

REGRESSION: DEF, INDUCTION & RECURSION

$X$  is definable over  $A$  is for some formula  $\varphi$ , with  $\text{Fr } \varphi \subseteq \{x, y_1, \dots, y_n\}$ , and some  $a_1, \dots, a_n$ ,

$$X = \{x : \varphi^{(A)}(x, a_1, \dots, a_n)\}$$

The idea:  $D(A) = \{X : X \text{ definable over } A\}$   
So  $D$  is like  $P$ , except just for <sup>intrinsically</sup> definable subsets.

$$L(0) = \emptyset$$

$$L(\alpha+1) = D(L(\alpha))$$

$$L(\lambda) = \bigcup_{\alpha < \lambda} L(\alpha) \quad \text{if } \lambda = \cup \alpha \neq 0$$

$$L = \bigcup \{L(\alpha) : \alpha \in \text{On}\}$$

- a proper class, the class of constructible sets

Actually, this doesn't work, because of the "any formula" clause. With that,  $L$  isn't defined by a single ZF-formula.

L2

$$\text{Proj}(A, R, n) = \{s \in A^n : \exists t \in R \quad t \upharpoonright n = s\}$$

↑  $R$  is a set of  $n+1$ -tuples.

$$\text{Diage}_\epsilon(A, n, i, j) = \{s \in A^n : s(i) \in s(j)\}$$

$$\text{Diag}_=(A, n, i, j) = \{s \in A^n : s(i) = s(j)\}$$

$$\left\{ \begin{array}{l} Df'(0, A, n) = \{\text{Diage}_\epsilon(A, n, i, j) : i, j < n\} \\ \quad \cup \{\text{Diag}_=(A, n, i, j) : i, j < n\} \\ \quad (\text{The } n\text{-tuples defined by } v_i \in v_j, v_i = v_j) \end{array} \right.$$

$$Df'(k+1, A, n) = \text{closure of } Df'(k, A, n)$$

under complement ( $\neg$ ),  
 intersection ( $\wedge$ ),  
 projection ( $\exists x$ )

$$\begin{aligned} &= Df'(k, A, n) \cup \{A^n - R : R \in Df'(k, A, n)\} \\ &\quad \cup \{R \cap S : R, S \in Df'(k, A, n)\} \\ &\quad \cup \{\text{Proj}(A, R, n) : R \in Df'(k, A, n+1)\} \end{aligned}$$

$$Df(A, n) = \bigcup_{k < \omega} Df'(k, A, n)$$

Lemma: If  $\phi(x_0, \dots, x_{n-1})$  is any fm1 then

$$\forall A \{s \in A^n : \phi^{(A)}(s(0), \dots, s(n-1))\} \in Df(A, n)$$

$$\mathcal{D}(A) = \{ X \subseteq A : \exists \text{new } \exists s \in A^n \exists R \in DF(A, n+1) \}$$

$$X = \{ x \in A : s^{\frown} (x) \in R \}$$

Then: Every <sup>subset of A</sup> <sub>set defined by a 1-formula, rel.</sub>  
to params in  $A$ , is in  $\mathcal{D}(A)$

$$\mathcal{D}(A) \subseteq P(A)$$

obvious

$$A \text{ transitive} \Rightarrow A \subseteq \mathcal{D}(A)$$

Let  $\phi$  be  $x \in v$

By lemma

$$\forall v \in A \quad \{ x \in A : x \in v \} \in \mathcal{D}(A)$$

$\uparrow$                        $\uparrow$   
 subset                      1-formula with  
 of  $A$                       parameter in

$\forall v \in A \quad v \in \mathcal{D}(A)$  when  $A$  is transitive.

$$X \subseteq A \text{ & } |X| < \omega \Rightarrow X \in \mathcal{D}(A)$$

If  $X = \{a_0, \dots, a_{n-1}\}$

use  $x = a_0 \vee \dots \vee x = a_{n-1}$

This proof is wrong!

$$(AC) \quad |A| \geq \omega \Rightarrow |\mathcal{D}(A)| = |A|$$

From  $|A^n| = |A|$

$$\forall \alpha \geq \omega, |\alpha| = |L(\alpha)|$$

Since  $\alpha \in L(\alpha)$ , we have  $|\alpha| \leq |L(\alpha)|$ .

$$L(\alpha+1) = D(L(\alpha)).$$

If  $|A| > \omega$ , then  $|D(A)| = |A|$ :

$$|D(A)| \leq \underbrace{\omega \otimes |A^n|}_{|A|} \otimes |Df(A, n+1)|$$

$$|Df(A, n)| \leq \omega \otimes |Df'(k, A, n)|, \text{ any } k.$$

$$|Df'(k, A, n)| \leq |A^n| = |A|.$$

On the other hand, even  $|R(\omega+1)| > |R(\omega)| = \omega$   
 $|R(\omega+\alpha)| = I_\alpha$

We show  $L$  is a model of ZF:

Extensionality:

$$\left( \forall x \forall y [ \forall z (z \in x \rightarrow z \in y) \wedge \forall z (z \in y \rightarrow z \in y) \rightarrow x = y ] \right)$$

holds by transitivity.

Similarly, we get  $\phi^{(\omega)}$  where  $\phi$  is  
Pairing, Union

Infinity because  $\omega \in L$

$\forall \alpha L(\alpha) + \text{transitive}, \forall \xi \leq \alpha L(\xi) \subseteq L(\alpha)$

Suppose true for all  $\beta < \alpha$ .

