1 December 2006
X Dai

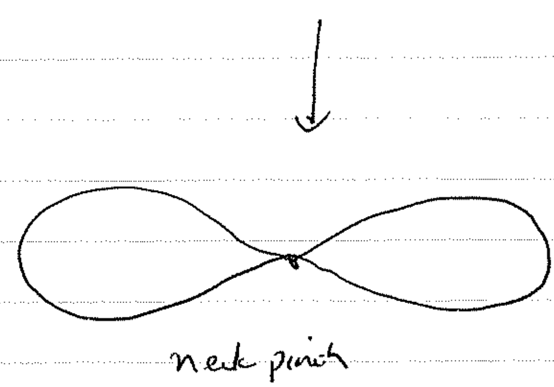
Introduction to Ricci Flow with Surgery

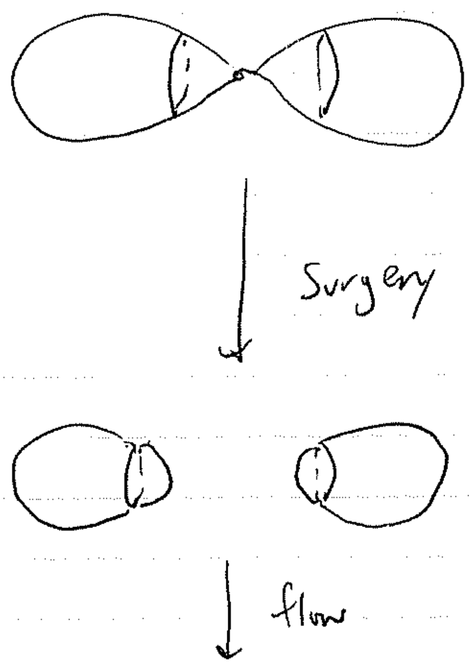
Ricci flow:
$$\begin{cases} \dfrac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \\ g(0) = g_0 \end{cases}$$

$g_0 \dashrightarrow$ "canonical metrics"

Problem: Finite time singularity occurs before you get what you want!

