Normal Functions & Disk Counting

Based on J. Walcher hep-th/0605162
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Quintic & Mirror

- quintic: $X = \Sigma F(x_1, \ldots, x_5) = 0 \subset \mathbb{C}P^4$ CY 3-fld
- "complexified Kähler moduli space:" $T \in H^2(X, \mathbb{C})$ w/ Im $\alpha$ Kähler class
- $q = e^{-\pi i \alpha} \in \mathcal{U} \subset H^2(X, \mathbb{C}/\mathbb{Z})$

- quintic-mirror singular
  - resolve $Y \rightarrow \tilde{Y} = \Sigma x_j^5 - 3t \prod x_j = 0 \subset \mathbb{C}P^2/(\mathbb{Z}_5)^3$
- ordinary ("cplx str") moduli spc has param $z = (-3t)^{-5}$
- Identification b/w two made w/ help of periods
  $\phi(z) = \int \Omega^2$, $\Gamma \in H_3(Y, \mathbb{Z})$, $\Omega_2$ hol. 3-form on $Y$

Diagram:

$z = \infty$
$z = 0$
$z = (5)^{-5}$

$\phi(z)$ satisfies algebra diffeq $D\phi = 0$, $D$ 4th order

Easy to find 1 power series soln near $z = 0$

$\phi_0(z) = \sum_{n=0}^{\infty} \frac{(5^n)!}{(n!)^5} z^n$

other 3 solns elusive
Formal power series
\[ \phi(z, x) = \sum_{n=0}^{\infty} (\text{mess}) \cdot z^{x+n} \]

\[ D(\phi(z, x)) = x^4 z^x \Rightarrow x^4 = 0 \text{ for soln} \]

\[ \Rightarrow \text{soln depends on } \mathbb{C}[x]/(x^4) \]
\[ \frac{(5x+1)(5x+2) \cdots (5x+4)}{(a+1)(a+2) \cdots (a+n)} \]
evaluated in winding poly of deg 5
\[ z^x = 1 + x \log z + \cdots \]

\[ \Rightarrow \phi(z, x) = \phi_0(z) + x \phi_1(z) + x^2 \phi_2(z) + x^3 \phi_3(z) \]

mirror map: \[ t = \frac{1}{2\pi i} \cdot \frac{\phi(z)}{\phi_0(z)} \]
\[ \Rightarrow z = \exp \left( \frac{\phi_0(z)}{\phi_0(z)} \right) \]

3 pt flat trilinear map on \( H^1(X, \mathbb{C}) \), \( H^1(Y, \mathbb{C}) \)

on \( X \): \( A, B, C \in H^1(X, \mathbb{C}) \)
\[ \langle \Theta_A \Theta_B \Theta_C \rangle = A \cdot B \cdot C + \sum_{0 \neq \eta \in H_1(X)} A(\eta) B(\eta) C(\eta) \cdot \frac{Z^\eta}{1-z^\eta} \]

counts # curves in \( \eta \) related to genus 0 hole

\[ \int_X A \wedge B \wedge C \]

Gromov-Witten

in \( \text{virt } \) of \( X \)
on mirror $Y$: $\alpha, \beta, \gamma \in H^i(Y, T^{(00)})$

$$\langle 0_\alpha \ 0_\beta \ 0_\gamma \rangle = \int \frac{\Omega_\alpha \Omega_\beta \Omega_\gamma}{\Omega_{\alpha+\beta+\gamma}} \text{ calc. from periods}$$

predictions of Candelas, de la Osa, Green, Parkes

2875 lines
609250 conics
317206375 twisted cubics

__Disk Counting__ so far, just closed strings, but w/ D-branes, open strings also important.

D-branes: $L \subseteq X$ special lagrangian submanifolds

instead of counting holo curves of fixed genus, should count even Riem. surf ending on $L$

$$X = \sum x_1^5 + x_2^5 + \cdots + x_5^5 = 0 \subseteq \mathbb{C}P^4$$

$$L = \sum x_1^5 + x_2^5 + \cdots + x_5^5 = 0 \subseteq \mathbb{R}P^4$$

$L$ topology of $\mathbb{R}P^3$

this compact contains Fermat quintic

some holo $g=0$ curves on $X$, defined over $\mathbb{R}$ these meet $L$, count these others in cplx conj. pairs

only do count for odd degree curves
actually need UC bundle on $L$ w/ flat connection $H_1(L, \mathbb{Z}) = \mathbb{Z}_2 \Rightarrow 2$ choices $L_+, L_-
abla$ in $(L_+)$ in $\text{Fuk}(X)$ same $\Rightarrow$ normal for

$$J(t) = \frac{t}{\delta} \pm \left( \frac{1}{d} \sum_\text{all $\eta \neq g$} \delta t \right)$$

open qrs - worst invariants $\delta$ degree $d$

domain null

tension for
null-separating $L_+$ and $L_-$ vacua boundary cond.

$L_+ - L_-$ trivial in $\text{Fuk}(X)$

$D^b(\text{Coh} Y) \rightarrow \text{Ker}$

Branes on Fermat quintic \xrightarrow{\text{mirror}} \text{Branes on Fermat minor} \xleftarrow{\text{mirror}}

$$(\sum_{i=1}^5 5g_i \Pi x_i) I_N = \begin{pmatrix} \text{matrix w/ entries} & \text{poly w/ entries} \\ \text{poly w/ entries} & \text{entries} \end{pmatrix}$$

$\Rightarrow$ mirror of $L_+ - L_- = \text{element of } D^b(\text{Coh} Y)$

$\text{ch}_2 (Z) = 0$ in $K$-theory

$\text{ch}^{13} (X) \neq 0$, represented by $C_+ - C_-$

two hole curves on $X$. 

\[\text{degree d in K-theory.}\]
\[ C \subset C = \partial P \quad \text{3-chain} \]

\[ y \in \{ x_3^5 + x_2^5 + \ldots + x_5^5 - 4x_1x_2x_3x_4 \} \]

\[ 0 = x_1^2 - 5x_2^2 - x_3^2 \]

\[ \forall \{x_1, x_2, x_3, x_4, x_5\} \]

\[ B \subset \text{well defined as element of } \text{Jacobi's over } \mathbb{Z}^3 \]

\[ \text{this only works when Fermat point } \]

\[ y = 2k \]

\[ \text{J}(z) = \{ 0, 2 \} \]
\[ \oint \omega_\pi = \frac{15}{16 \pi^2} \sqrt{z} \]

\[ \oint \omega_r = 0 \]

\[ \oint \omega_\pi = 0 \Rightarrow \oint \omega_r = 0 \quad \text{and} \quad \oint (t) = \ldots \]