

S Y M B O L S :

variables: v_0, v_1, v_2, \dots

logical constants:

$\wedge \quad \neg \quad \exists \quad =$

non-logical constant: \in

parentheses: ()

EXPRESSION: a string of symbols

F O R M U L A S :

(i) $v_i \in v_j$ $v_i = v_j$ are atomic formulas

(ii) If ϕ, ψ are formulas, so are

$(\phi) \wedge (\psi)$ $\neg (\phi)$

(iii) If ϕ is a formula and v_i a variable
then

$\exists v_i (\phi)$

is a formula.

(iv) Nothing is a formula except as required by (i), (ii), (iii).

Abbreviations:

$$\forall v_i (\phi) \quad \text{for} \quad \neg(\exists v_i (\neg(\phi)))$$

$$(\phi) \vee (\psi) \quad \text{for} \quad \neg((\neg(\phi)) \wedge (\neg(\psi)))$$

$$(\phi) \rightarrow (\psi) \quad \text{for} \quad (\neg(\phi)) \vee (\psi)$$

$$(\phi) \leftrightarrow (\psi) \quad \text{for} \quad ((\phi) \rightarrow (\psi)) \wedge ((\psi) \rightarrow (\phi))$$

Omit parentheses when convenient.

$$v_i \neq v_j \quad \text{for} \quad \neg(v_i = v_j)$$

$$v_i \notin v_j \quad \text{for} \quad \neg(v_i \in v_j)$$

Example:

$$\exists v_0 \exists v_1 (v_1 \in v_0 \wedge \forall v_3 (v_3 \notin v_1))$$

subformula : substring which is a formula

v_i is bound at a certain place in ϕ :

that place is within a subformula
of form $\exists v_i (\psi)$

v_i is free at a place in ϕ :

it occurs there, but is not bound there.

v_i is free in ϕ :

it is free at some place in ϕ .

x, y, u, A, B, \dots stand for variables.

Substitution:

$\varphi(x/y)$ is the result of replacing x by y at each free occurrence of x in φ .

$\varphi(x_1/y_1, \dots, x_n/y_n)$

A substitution $\varphi(x/y)$ is legitimate only if y is not bound at any place in φ where x occurs free.

φ is $\forall v_i (v_i \notin v_o)$

$\varphi(v_o/v_i)$ is $\forall v_i (v_i \notin v_i)$

This is not legitimate.

We always assume substitutions are legitimate. Bound variables are to be changed if necessary. Note that

$\forall v_2 (v_2 \notin v_o)$

says the same thing about v_o as φ does, and now the substitution is legitimate:

$\forall v_2 (v_2 \notin v_i)$

$\phi(x_1, \dots, x_n)$

- emphasizes that x_1, \dots, x_n may be among the free variables of ϕ
- establishes a local convention for substitution:

$$\phi(u_1, \dots, u_n) \text{ is } \phi(x_1/u_1, \dots, x_n/u_n)$$

Sentence: formula with no free variables.

Domain of discourse: the "universe"
the allowable values of variables.

V = the domain of discourse

variables can be assigned values in V

\in is interpreted as a relation on V

$=$ means "is the same as"

Formulas are satisfied, or not, by assignments.

Sentences are true or false

So if V is the universe of sets,

$$\forall v_i (v_i \notin v_0)$$

is satisfied iff v_0 is assigned \emptyset .

$$\exists v_0 \forall v_i (v_i \notin v_0)$$

is a true sentence.

$S \vdash \phi$

means: there is a formal deduction of the sentence ϕ from the set of sentences S .

"formal deduction" has a precise,
purely combinatorial

definition.

$S \vdash \phi$ iff for some finite $S_0 \subset S$, $S_0 \vdash \phi$. (COMPACTNESS)

ϕ is logically valid: $\vdash \phi$ (i.e., $\phi \vdash \phi$)

universal closure of ϕ :

$\forall x_1 \dots \forall x_n \phi$

where x_1, \dots, x_n are all free vbls of ϕ .

$\vdash \phi$ means $\vdash \forall x_1 \dots \forall x_n \phi$.

