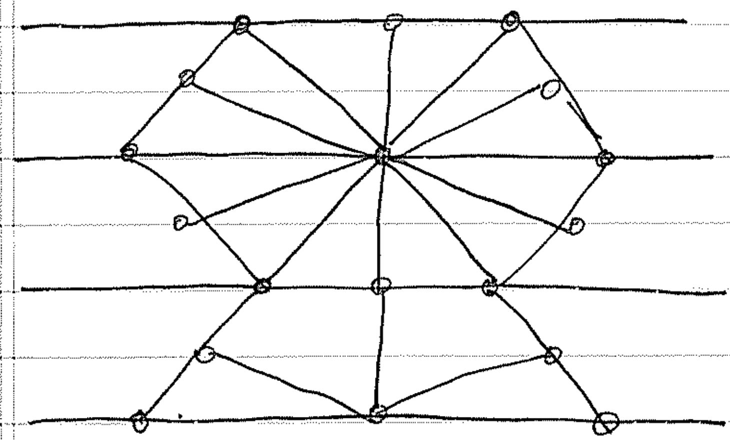
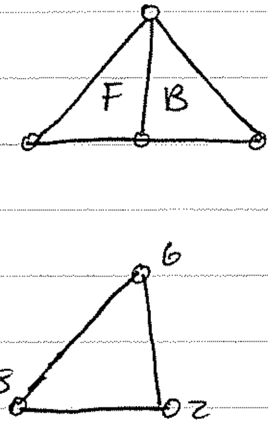
Hyperbolic Structures

Euclidean Wallpaper



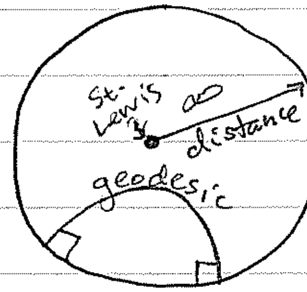


Fundamental Domain

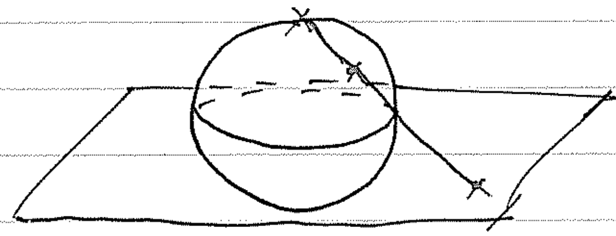
The Hyperbolic Plane

unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$
+ metric

"the TWA space"
"the sphere of radius $\sqrt{-1}$"



Transport metric on sphere of radius R onto plane by stereographic proj.

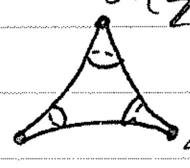


$$ds = dx/(1+\lambda\|x\|^2)$$

Gaussian curvature $\lambda = 1/R^2$

$\lambda = -1$ gives hyperbolic plane

3 ~ $\mathbb{Z}_3$ rot. sym.



$\longleftarrow$ $S^2(3,4,4)$

4 ~ $\mathbb{Z}_4$ rot. sym.

ie $\mathbb{Z}_4$ rot. sym.

Guess for smallest volume manifold is
Weeks manifold, vol $= 0.9427$
recently, Agol-Dunfield showed ~~xx~~ vol > 0.67

transcendental integral need to calculate:
$$\Pi(\theta) = -\int_0^\theta \log|2\sin\theta| d\theta$$

note) if have 2 distinct manifolds, if they have different volumes, calculating, you will eventually find ~~xxx~~ a difference, but if they are the same, you will never know.

General Remark 8 Thurston geometries. Even if structure is not unique, the Moduli space is finite dimensional.

Finite dimensional moduli space
$G =$ Lie group, finite dim
$X^n =$ homogeneous space, G acts on X
$M^n =$ closed manifold

(G, X) structure $M = X/\Gamma$ ~ discrete ~~x~~ subgroup of G

deformation space is open subset of Alg. Variety

$$\rho_0 : \pi_1(M) \longrightarrow \Gamma \subset G = \text{matrix group}$$

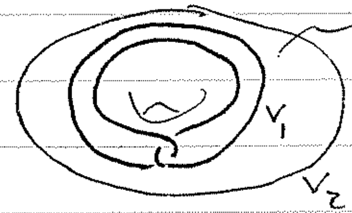
$$g_1, \ldots, g_n \longmapsto M_1, \ldots, M_n$$

$\mathrm{Hom}(\pi_1(M), G) = \langle M_1, \dots, M_n ; M_{i_1}^{\pm 1} \cdots M_{i_k}^{\pm 1} = I \rangle$

Whitehead 3-Manifold

$W \neq \mathbb{R}^3$, but $W \times \mathbb{R} = \mathbb{R}^4$

$W = \bigcup_{n>0} V_n$



W is not tame

def: n-mfld M is tame if $\exists$ compact n-mfld N st $M \setminus \partial M = \mathrm{int}(N)$.

tameness thm (Agol, Calegari-Gabai)
If M^3 hyperbolic, $\pi_1(M)$ finitely generated then M is tame.

