

Sets -

Set: collect. of objects.

i.e., primitive notion, \in

$$A = B \Leftrightarrow \forall x(x \in A \Leftrightarrow x \in B)$$

$$A \subseteq B, A \neq B$$

$$A = B \Leftrightarrow A \subseteq B \text{ & } B \subseteq A \quad \text{For sets}$$

$\cap, \cup, -$

\emptyset

{a}, {a, b}, ...

{x : P(x)} P is a property.

$$\emptyset = \{x : x \neq x\}$$

$$\{a, b\} = \{x : x = a \text{ or } x = b\}$$

Possible ur-elements Nat'l numbers?
— We'll avoid ur-elements — everything is a set!

$$A \cup B = B \cup A$$

$$\text{If: } x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B \Leftrightarrow x \in B \text{ or } x \in A \Leftrightarrow x \in B \cup A$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\Rightarrow x \in A \cup (B \cap C) \Rightarrow x \in A \text{ or } x \in B \cap C \quad A + BC = (A+B)(A+C)$$

$$x \in A \Rightarrow x \in A \cup B \text{ and } x \in A \cup C \Rightarrow x \in (A \cup B) \cap (A \cup C)$$

$$x \in B \cap C \Rightarrow x \in B \text{ & } x \in C \Rightarrow " = "$$

\Leftarrow If $x \notin A$, then

if $x \notin A$, then $x \notin B$ and $x \notin C$

Abstractions &

set-theoretic models.

Ordered pair, (a, b)

- Cartesian coords.

- ordered triples, etc.

Venn diagrams

$$A \cdot (B + C) = AB + AC$$

$$A + BC = (A+B)(A+C)$$

Prob: Make
a list of
basic properties
of $\cup, \cap, -, \cdot$
how they interact
e.g. $A - (B \cup C)$

Set-theoretic model for concept
of ordered pair.

$$(a, b) = \{\{a\}, \{a, b\}\}$$

Req'd prop:

$$(a, b) = (a', b') \Leftrightarrow a = a' \text{ & } b = b'$$

\Rightarrow $a = b$: $(a, b) = \{\{a\}\}$, only one elt, which is
a singleton.

$$\{a'\} \in (a, b), \text{ so } \{a'\} = \{\{a\}\}; \text{ hence, } a = a'$$

$$\{a'; b'\} \in (a, b), \text{ so } \{a'; b'\} = \{\{a\}\}, \text{ so } b' = a'$$

$a \neq b$ Obtain $a' \neq b'$ (else, $\{a', b'\}$ has only one elt.)

Write out
details.

Problems: What's an ordered triple?

$$\text{Maybe } (a, b, c) = (a, (b, c))$$

$$\text{Get } (a', b', c') = (a, b, c) \Leftrightarrow a = a' \& b = b' \& c = c'.$$

But can't tell triples from 2-tuple,
'cause every triple is a 2-tuple.

Would $(a, b, c) = \{\{a\}, \{\{a\}, b\}, \{\{a\}, \{a, b\}, c\}\}$ work? $\begin{matrix} a = b \\ "c" \end{matrix}$

Cases: $a = b = c$, $a \neq b = c$, $a = b \neq c$, $a \neq b \& a \neq c \& c \neq b$

Propose another
model of the ordered
pair.

$A \times B$

Function concept.

$f(x)$ is somewhat motivated by
alg. expression.

But: all agree

$$f(x) = x^2 \quad g(y) = y^2$$

define same function, $f = g$.

Set-theoretic model:

f = set of ordered pairs, with property

$$\left. \begin{array}{l} \langle x, y_1 \rangle \in f \\ \langle x, y_2 \rangle \in f \end{array} \right\} \Rightarrow y_1 = y_2$$

$f(x)$, $\text{dom}(f)$, $\text{ran}(f)$, $\text{fld}(f)$

$$f \upharpoonright A = \{ \langle x, y \rangle : \langle x, y \rangle \in f \wedge x \in A \} = f \cap (A \times \text{ran}(f))$$

$$f''A = \{ y : \exists x \in A \ f(x) = y \} = \text{ran}(f \upharpoonright A)$$

(aka $f[A]$)

1-1, onto, into, surjection, injection

Relation — also set op's.

model for $<$, \leq , $>$, \geq , etc.

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The first successes of set theory: cardinality

\mathbb{N} = set of natural numbers

\mathbb{R} = set of reals.