Pos. curv. $\longrightarrow$ shrinking
Neg curv. $\longrightarrow$ expanding

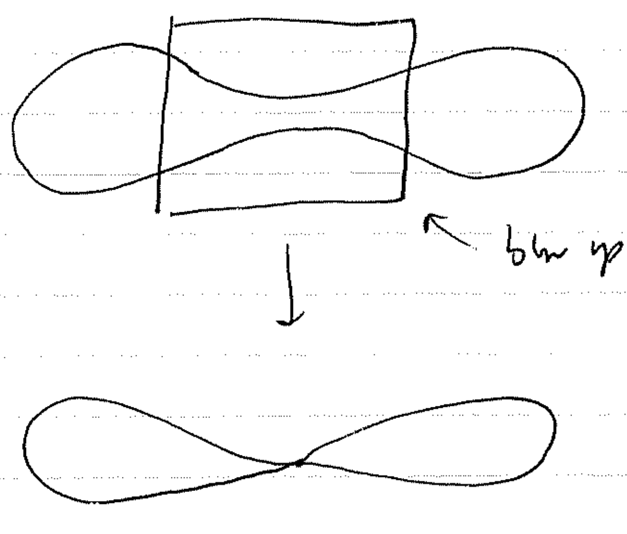


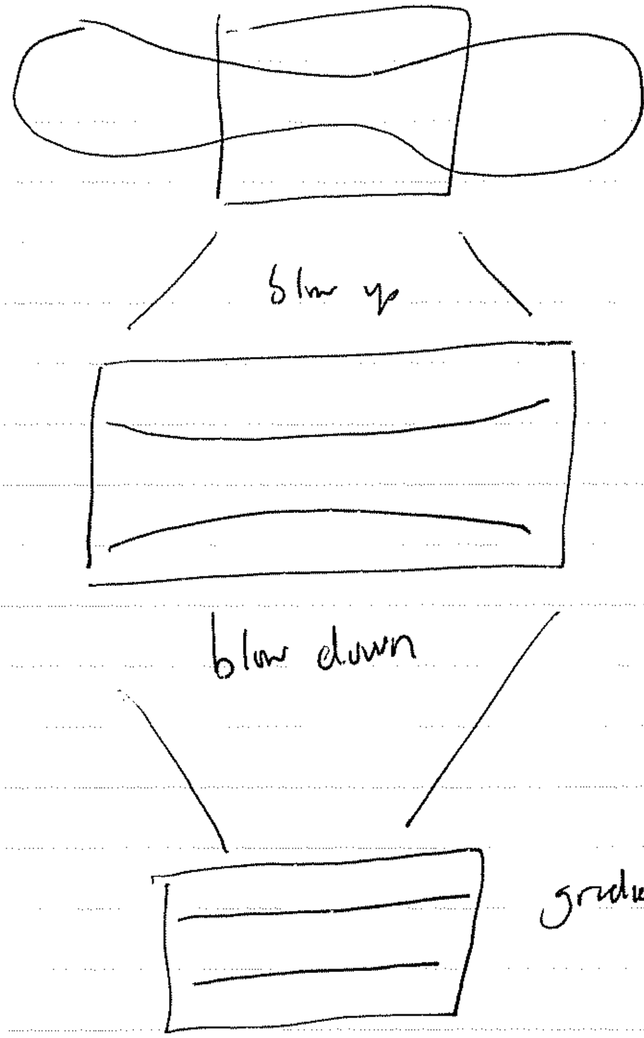


T = max. time of existence

<u>Hamilton</u> $T < +\infty \Rightarrow \lim_{t \to T} \max |Rm(t)| = +\infty$

i.e. finite time sing. preceded by high curvature regions.





ancient solutions.

classical Riemannian geometry

gradient shrinking solitons;
classify these
⇓
classify ancient solutions
⇓
high curvature region

Perelman's 1st breakthrough: no local collapsing
+ curvature

"ancient κ-solutions".

§1. Geometric limits and no local collapsing

Cheeger - Gromov cpt ness thm

$\kappa > 0$

Say $B(x,r) \subseteq M$, Riemannian, is

κ - noncollapsed if

$$|Rm| \leq r^{-2} \Rightarrow vol(B(x,r)) \geq \kappa r^n$$

$$|Rm| \leq r^{-2} + vol \geq \kappa r^n \Rightarrow inj \geq C_0(r,\kappa)$$

Say (M,g) is κ - noncollapsed on the scale ρ (>0)

if $\forall$ $B(x,r)$, $r < \rho$ is κ - noncollapsed

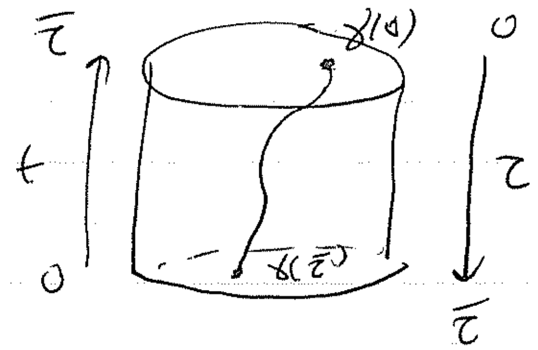
Perelman $g_{ij}(t)$ Ricci flow $t \in [0,T)$

$T < +\infty \Rightarrow \exists\ \kappa = \kappa(g_0, T) > 0$ s.t.

$g_{ij}(t)$ is κ - noncollapsed on the scale $T^{1/2}$

$\leftarrow$ monotonic quantities, reduced volume (= weighted volume)

Parabolic length: fix $(x_0, \overline{\tau}) \in M \times [0,T)$



$\gamma(\tau)$ path:

$\gamma(0) = x_0$

$\gamma(\bar\tau) = x$

Scalar curv.

parabolic length:

$$\mathcal{L}(\gamma) = \int_0^{\bar\tau} \sqrt{\tau}\left(R_{g(\bar\tau-\tau)}(\gamma(\tau)) + |\gamma'(\tau)|^2_{\cdot} \right) d\tau$$

$g(\bar\tau - \tau)$

$$L(x, \bar\tau) = \inf_{\substack{\gamma(0)=x_0 \\ \gamma(\bar\tau)=x}} \mathcal{L}(\gamma)$$

$$l(x, \bar\tau) = \frac{1}{2\sqrt{\bar\tau}} L(x, \bar\tau) \quad \text{reduced distance}$$

reduced volume

$$\tilde{V}(\bar\tau) = \int_M (\bar\tau)^{-n/2} e^{-l(x,\bar\tau)} \underbrace{dx}_{g(t)}$$

