Normal Functions and Disk Counting

David R. Morrison

Geometry, Topology, and Physics Seminar
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Based on

- J. Walcher, *Opening mirror symmetry on the quintic*, `arXiv:hep-th/0605162`
- DRM and J. Walcher, *D-branes and normal functions*, `arXiv:0709.4028 [hep-th]`.

The Quintic and its Mirror

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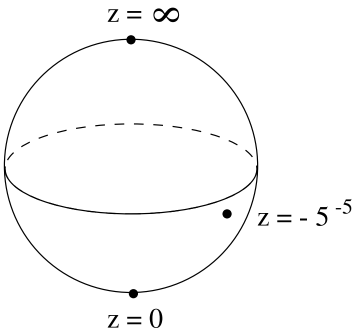
- ordinary ("complex structure") moduli space has parameter $z = (-5\psi)^{-5}$
- the identification between the two is made with the help of periods $\Phi(z) = \int_\Gamma \Omega_z$, for $\Gamma \in H_3(Y, \mathbb{Z})$ and Ω_z a holomorphic 3-form on Y_z


z = ∞
z = - 5 -5
z = 0

Periods and the Mirror Map

$\Phi(z)$ satisfies an algebraic differential equation $\mathcal{D}\Phi = 0$, where, for an appropriate choice of Ω_z,

$$\mathcal{D} = \left(z\frac{d}{dz}\right)^4 - 5z\left(5z\frac{d}{dz}+1\right)\left(5z\frac{d}{dz}+2\right)\left(5z\frac{d}{dz}+3\right)\left(5z\frac{d}{dz}+4\right)$$

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It is easy to find a single power series solution near $z = 0$:

$$\Phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}\, z^n$$

but the other three solutions are elusive.

The recursion relations implied by the equation lead one to a formal power series of the form

$$\Phi(z,\alpha)=\sum_{n=0}^{\infty}\frac{(5\alpha+1)(5\alpha+2)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}\,z^{\alpha+n}\,;$$

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one finds that $\mathcal{D}(\Phi(z,\alpha)) = \alpha^4 z^\alpha$ and so we must have $\alpha^4 = 0$ in order to obtain a solution. In fact, the formal solution can be interpreted with α taken from the ring $\mathbb{C}[\alpha]/(\alpha^4)$ as follows: each coefficient

$$\frac{(5\alpha+1)(5\alpha+2)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}$$

can be evaluated in that ring, and written as a polynomial in α of degree 3; moreover, z^α can be expanded as $1 + \alpha \ln z + \frac{1}{2}\alpha^2(\ln z)^2 + \frac{1}{6}\alpha^3(\ln z)^3$.

Thus, we can write

$$\Phi(z,\alpha) = \Phi_0(z) + \alpha\Phi_1(z) + \alpha^2\Phi_2(z) + \alpha^3\Phi_3(z),$$

where each $\Phi_j(z)$ is a polynomial in $\ln z$ of degree j whose coefficients are formal power series in z; by construction, $\mathcal{D}\Phi_j(z) = 0$.

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where each $\Phi_j(z)$ is a polynomial in $\ln z$ of degree j whose coefficients are formal power series in z; by construction, $\mathcal{D}\Phi_j(z) = 0$. The *mirror map* is the identification of the complexified Kähler moduli space of X with the complex moduli space of Y via

$$t = \frac{1}{2\pi i}\frac{\Phi_1(z)}{\Phi_0(z)},$$

or

$$q = \exp(\Phi_1(z)/\Phi_0(z)).$$

Three-point Functions

A key aspect of the physics is captured by the so-called *topological correlation functions*, among which is the "three-point function," a trilinear map on $H^{1,1}(X, \mathbb{C})$, resp. $H^1(Y, T_Y^{(1,0)})$.

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$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \rangle = A\cdot B\cdot C + \sum_{0 \neq \eta \in H_2(X,\mathbb{Z})} A(\eta)B(\eta)C(\eta)N_\eta \frac{z^\eta}{1-z^\eta},$$

where N_η counts the number of genus zero holomorphic curves in the class η, and is closely related to the Gromov–Witten invariant of X.

On the mirror quintic Y, given $\alpha, \beta, \gamma \in H^1(Y, T_Y^{(1,0)})$, the three-point function takes the form

$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle = \int_Y \nabla_\alpha(\Omega \lrcorner \beta) \wedge (\Omega \lrcorner \gamma),$$

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and this can be readily calculated from the periods. Comparing the two yields the famous predictions of Candelas, de la Ossa, Green and Parkes: the generic quintic threefold has 2875 lines, 609250 conics, 317206375 twisted cubics, and so on.

Disk Counting

The story so far has been about closed string theory. But in the presence of D-branes, it is now understood that open strings also play a rôle. The relevant D-branes on the quintic threefold are special Lagrangian submanifolds L of X, and open strings are expected to end on such a submanifold. Instead of counting holomorphic curves of fixed genus, the open string theory should count open Riemann surfaces whose boundary lies on L (again, hopefully, of fixed genus).

The specific special Lagrangian L which we use is the set of real points of a quintic threefold defined over $\mathbb{R}$. In, fact, since there are many connected components of the moduli of such real quintic threefolds (and many things about L, including its topology, depend on the component), we specialize further to the component containing the real Fermat quintic:

$$X = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{CP}^4,$$

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This L is known to have the topology of $\mathbb{RP}^3$. Some of the holomorphic curves of genus zero on X are defined over $\mathbb{R}$; others come in complex conjugate pairs. The ones defined over $\mathbb{R}$ meet L and are divided by it into a pair of disks: it is these disks which we wish to count.

Just to preview the results, we will be able to do the count only for curves of odd degree, and the answer will be: for degree 1, there are 1430 complex conjugate pairs and 15 invariant curves, leading to 30 disks; for degree 3 there are 158602805 complex conjugate pairs and 765 invariant curves, leading to 1530 disks; and so on.

The background for doing open string theory requires one additional piece of data, in addition to the special Lagrangian submanifold L: it requires a $U(1)$ bundle to be specified on L, with flat connection. Since $H_1(L, \mathbb{Z}) = \mathbb{Z}_2$, there are two choices for this data; we will use L_+ and L_- to denote the special Lagrangian, equipped with such a choice.

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$$\mathcal{T}(t) = \frac{t}{2} \pm \left(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} n_d q^{d/2} \right),$$

where $q = e^{2\pi i t}$ and n_d are the open Gromov–Witten invariants counting disks of degree d.

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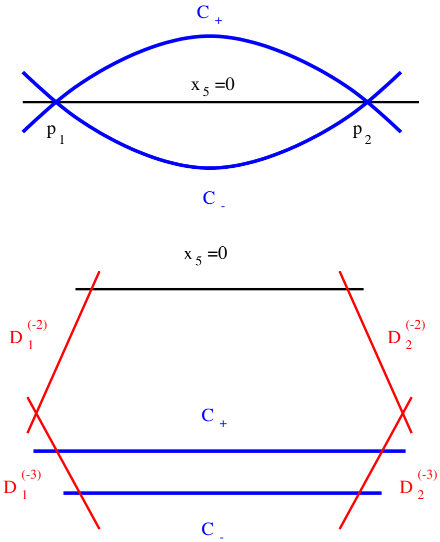
where $q = e^{2\pi i t}$ and n_d are the open Gromov–Witten invariants counting disks of degree d. Physically, $\mathcal{T}$ represents the domain wall tension for a domain wall separating vacua corresponding to L_+ and L_- boundary conditions.



C +
x 5 =0
p 1
p 2
C -
x 5 =0
D 1 (-2)
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