$A \approx B : \exists f : A \xrightarrow{\text{onto}} B$ equivalence. $A \approx_f B$

$A \lesssim B : \exists f : A \xrightarrow{\text{into}} B$ or: same cardinality,
equal power

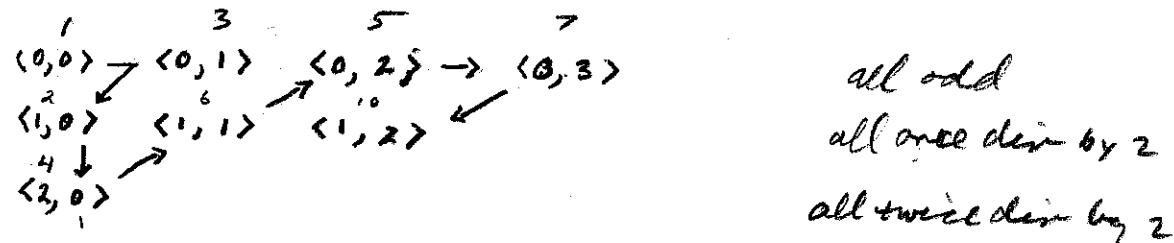
SCHRODER-BERNSTEIN $A \lesssim B \& B \lesssim A \Rightarrow A \approx B$

$A = \{x_1, x_2\}$ with $x_1 \neq x_2$,

$A \approx B$ iff $B = \{y_1, y_2\}$ for some y_1, y_2 with $y_1 \neq y_2$

Strange fact:

$$\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$$



or:

$$f(\langle m, n \rangle) = \begin{matrix} 2^m \cdot (2n+1) \\ 2 \cdot 0 + 1 \\ 2 \cdot 0 + 2 = 3 \\ \dots \end{matrix} - 1 \quad \begin{matrix} 1 \\ 3 \\ 5 \\ 7 \end{matrix}$$

$$2^0 = 1 \quad \langle 0,0 \rangle \quad \langle 0,1 \rangle \quad \langle 0,2 \rangle \quad \langle 0,3 \rangle \quad \begin{matrix} 0 \\ 2 \\ 4 \\ 6 \\ 8 \end{matrix}$$

$$2^1 = 2 \quad \langle 1,0 \rangle \quad \langle 1,1 \rangle \quad \langle 1,2 \rangle \quad \begin{matrix} 1 \\ 5 \\ 9 \end{matrix}$$

$$2^2 = 4 \quad \langle 2,0 \rangle \quad \langle 2,1 \rangle \quad \langle 2,2 \rangle \quad \begin{matrix} 3 \\ 11 \\ 19 \end{matrix}$$

$$2^3 = 8 \quad \langle 3,0 \rangle \quad \begin{matrix} 7 \end{matrix}$$

worse: let $A = \{2^n : n \in \mathbb{N}\}$.

Then $A \not\approx \mathbb{N}$

so: The whole is equal to a proper part of itself.

An even larger infinity:

$\mathbb{R} \not\approx \mathbb{N}$!

Suppose $\mathbb{N} \approx [0, 1]$

$$f(0) = .a_{00} a_{01} a_{02} \dots \quad \text{don't end with all } 9's.$$

$$f(1) = .a_{10} a_{11} a_{12} \dots$$

:

Define $b_i = \begin{cases} 0 & \text{if } a_{ii} \neq 0 \\ 1 & \text{if } a_{ii} = 0 \end{cases} \quad b = .b_0 b_1 \dots$

Then $b \in [0, 1]$ but $b \notin \text{ran}(f)$.

$\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$!

$f(.a_0 a_1 a_2 \dots, .b_0 b_1 b_2 \dots)$ write in binary.
ren.

$= .a_0 b_0 a_1 b_1 \dots$ interp. in ternary
or decimal.

— shows $\mathbb{R} \times \mathbb{R} \not\approx \mathbb{R}$; obv $\mathbb{R} \not\approx \mathbb{R} \times \mathbb{R}$

Always: $A \not\approx \mathcal{P}(A)$

let $f: A \xrightarrow{\text{onto}} \mathcal{P}(A)$ Then f isn't onto.

consider $B = \{a : a \notin f(a)\}$. Suppose $B = f(b)$.

$b \in B \Rightarrow b \notin f(b) \Rightarrow b \notin B$. $\therefore B \notin \text{ran}(f)$