Thm $S \vdash \phi$ iff ϕ is true under every interpretation which makes all the sentences of S true. *

S is inconsistent: $S \vdash \phi$ and $S \vdash \neg \phi$

* If we allow infinitistic methods in the metatheory, this can serve as a definition of \vdash . Then the Compactness Theorem, $S \vdash \phi \Leftrightarrow S_0 \vdash \phi$ for some finite $S_0 \subseteq S$, becomes a deep result.

$$\rho(X) = \{Y : Y \subseteq X\}$$

ZERMELO:

- Let the domain of discourse be all sets obtained by starting with the empty set, and applying $\rho()$
- Axiomatize this domain.

$$V_0 = \emptyset$$

$$V_1 = \rho(\emptyset) = \{\emptyset\}$$

$$V_2 = \rho(V_1) = \{\emptyset, \{\emptyset\}\}$$

:

$$V_w = V_0 \cup V_1 \cup V_2 \cup \dots$$

$$V_{w+1} = \rho(V_w)$$

:

Avoid the paradoxes by rejecting unrestricted comprehension, i.e.,

$$\{x : \phi(x)\}$$

doesn't always exist.

Sets have to be constructed by other means.

ZFC - P

$$A = \{x : \phi\}$$

Justification of this as a definition:

$$\begin{array}{c} S \\ \vdash \\ \text{ZFC-P} \\ \text{for now} \end{array} \vdash \exists y \forall x (x \in y \leftrightarrow \phi) \quad \underbrace{\text{(y not free}}_{\text{in } \phi})$$

use universal closure
if other free vbls are present.

After justification, we can use the defined object as a constant, or in case it has free variables, as an operator on sets.

Statements involving defined objects as above, are regarded as abbreviations for statements in the official language.
(See Ch 1, §13).

" $\exists y$ " in the justification above can be replaced by " $\exists! y$ " w.l.o.g. Once the existence is proved, extensionality takes care of uniqueness.

When we say " $\{x : \phi\}$ exists", we mean that the justification statement can be proved.

Standard method for justification:

- Construct, by previously justified means, a set A such that

$$(*) \quad \vdash \varphi(x) \rightarrow x \in A$$

- The Comprehension Axiom gives

$$\exists y \forall x [x \in y \leftrightarrow x \in A \wedge \varphi(x)]$$

which together with $(*)$, yields

$$\exists y \forall x [x \in y \leftrightarrow \varphi(x)]$$

$$\emptyset = \{x : x \neq x\}$$

Justification: as above, note

$$\vdash x \neq x \rightarrow x \in A \quad \text{by logic alone}$$

since $\vdash \neg(x \neq x)$.

General Comprehension doesn't hold:

$$\neg \exists z \forall x (x \in z)$$

$$\forall x (x \in z)$$

$$\{x : x \in z \wedge x \neq x\} \xrightarrow{\text{exists by comprehension}} \{x : x \neq x\} \text{ exists}$$

But

$$A = \{x : x \neq x\} \rightarrow (A \in A \leftrightarrow A \notin A)$$

and $A \in A \leftrightarrow A \notin A$ is logically false.

$\therefore \{x : x \neq x\}$ doesn't exist.

$$x \subset y \quad \text{is} \quad \forall u (u \in x \rightarrow u \in y)$$

$$\{x, y\} = \{u : u = x \vee u = y\} \quad (y \text{ is not the same variable as } x)$$

Justification: Pairing and Comprehension

$$\{x\} = \{x, x\}$$

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}$$

Lemma $\langle x, y \rangle = \langle u, v \rangle \rightarrow x = u \wedge y = v$

$$\cup x = \{y : \exists z (y \in z \wedge z \in x)\}$$

$$\begin{aligned} \cap x &= \{y : \forall z (z \in x \rightarrow y \in z)\} \quad \text{if } x \neq 0 \\ &= \emptyset \quad \text{if } x = 0 \end{aligned}$$

(We should only use $\cap x$ when we know $x \neq 0$)

$$x \cap y = \{z : z \in x \wedge z \in y\} = \cap \{x, y\}$$

$$x \cup y = \{z : z \in x \vee z \in y\} = \cup \{x, y\}$$

$$x \setminus y = \{z : z \in x \wedge z \notin y\}$$