alpha$

5): Let $f(a) = \tau \langle A_a, R \rangle$
 onto: see def $<$
 i-1: $\langle A_a, R \rangle \cong \langle A_b, R \rangle$ would mean
 one \cong init seg of itself
 $a_1 R a_2 \Rightarrow A_{a_1} = (A_{a_2})_{a_1}$, so $f(a_1) < f(a_2)$

6) Any set of ordinals is WO by $<$.

7) Let A be set ordinals \Rightarrow

$\beta < \alpha \in A \Rightarrow \beta \in A$

Then $\alpha \in A \Rightarrow \alpha < \tau \langle A, < \rangle$

6) Let $\alpha \in A$. If α not first, look at
 $\{ \beta : \beta < \alpha \ \& \ \beta \in A \} \subseteq \bar{\alpha}$

7) $\alpha \in A \Rightarrow A_\alpha = \bar{\alpha}$. Since A is WO,
 it has type ξ . By def $<$, $\alpha < \xi$

SKIPPED AHEAD TO BURALI-FORTI

Least Element Principle:

$\exists \alpha Q(\alpha) \Rightarrow \exists \alpha (Q(\alpha) \wedge \forall \beta < \alpha \rightarrow \neg Q(\beta))$

Principle of Transfinite Induction

$\forall \alpha (\forall \beta < \alpha Q(\beta) \Rightarrow Q(\alpha)) \Rightarrow \forall \alpha Q(\alpha)$

In L.E.P., replace Q by $\neg Q$
 and take contrapositive.

Classification of ordinals:

- 0
- Successor $\alpha = S(\nu\alpha)$
- Limit $\alpha \neq 0 \ \& \ \alpha = \cup \alpha$

PTI:

$Q(0)$
 $\forall \alpha (Q(\alpha) \Rightarrow Q(S\alpha))$
 $(\forall \alpha \text{ limit}) \forall \beta < \alpha (Q(\beta) \Rightarrow Q(\alpha))$ } $\Rightarrow \forall \alpha Q(\alpha)$

$On = \{ \alpha : \alpha \text{ is an ordinal} \}$

By 6), On is W.O.

By 7), $\alpha \in On \Rightarrow \alpha < \xi = \tau \langle On, < \rangle$

But $\xi \in On$. $\therefore \xi < \xi$, and $\{ \xi \}$
 has no first.

CONTRADICTION!

Burali-Forti Paradox

On isn't a set.

Set - theoretic model of ordinals

ordinal: A transitive well-founded set totally ordered by \in

α, β, \dots represent ordinals

$$\alpha < \beta \iff \alpha \in \beta$$

Various Properties:

$$\alpha \in \beta \implies \alpha \leq \beta$$

$\alpha \cap \beta$ is an ordinal

$\alpha \cup \beta$ is an ordinal

$$\alpha \in \beta \text{ or } \alpha = \beta \text{ or } \beta \in \alpha$$

A set of ordinals $\implies \cup A$ is ordinal

$S(\alpha)$ is ordinal

$$\alpha < \beta \implies S(\alpha) \leq \beta$$

Successor $\bar{\alpha} = S(\cup \alpha)$, Limit $\alpha = \cup \alpha \neq 0$

Possibly, give PTI, LEP

Transitive:

$$\forall y (y \in x \implies y \subseteq x)$$

Well-founded:

$$\forall u \subseteq x [u \neq \emptyset \implies \exists z \in u \ z \cap u = \emptyset]$$

Totally ordered by \in

$$\forall u, v \in x [u \in v \text{ or } u = v \text{ or } v \in u]$$

Zermelo's conception.

Also: Other views

Type theory