Calabi-Yau Singularities II

\[ \pi \downarrow \quad K_X = 0 \]
\[ p \in X \quad X \setminus \text{Sing } X = X \setminus \pi^{-1}(\text{Sing } X) \]

has a holomorphic form double dual and better behaved \( \pi \)-connection \( \text{Gorenstein} \)

sheaf \( (i_* W) \setminus \text{Sing } X \) should be locally free of rank \( 1 \) with nowhere-vanishing section

Technical condition: Cohen-Macaulay \( (\omega_X^*) \) is concentrated in a single degree

"Canonical": A blow-up

\[ Y \quad \eta \downarrow \quad (\eta \setminus (i_* \omega_X)^{**}) \] has at most
\[ X \quad \text{only zeros, no poles} \]

doub \( X \) has canonical singularities if it is Cohen-Macaulay, \( \omega \)-Gorenstein, and \( \eta^* (i_* \omega_X^{**n}) \) is regular for some resolution of singularities.

Question: Does every Gorenstein canonical singularity \( p \in X \) have a resolution \( \pi: \tilde{X} \to X \) s.t. \( \pi^* (\omega_X^{**}) \) has no zeros near \( \pi^{-1}(p) \)?

Answer: Yes in dim 2
No in dim > 2
In dim 3, answer is "yes, unless..."
dim 2

Cor. canan. sing = rational double pts
\[ C^2 / G, \quad G \leq SU(2) \text{ finite} \]

In dim 2, \( \exists \) a well defined minimal resolution of singularities

\[ \xleftarrow{\sim} \frac{\mathbb{C}^2}{G} \]

minimal resolution & RDP is a collection of \( E_i \)
\[ E_i^2 = -2, \quad K X \cdot E_i = 0 \]
\[ i \neq j, \quad E_i \cdot E_j \in \mathbb{Z} \quad (E_i \cdot E_j) \text{ is negative definite} \]

\[ 2g - 2 = K \cdot C + C^2 \quad \Rightarrow \text{genus 0} \]

\[ \Rightarrow \text{simply laced Dynkin diagrams} \]

\[ A_n \quad E_1 \quad E_2 \quad \cdots \quad E_n \quad G = \mathbb{Z}_n \leftrightarrow A_{n-1} \]

\[ D_n \quad E_1 \quad E_2 \quad \cdots \quad E_n \quad G = \mathbb{Z}_n \leftrightarrow D_n \]

\[ E_n \quad \cdots \quad \leftrightarrow \quad n = 6, 7, 8 \quad G = \mathbb{Z}_n, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \leftrightarrow E_6, E_7, E_8 \]

lowest degree:

\[ \mathbb{Z}_n : x y + z^n = 0 \]
\[ E_6 : x^2 + y^3 + z^4 = 0 \]
\[ E_8 : x^2 + y^3 + z^5 = 0 \]
**Bottom-up approach**

- For $x \in X$, try to blow up in a way which does not introduce zeros into $\pi^*(\mathcal{A}_X)$.

  $$x^2$$

  $$xy + z^n = 0$$ Non-unique origin:

  $$x, y_1 + z\chi^n = 0$$

  Blow up rational double point $P$, result has no zeros of $\pi^*(\mathcal{A}_X)$ and only RDP on blow-up.

**dim 3** (Miles Reid 1980)

1. Singularity might not be isolated, but

   $$\xi \to C$$

   **normalization** (always introduces poles) in codim 1

2. The general surface through curve of singularities has a RDP.

![Diagram of singularity classification](image)

**Singular curve, ADE classification (+ monodromy)**

- $B_n$
- $C_n$
- $D_n$
- $E_6$
- $E_7$
- $E_8$
\[ xy + z^2 = 0 \rightarrow x_1, y_1 + z_1^{n-2} = 0 \]

Blowup the ideal of that component of $\Sigma_{xy} X$

@ generic \( r \), get partial res. of RDPs, monodromy invariant

**Step 1:** resolve all curves of RDPs by standard blowups

1. All remaining rings are isolated.

   **Case 1:** there is a blowup

   \[ \pi : \tilde{X} \rightarrow X \text{ s.t. } \pi^{-1}(p) \text{ contains a divisor } D \]

   \[ p \in X \text{ on which } \pi^*(\omega_X) \text{ does not vanish identically} \]

2. \( \pi^* \omega_X \) vanishes along divisors

\[ xy + zu = 0 \quad x = z = 0 \quad \text{or} \quad x = u = 0 \]

\[ \begin{cases} \circ & \circ \end{cases} \]

**Case 2:** blunts containing divisors,

\[ \pi^* \omega_X \] vanishes along divisors

**Case 1:** Consider the tangent cone to \( X \) at \( p \)

\( S \) \text{ s.t. } \(-K_S\) is ample

\[ -K_S \text{ is ample} \]

\[ 1 \leq \deg S = (-K_S)^2 \leq 9 \]

\[ \begin{cases} \text{might be reducible, have singulatities } & \text{RDPs} \\
\text{simple elliptic, might be non-normal} & \text{non-normal} \end{cases} \]

Warning: after such blowup, many again be non-isolated sing.
Step 2 case 21 Reid shows: the general surface $S$ through $P$ has a rational double sing. (cK)

"terminal"

some of these admit local CY resolution and
some do not

$x^2 + y^2 + z^2 + t^k = 0$ terminal

local CY resolution $\Leftrightarrow k$ even

this step doesn’t introduce other singularities of other
types, so last step

Top Down

$Y = \text{(non-sing) CY manifold}$

$X = \text{Kähler cone}$

$\left[ \mathcal{Z} \right] \to \partial K$

expect a family of metrics in which

$\mathcal{Z} \subseteq Y$ shrinks to zero size, leaving

$X = \text{singular CY } \partial K$ locally polyhedral,

$\left[ \mathcal{Z} \right] \to \text{generic pt of } \partial K$ but may have accumpts;
don’t know what to do here

$\exists \text{ single } Y \in \mathcal{H}_2 \text{, } \mathcal{Z} = \left[ \mathcal{C} \right]$,

$\sum_{\mathcal{C}} \mathcal{Y}$

$\left[ \mathcal{C}_1 \right] = \mathcal{Y}$ \text{ t w finite set alg.} \text{ dim } \geq 2

$P, Q \in Y \Rightarrow \pi_O(P) = \pi_O(Q) \in \mathcal{X}$

$P, Q \in \mathcal{C}$, $\left[ \mathcal{C} \right] = Y$
Cases:
1. \( Y \cong \mathbb{C} \rightarrow \rho \in X \) like step 2 case 1

2. \( \exists C \ni Y \rightarrow \rho \in X \) step 1

3. \( \exists C \ni Y \) step 2 case 1

Bardenbargher has him about step 2 case 2 coming soon!

References

M. Reid, Canonical 3-folds, Angers (1979)
P. Wilson, Inventiones 107 (1992) 561-583