Ex $\mathbb{R}^n$ $g(t) = g_0$, standard metric

$$\Rightarrow l(x, \bar\tau) = \frac{|x - x_0|^2}{4\bar\tau} \quad \text{Gaussian dist}$$

$$\tilde{V}(\bar\tau) = \ldots$$

$$\tilde{V}(\bar{\tau}) = \int (\bar{\tau})^{-n} \underbrace{e^{-|x-x_0|^2/4\bar{\tau}}}_{\text{heat kernel}} dx$$

$$\equiv (4\pi)^{-n/2}$$

Perelman $\tilde{V}(\bar{\tau})$ is monotonically non-increasing in $\bar{\tau}$

(so monotonically non-decreasing in t)

Moreover $\tilde{V}(\bar{\tau}) \equiv \text{const} \iff g(t)$ gradient shrinking soliton

§2. Blow up and down

Restrict to dim = 3

- Hamilton rounding : $Ric(g_0) > 0 \Rightarrow g(t)$ get rounder and rounder

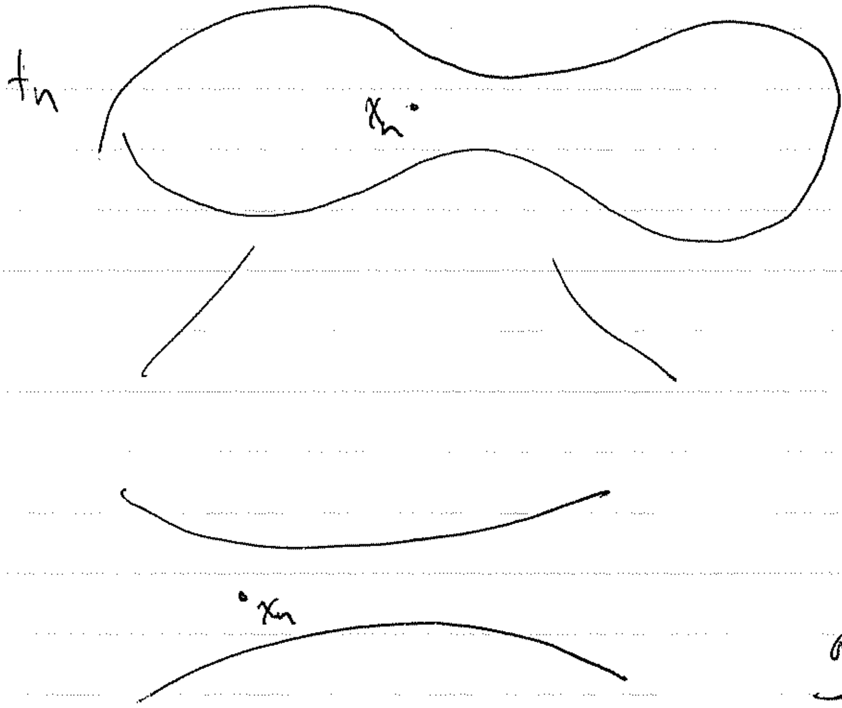
- Hamilton-Ivey curvature pinching:

neg. curva $\ll$ most positive curva. as $t \to T$

$\Rightarrow \max_M R(t) \to +\infty$ as $t \to T$

and scalar curva. controls all curva.

$\exists \ t_n \to T, \ x_n \in M$ s.t. $R_n = R(x_n, t_n) \to +\infty$



$g(t) \to \lambda^2 g\left(\frac{t}{\lambda^2}\right)$

Rescale

$g_n(t) = R_n \, g\left(t_n + \frac{t}{R_n}\right)$

$t \in [-t_n R_n, (T-t_n)R_n)$

$\downarrow$

$-\infty$

$g_n(t) \to \bar{g}(t)$ smooth limit

Smooth Ricci flow $t \in (-\infty, 0]$

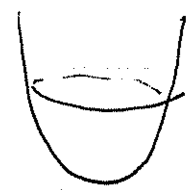
- κ-noncollapsed on all scales (if $x_n \to x$)
- sectional curva ≥ 0
- nonflat

ancient κ-solution

maybe non-compact

Example: Cigar soliton (Witten black hole)

