

DEFINE  $\text{p IF } \tau_1 = \tau_2$  FOR ATOMIC FORMULAS.

(a)  $\text{p IF } \tau_1 = \tau_2 \iff$

$$\forall \pi \in \text{dom}(\tau_1) \cup \text{dom}(\tau_2) \quad \forall g \leq p \quad (g \text{ IF } \pi \in \tau_1 \iff g \text{ IF } \pi \in \tau_2)$$

(b)  $\text{p IF } \tau_1 \in \tau_2 \iff$

$$\forall r \leq p \quad \exists g \leq r \quad \exists \langle \pi, s \rangle \in \tau_2 \quad (g \leq s \wedge g \text{ IF } \pi = \pi)$$

Is this a legitimate recursive defn?

Use clause (b) to replace the  $g \text{ IF } \pi \in \tau_i$  in clause (a). Then we'll get a defn. of  $\text{p IF } \tau_1 = \tau_2$  only; on the right, there will be expressions of form  $r \text{ IF } \pi = \pi'$  but only with

$$\max(\text{rank } \pi, \text{rank } \pi') < \max(\text{rank } \tau_1, \text{rank } \tau_2).$$

FOR OTHER FORMULAS:

$$\text{p IF } (\phi \wedge \psi) \iff \text{p IF } \phi \wedge \text{p IF } \psi$$

$$\text{p IF } \neg \phi \iff \forall g \leq p \quad g \text{ IF } \phi$$

$$\text{p IF } \exists x \phi(x) \iff \underbrace{\forall r \leq p \quad \exists g \leq r \quad \exists \sigma \quad g \text{ IF } \phi(\sigma)}$$

$\{g : \exists \sigma \quad g \text{ IF } \phi(\sigma)\}$  is  
dense below p

NOTE: FOR ATOMIC  $\phi$ ,  $p \Vdash \phi$  IS ABSOLUTE.

THESE ARE EQUIVALENT:

$$(1) p \Vdash \phi$$

$$(2) \forall s \leq p \ s \Vdash \phi$$

(3)  $\{s : s \Vdash \phi\}$  is dense below  $p$ .

Proof. First, for atomic  $\phi$ :

$$(1) \Rightarrow (2) : \text{say } s \leq p$$

If  $\phi$  is  $\tau_1 = \tau_2$ , just note  $g \leq s \leq p \rightarrow g \leq p$ .

If  $\phi$  is  $\tau_1 \in \tau_2$ , suppose  $r \leq s$ . Then  
 $r \leq p$ , so  $\exists g \leq r$ , etc.

$$(2) \Rightarrow (3) \text{ Trivial}$$

$$(3) \Rightarrow (1) \text{ Do this for } \tau_1 = \tau_2, \tau_1 \in \tau_2$$

Simultaneously by induction over the ordering of pairs of ordinals, applied to  $\langle \text{rank } \tau_1, \text{rank } \tau_2 \rangle$

Suppose  $\{s : s \Vdash \tau_1 = \tau_2\}$  is dense below  $p$ .

Let  $\pi \in \text{dom } \tau_1$ ,  $g \leq p$ . Suppose  $g \Vdash \pi \in \tau_1$ .  
Then

- $\rho \Vdash \tau_1 = \tau_2 \iff \rho \Vdash \tau_2 = \tau_1$
- $\rho \Vdash \tau_1 = \tau_2 \wedge \rho \Vdash \tau_2 = \tau_3 \Rightarrow \rho \Vdash \tau_1 = \tau_3$

Let  $\pi \in \text{dom } \tau_1 \cup \text{dom } \tau_3$ ,  $g \leq \rho$

$$g \Vdash \pi \in \tau_1 \iff \begin{array}{c} g \Vdash \pi \in \tau_2 \\ \downarrow \\ \rho \Vdash \tau_1 = \tau_2 \end{array} \iff \begin{array}{c} g \Vdash \pi \in \tau_3 \\ \downarrow \\ \rho \Vdash \tau_2 = \tau_3 \end{array}$$

- $\rho \Vdash \tau_1 = \tau_2 \wedge \rho \Vdash \pi \in \tau_1 \Rightarrow \rho \Vdash \pi \in \tau_2$

Pick any  $n \leq \rho$ .

Get  $g \leq n$ ,  $\langle \pi', s' \rangle \in \tau_1$  such that

$$g \leq s' \text{ and } g \Vdash \pi = \pi'$$

But  $\pi' \in \text{dom } \tau_1$ , so

$$g \Vdash \pi' \in \tau_1 \iff g \Vdash \pi' \in \tau_2$$

So there is a  $g' \leq g$  and a  $\langle \pi'', s'' \rangle \in \tau_2$

with  $g' \leq s''$ , and  $g' \Vdash \pi' = \pi''$

Then  $g' \Vdash \pi = \pi''$

We thus have

$$g' \leq n$$

$$\langle \pi'', s'' \rangle \in \tau_2$$

$$g' \leq s''$$

$$g' \Vdash \pi = \pi''$$

Hence,  $\rho \Vdash \pi \in \tau_2$

$$g \Vdash \pi_1 \in \tau_2 \Rightarrow \exists \langle \pi_2, s_2 \rangle \quad s_2 \in g \wedge s_2 \Vdash \pi_1 = \pi_2$$

