work in progress w/ E. Witten

* understand representations of $G_R \subset G_C$ in terms of geometry (D-branes)

Toy model: top. twist of $N=4$ SYM

motivation: real form of $G_C$ (GL-twist)

take GL-twist of $N=4$ SYM on $M_4 = W \times I$, $I = [0;1]$

boundary cond.

$b.c.c.$

$F_A^+ \big|_W = 0$

boundary cond. preserve topol. susy

theory $\xrightarrow{\sim} 3d$ TQFT on $W$

Chern-Simons gauge theory w/ $g-g'$ in $G$
Interesting question: how does S-duality act on Chern-Simons theory?

Non-local operators in 4d theory supported on two-dimensional surface $D \subset M_4$
e.g., $D = y \times I$

$\implies$ Line operator supported on $y$
Wilson line $W_R(y)$
$R$ rep. of $G$

Non-local operators in 4d theory parameterized by data very different from representations
$(\implies (\alpha, \beta, y, \eta) \text{ in work of Gukov/Witten '06})$

Generalization:
Take $W = R \times C$
$\text{ from } \mathcal{W}$
and $y = R \times \{ \text{pt} \}$
Hamiltonian approach ⇒ Hilbert space

replace C by disc w/ puncture at origin ⇒ ℌ = representation space

in 4d gauge theory here

\[ \Sigma = R \times \mathbb{I} \]

\( M = \Sigma \times C \)
4d gauge theory on $M = \Sigma \times C$

4d topol. $\sigma$-model $\Sigma \to \mathcal{M}_H(G, C)$

moduli space of sol. to Hitchin equ. on $C$

$\mathcal{H} = \text{Hom}(\mathcal{B}, \mathcal{B}')$

= space of open string states between $\mathcal{B}$ and $\mathcal{B}'$ on $\mathcal{M}_H$

for applications to representation theory

$\mathcal{C} = D^\ast$ (punctured disc) with a surface operator

$\mathcal{M}_H \simeq \mathcal{T}^\ast(G/\Pi)$

Eq. for $G = SU(2)$ $\mathcal{M} = \mathcal{T}^\ast S^2$

$\mathcal{B}' = \{ \phi = 0 \} =$ brane supported on $(G/\Pi) \subset \mathcal{M}_H$

$\mathcal{B}$ c. = canonical coisotropic brane

$\mathcal{M}_H = \mathcal{T}^\ast(G/\Pi)$ hyperkähler: $I, J, K$

Kähler-forms: $\omega_I, \omega_J, \omega_K$

$\sigma$-model could be $A$-model or $B$-model (related by mirror-sym)
focus on $A$-model for $\omega = \omega_k$

A-branes:

- Lagrangian A-branes on $M_H$
  
  $B'$ is such a lag. A-brane

- A-branes w/ curvature $F$
  
  of Chern - Paton bd.

  $(F \cdot \omega^{-1})^2 = -\mathbb{I}$

  ($\Rightarrow$ Kapustin, Okl.)

$B_{c.c.} = \text{space-filling A-brane on } M_H$

$F = \omega^{A}$

in $A$-model on $M_H = T^*(G/\Pi)$

$H = \text{Hom} (B_{c.c.} , B')$

in this example (for these $B_{c.c.}$ and $B'$)

$B_{c.c.}$ and $B'$ are both of type $(A, B, A)$

$\Rightarrow$ space of open string states

$H = \text{Hom} (B_{c.c.} , B')$

= hol. sections of vector bundle on $G/\Pi$

(Borel - Weil - Bott - theory)
in general consider $A_K$-model (pick $\omega_K$) on $M_H = T^*(G/\Pi)$

$A_K$-branes $\leftrightarrow$ representations of $G_{IR}$ of $G_c$

**Remark:** $M_H \cong T^*(G/\Pi) \cong \text{reg. coadjoint orbit of } G_c$

$G$ acts on $G/\Pi = SU(2)/U(1)$

all $A_K$-branes: too large set

$\Rightarrow$ instead consider:

$A$-branes (Lagrangian) invariant under $K_{IR} \subset G_{IR}$

---

Ex: for $sl(2)$:

representations of $G_{IR} = SL(2, \mathbb{R})$

$M_H = T^*S^2$ A-branes under the maximal cpt $K_{IR} = SO(2)$

Gibbons-Hawking:

$$ds^2 = H \cdot (dx^2) + H^{-1} (dx + A)^2$$

$x \in \mathbb{R}^3$, $x \in [0, 2\pi)$

$$H = \frac{1}{\sqrt{x^2 + x_0^2}} + \frac{1}{\sqrt{x^2 + x_0^2}}$$

\[\nabla \cdot A = \nabla H\]
\[ \vec{x}_0 = (\alpha, \beta, \gamma) \]

\[ \vec{\omega} = (\omega_I, \omega_J, \omega_K) \]
\[ = (d\chi + A^3) \cdot d\vec{x} - \frac{i}{2} H \, d\vec{x} \times d\vec{x} \]

\[ \omega_k \bigg|_{B^1} = 0 \iff \text{contained in a plane transverse to z-direction (k-dried)} \]
\[ \mathbb{R}^2 \subset \mathbb{R}^3 \]

\[ \Rightarrow \] classification of \( SL(2; \mathbb{R}) \) irreducible reps
\[ \Rightarrow \] classifications of \( K_{\mathbb{R}} \)-inv. A-branes

\[ \mathbb{R}^2 \]
possibilities

\[ \mathbb{R}^2 \]  2 centers contained in plane

sphere
finite
degm reps.
of SL(2; \mathbb{R})

1 center contained

discrete series

no centers

(\rightarrow \text{ principle series representations )}

obtained by quant.
of clyndrs

(up to symplectomorphi)