$\mathbb{R}^2$, $g = \dfrac{dx^2+dy^2}{1+x^2+y^2}$



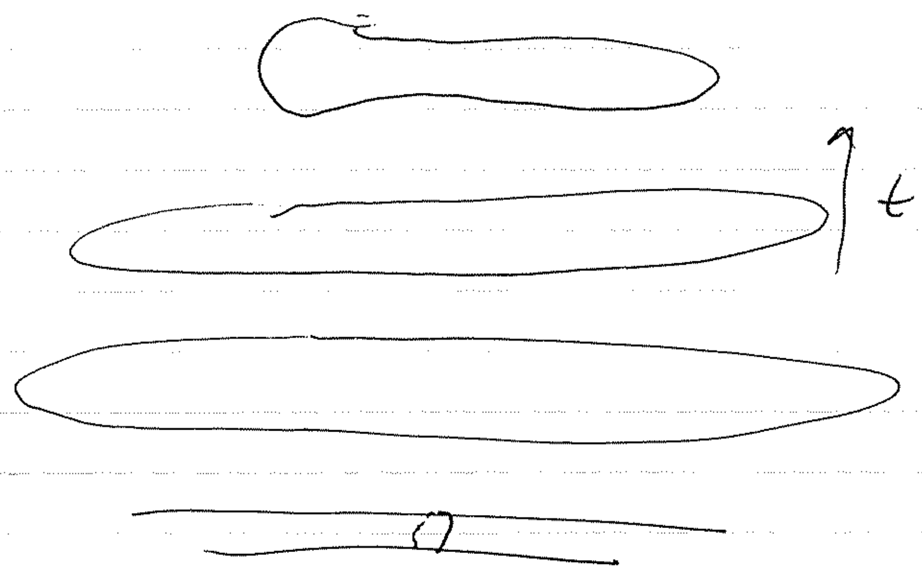
$S^1 \times \mathbb{R}$

Example: Cigar soliton $\times \mathbb{R}$

is an ancient solution in dim 3

but not noncollapsed on all scales!

Turns out: ancient κ-solns have asymptotic profile at $-\infty$



$\bar{g}(t)$ = ancient κ-solution

$t_k \to -\infty$

$$\bar{g}_k(t) = \frac{1}{(-t_k)}\, \bar{g}((-t_k)t)$$

$t \in (-\infty, 0]$

$\bar{g}_k(t) \longrightarrow \bar{g}_\infty(t)$ Smooth limit -

asymptotic soliton.

$\tilde{V}$ scale invariant + monotone

$\Downarrow$

$\tilde{V}$ of $\bar{g}_\infty(t)$ is constant.

$\Rightarrow$ gradient shrinking soliton

§3. Classification of gradient shrinking solitons

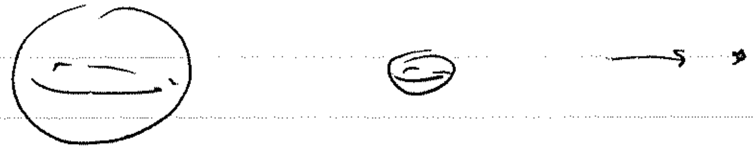
+ canon. nbds

$g(t)$ $\quad g = g(-1) \Rightarrow Ric(g) + Hess(f) - \frac{1}{2}g = 0.$

$\uparrow$ potential

Examples

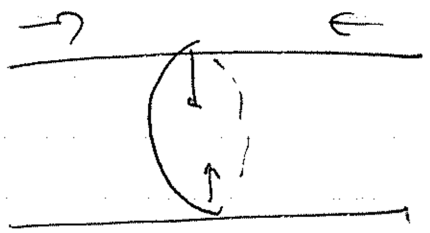
1) (S^3, g_0) standard, $g(t) = 4(-t)g_0$



on S^2

2) Cylinder $S^2 \times \mathbb{R} \ni s$ $\quad g(t) = 2(-t)g_0 + ds^2$

$f = \frac{s^2}{4}$

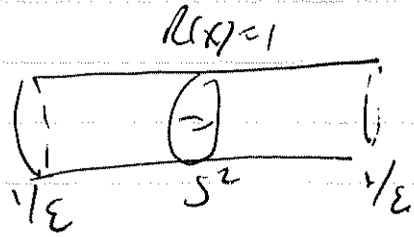


(Up to finite quotient) these are all the gradient shrinkers solitions i.e. S^3/Γ or $S^2\times\mathbb{R}$ or $S^2\times\mathbb{R}/\mathbb{Z}_2$

$\rightarrow$ ancient κ-sol^{n}s.