D is dense below p:

$$\forall g \leq p \quad \exists r \leq g \quad r \in D.$$

$$\forall n \leq r \quad \exists g \leq n \quad [g \leq s_1 \rightarrow \exists \langle \pi_2, s_2 \rangle \in \tau_2 \quad g \leq s_2 \wedge g \Vdash \pi_1 = \pi_2]$$

$$\forall \langle \pi_1, s_1 \rangle \in \tau_1 \quad \forall r \leq p \quad \exists g \leq r \quad g \not\leq s_1 \vee \exists \langle \pi_2, s_2 \rangle \in \tau$$

$$g \leq s_2 \wedge g \Vdash \pi_1 = \pi_2$$

DEFINABILITY OF  $\vdash$  & FOR ATOMIC  $\phi$ .

$$\forall G_{\text{generic}} [p \in G \rightarrow \tau_{1G} = \tau_{2G}] \iff p \Vdash \tau_1 = \tau_2$$

Say  $\langle \pi_1, s_1 \rangle \in \tau$ . Consider all  $G$  with  $p, s_1 \in G$ . For each of these, there will be a  $\langle \pi_2, s_2 \rangle \in \tau_2$  with  $s_2 \in G$  and  $\tau_{1G} = \tau_{2G}$

$$p, s_1 \in G \iff \exists g \in G \quad g \leq p \wedge g \leq s_1$$

Imagine constructing  $G$ : Having put  $p, s_1$  into  $G$ , no matter how one proceeds further, a  $g$  is put in  $G$  with this property:

$$\exists \langle \pi_2, s_2 \rangle \in \tau_2 \quad g \leq s_2$$

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$$p \Vdash \tau_1 = \tau_2 \Rightarrow \forall \langle \pi_1, s_1 \rangle \in \tau_1 \quad \forall g \leq p, s_1 \quad g \Vdash \pi_1 \in \tau_2$$

What is  $g \Vdash \pi_1 \in \tau_2$ ?

Whatever  $G$  we get, there must be a  $\langle \pi_2, s_2 \rangle \in \tau$  s.t.  $s_2 \in G$ ,  $\tau_{1G} = \tau_{2G}$

~~ZF~~  $P + \neg P$  is consistent, if

ZF - inf is consist if ZF is

ZF  $\vdash \phi^{(R(\omega))}$ , for  $\phi$  except Inf

ZF  $\vdash (\neg \text{Inf})^{(R(\omega))}$

ZF  $\vdash \forall x \in M \forall y \in M \exists z \in M \forall w \in M (w \in z \leftrightarrow w \in x \vee w \in y)$

$\forall w$  cause  $w \in z \rightarrow w \in M$   
 $w \in x \vee w \in y \rightarrow w \in M$   
M trans

$\alpha \leq \beta \quad x \in R(\alpha) \wedge y \in R(\beta) \Rightarrow x, y \in R(\beta)$

$\Rightarrow x \cup y \in R(\beta)$  (by ind on  $\beta$ )

$\therefore ZF \vdash \forall x \in M \forall y \in M \exists z \in M \forall w ( \quad )$

Meta math result:

If  $ZF^-$  is consistent, so is  $ZF$ .

Union axiom:

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w \in x \vee w \in y)$$

$\cup$  is absolute:

$$\cup = \cup^{(M)}$$

i.e.

$$x, y \in M \rightarrow [x \cup y = x \cup^{(M)} y]$$

$$\rightarrow \forall z [z \in x \cup y \leftrightarrow z \in x \cup^{(M)} y]$$

But what does  $z \in x \cup^{(M)} y$  mean?

$$\begin{aligned} z \in x \cup y &\leftrightarrow z \in \{z' : z' \in x \vee z' \in y\} \\ &\leftrightarrow z \in x \vee z \in y \end{aligned}$$

$$z \in x \cup^{(M)} y \leftrightarrow z \in \{z' \in M : z' \in x \vee z' \in y\}$$

## Inner model

Class  $M$  (formula)

[Relation  $\in$ ]

Relativize  $\varphi^{(M)}$

$ZF \vdash \varphi^{(M)}$  for axioms  $\varphi$  of  $ZF$

- ok in case  $M = \text{reg}$

$$\begin{cases} R(0) = \emptyset \\ R(\alpha+1) = R(\alpha) \cup P(R(\alpha)) \\ R(\lambda) = \bigcup_{\alpha < \lambda} R(\alpha) \end{cases} = P(R(\alpha))$$

$\text{reg}(x) \Leftrightarrow \exists \alpha x \in R(\alpha)$

Absoluteteness:

$\vartheta$  is absolute (wrt  $M$ )

$ZF \vdash \vec{x} \in M \rightarrow [\vartheta(\vec{x}) \leftrightarrow \vartheta^{(M)}(\vec{x})]$

$ZF \vdash \vec{x} \in M \rightarrow [\tau(\vec{x}) = \tau^{(M)}(\vec{x})]$