Salvatore's Game

I have this friend Sal Lestofante. Well, not a friend exactly, let's just say an acquaintance. Sal is always coming around with new games to play. Here's one he came up with a few days ago. He said "I've come across a new game! It's played with two dice, just like craps. But instead of adding the numbers, you subtract them, the smaller from the larger. The possible answers are 0, 1, 2, 3, 4, 5. Look, we'll each put up $1, and we'll each take half the numbers. I'll take 0, 1, 2, and you take 3, 4, 5. Whoever has the number that comes up wins the $2."

What do you think about this?

After a little while, I told Sal, "Look, I don't like this game. It isn't as fair as it looks. You win much more often than I do." Surprise—he agreed with me! He said "Yeah, I think you may be right. I'll mull it over, and see if I can come up with a better set of rules."

Funny thing about Sal, he doesn't seem to have any visible means of support. All he wants to do is play these weird gambling games.

So next day he came around again. He said "I fixed it! You put up $2, and I'll put up $3. Same playing rules as before, but now you get more than I do when you win."

What do you think of this one?

The probability of an event is the percentage of times the event occurs in a large number of trials.

More precisely, it's the *limit* of the percentage observed as the number of trials is increased without bound. For example, if you roll a normal six-sided die, the probability of any one number is 1/6. If you actually do it, you might find, say, one six in ten rolls, 20 in 100 rolls, 162 in 1000 rolls, and so on—the fraction of rolls approaches 1/6.

With two dice, we get a similar result—each possible pair of numbers is just as likely as any other; there are 36 pairs (1,1), (1,2), ..., (1,6), (2,1), ..., (6,6). The probability of any pair is 1/36.

But if we're interested in a more complex event, such as the sum of the two dice, we have to add up the probabilities of the individual equally likely events that form the one we're interested in. For example, there are four pairs that add up to 5: (1,4), (2,3), (3,2), (4,1). Those pairs are equally likely, each one occurs 1/36 of the time, so the probability of 5 as the pair sum is $4 \times 1/36$, or 1/9.

Sometimes there's a value associated with an event. This is the case with Sal's proposed game. In that game, there's a value (to each player) associated with the difference of the dice. Sal wins one dollar if the difference is 0, 1, or 2; there are a total of 24 ways to achieve these differences. Sal loses a dollar if the difference is 3, 4, 5; there are only 12 ways this happens. In a large number of plays, Sal wins 2/3 of the time. I, the chump, win only 1/3 of the time.

In many plays of the game, Sal should win, on average, 2/3 of the time, whereas I should expect to win only 1/3 of the plays. Each win is worth \$1, each loss is -\$1. Look at it this way:

$$\begin{gathered}\% \text{ of trials Sal wins} \times \text{win value} \\ \text{minus} \\ \% \text{ of trials Sal loses} \times \text{loss value} \\ = 2/3 \times \$1 - 1/3 \times \$1 \\ = \$1/3.\end{gathered}$$

This works out to a positive \$1/3. In 100 plays, Sal can expect to collect \$67, and lose \$33. Net profit: \$33. This is also my net loss. The *expected value* of the game is 1/3 for Sal, and -1/3 for me. His average gain per play is \$1/3.

In Sal's revised game, he would put up \$3 to my \$2. This time the value to Sal would be $2/3 \times \$2 - 1/3 \times \$3 = \$4/3 - \$1 = \$1/3$. In 100 plays, he should collect \$133, but will lose \$100. Net profit: \$33, and again, this is my loss. Notice however that even though he nets the same amount, he's putting twice as much at risk.

In a fair game, Sal should only expect to get back as much as he put in. But he can't make a living that way.

Can you suggest modifications to Sal's game that would make it fair?