OK for  $\alpha = 0$  or  $\omega$  limit; say  $\alpha = \beta + 1$

$$L(\alpha) = D(L(\beta)) \subseteq P(L(\beta))$$

$L(\beta)$  trans., so  $L(\beta) \subseteq L(\alpha)$

$$x \in L(\alpha) \Rightarrow x \in P(L(\beta)) \Rightarrow x \in L(\beta).$$

$\therefore L(\alpha)$  trans.

$$\xi < \alpha \Rightarrow \xi < \beta \vee \xi = \beta$$

$L(\xi) \subseteq L(\beta) \subseteq L(\alpha)$ .

$$L(\alpha) \in L(\alpha + 1)$$

$$L(\alpha) = \{x \in L(\alpha) : (x = x)^{L(\alpha)}\} \in D(L(\alpha))$$

$$\forall \alpha L(\alpha) \subseteq R(\alpha)$$

$$L(\alpha + 1) = D(L(\alpha)) \subseteq P(L(\alpha))$$

if  $L(\alpha) \subseteq R(\alpha)$ , then

$$P(L(\alpha)) \subseteq P(R(\alpha)) = R(\alpha + 1)$$

$$\forall \alpha \in \text{On } L(\alpha) \cap \text{On} = \alpha$$

Trivial except when  $\alpha = \beta + 1$

$$\beta = L(\beta) \cap \text{On} = \{x \in L(\beta) : x \in \text{On}\}$$

$$= \{x \in L(\beta) : (x \in \text{On})^{L(\beta)}\} \quad \text{by absoluteness}$$

$\in D(L(\beta))$  by lemma  
 $\Downarrow L(\alpha)$ .

$$\therefore \alpha = \beta \cup \{\beta\} \subseteq L(\alpha) \cap \text{On}. \quad \text{If } \xi \in L(\alpha) \cap \text{On}, \text{ then}$$

in + above

$$\xi = \beta \text{ or } \xi \in \beta$$

(Power Set)<sup>(L)</sup>

$$\left( \forall x \exists y \forall z (z \subseteq x \rightarrow z \in y) \right)^{(L)}$$

$$\forall x \in L \exists y \in L \forall z \in L ((z \subseteq x)^{(L)} \rightarrow (z \in y)^{(L)})$$

$$\forall x \in L \exists y \in L \forall z \in L (z \subseteq x \rightarrow z \in y)$$

I.e., for  $x \in L$ , show

$$P_L(x) = \{ z \subseteq x : z \in L \} \in L$$

For  $z \in L$ , let  $f(z) = \text{least } \alpha : z \in L(\alpha)$

$$\text{Let } \beta = \bigcup \{ f(z) : z \subseteq x \wedge z \in L \}.$$

Thus  $P_L(x) \subseteq L(\beta)$ .

$$\begin{aligned} P_L(x) &= \{ y \in L(\beta) : \forall z (z \in y \rightarrow z \in x) \} \\ &= \{ y \in L(\beta) : \forall z \in L(\beta) (z \in y \rightarrow z \in x) \} \\ &\quad \uparrow \\ &\quad \text{by trans.} \\ &= \{ y \in L(\beta) : (z \subseteq x)^{L(\beta)} \} \end{aligned}$$

Thus,  $P_L(x) \in D(L(\beta)) = L(\beta+1)$

L7

Exactly as with the  $R(\alpha)$ ,

The class of  $\alpha$  such that  $\varphi$  is  $L(\alpha)$ -absolute (or absolute for  $L(\alpha), L$ ) is closed unbdd.

Comprehension: Given  $\psi(x, z, v_1, \dots, v_n)$  we need

$$\forall z, v_1, \dots, v_n \in L \{x \in z : \psi^{(L)}(x, v_1, \dots, v_n)\} \in L$$

The way  $D(L(\alpha))$  is defined, we get

$$\forall z, v_1, \dots, v_n \in L(\alpha) \{x \in z : \psi^{(L(\alpha))}(x, v_1, \dots, v_n)\} \in L(\alpha+1)$$

The absoluteness of  $\psi$  for  $L(\alpha), L$  allows replacing  $L(\alpha)$  by  $\psi^{(L)}$

$$(V=L)^{(L)}$$

Easy; show  $L^{(L)} = L$

AC:

Recursively define orders  $\Delta_\alpha$  of  $L(\alpha)$ .

$\Delta_\alpha$  induces lexicographic order on  $L(\alpha)^n$ , each  $n$ . Use these to order  $Df(L(\alpha), n)$ .

(8)

## Remarks on Replacement & Comprehension:

If we use this form of Replacement

$$\forall x \exists ! y \phi(x, y) \rightarrow \forall u \exists v \forall y (y \in v \leftrightarrow \exists x \in u \phi(x, y))$$

(may have parameters  $w_1, \dots, w_n$ )

$$\rightarrow \forall u \exists v \quad v = \text{"range of } \phi \text{ on } u"$$

Then there is no need for Comprehension:

Given  $A$  and  $\psi$ : If  $\neg \exists u \in A \psi(u)$ , let  $v = \emptyset$

If  $u_0 \in A$  and  $\psi(u_0)$ , let

$$\phi(x, y) = \begin{cases} y = u_0 \wedge \neg(x \in A \wedge \psi(x)) \\ y = x \wedge x \in A \wedge \psi(x) \end{cases}$$

Then  $\forall x \exists ! y \phi(x, y)$ . From replacement,  
the range of  $\phi$  on  $A$  is a set, and it's

$$\{x \in A : \psi(x)\}$$

To make things easier, I'll just treat  
ZF as defined with this form of  
Replacement and no Comprehension.