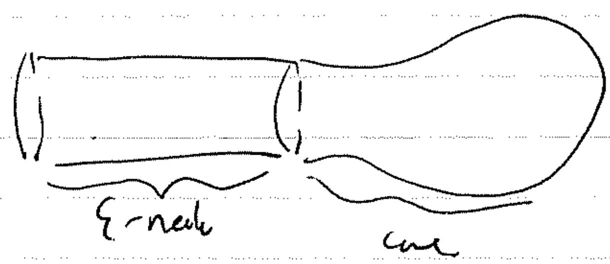
every pt. will be in one of canon. nbds

i) ε neck



ε-close in $C^{[1/\varepsilon]}$ top to $S^2\times(-\varepsilon^{-1},\varepsilon^{-1})$

ii) ε-cap



bdd geometry
$\simeq B^3$ or $\mathbb{RP}^3-\{pt\}$

(C,ε)-cap.

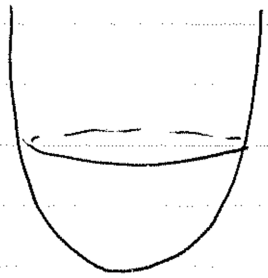
iii) C-component $\sim S^3$ or $\mathbb{RP}^3$

iv) ε-round S^3/Γ,
ε-close to

high curvature regions

$\subseteq$ canonical nbhds

§5. Standard sol^{n}s and surgery construction



$\xrightarrow{\text{evolute}}$ max $T = 1$.

Sect-curva > 0

$R_{min}(t) \geq \frac{c}{1-t}$

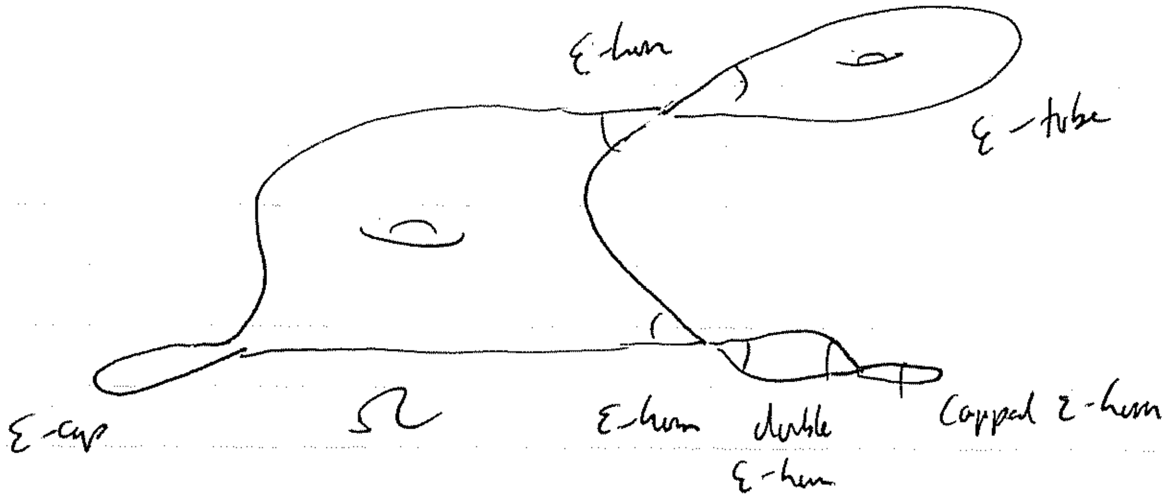
all high curvature region as $t \to 1$

Structure at 1st singular time

$M = \{\text{bdd curva region}\} \cup \{\text{high curvature region}\}$

Thick $\qquad\qquad\qquad$ thin

$g(t) \downarrow \bar{g}$ incomplete $\qquad \xrightarrow{t \to T} \Omega \qquad \cap|$ canonical nbhds


ε-horn
ε-tube
ε-cap
Ω
ε-horn
double
ε-horn
capped ε-horn