Recognition Problem Perelman $\Rightarrow$ solved
can decide if 2 closed orientable 3-manifolds are the same or not.

for S^3: A. Thompson, for hyperbolic: J. Manning, fibration: T. Li
algorithmic: Jaco, Rubenstein

Uniqueness of decomposition

dim 2)

closed Cpct manifolds form commutative monoid,
add = connected sum

$$= \left\langle S, P, T : \begin{array}{l} S \text{ is identity} \quad S+X=X \\ 3P = P+T \end{array} \right\rangle$$

high dim) $S^7 = M_1 \# M_2$ ~ exotic 7-spheres
$= (M_1 \# M_2) \# \dots \# (M_1 \# M_2)$

Kneser (30s) closed 3-manifold has a prime decomposition

prime) not a non-trivial connected sum

irreducible) every sphere bounds a ball

irreducible $\Rightarrow$ prime $\Rightarrow$ $\left(\begin{array}{l} \text{irreducible or} \\ S^1 \times S^2 \text{ or } S^1 \tilde{\times} S^2 \end{array} \right)$

Milner (60) Prime decomp (closed M^3) pieces are unique

$$(S^1 \times S^2) + (S^1 \tilde{\times} S^2) = 2(S^1 \tilde{\times} S^2)$$

not orientable

not unique up to isotopy

Torus decomposition| cut along tori into pieces that are either

(1) Siefert Fibration Spaces

(2) Hyperbolic

(or (3) Solv: orbifold bundle, generic fiber T^2)

Jaco Shalen Johansson

JSJ decomposition| use the least number of tori

thm| M prime $\Rightarrow$ JSJ unique up to istopy

Graph Manifolds

def| (Waldhausen '68)

Graph 3-mfld = $\bigcup$ (circle bundles) glued along boundary tori

in fact, can restrict to

$S^1 \times D^2$, $S^1 \times$ (Pants), $S^1 \tilde{\times}$ (Mobius band)

thm| M^3 closed $\Rightarrow$ $M = G \cup N$

graph   tori incompres.   hyperbolic

G, N are unique

If M irreducible $\Rightarrow$ isotopy uniqueness

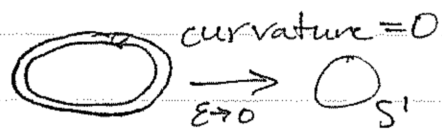
thm| M Graph $\Leftrightarrow$ M has Thurston decomp. w/ no hyperbolic pieces $\Leftrightarrow$ M collapses w/ bounded curvature

Geometrization Conj / Thm| (Thurston / Hamilton, Perelman, ...)
M closed orientable $\Rightarrow M$ has a prime decomp. into pieces, which have a JSJ decomp. into pieces each of which has Thurston geometries. this is called Thurston decomp.

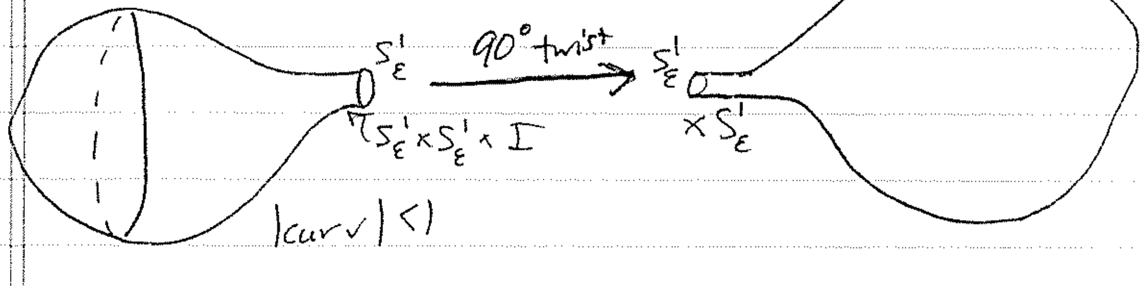
Uniqueness| of Geometry? No
of topology? Yes

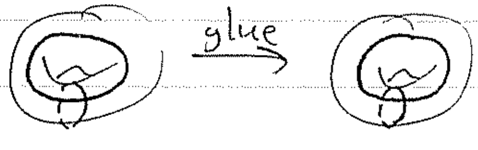
Collapse w/ bounded curvature
M^n collapses w/ bounded curvature : $\exists$ a sequence of Riemannian metrics on M st
(1) $|\text{all curvatures}| \leq 1$
(2) $\forall x \in M \quad inj(x) \longrightarrow 0$

eg| Torus $S^1 \times S^1_\varepsilon$



eg| S^3 collapses





$|inj| = \varepsilon$

this is the Genus 1 - Heegaard split of S^3

Same idea works in general w/ more complicated cycles, long interval

Cheeger - Gromov | (collapsing thm)
If M^3 closed collapses, then it is graph