3D $\mathcal{N} = 4$ Supersymmetric Gauge Theories and Hyperkähler Metrics

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$3D \mathcal{N} = 4$ Supersymmetric Gauge Theories and Hyperkähler Metrics
Scalar Theory

Circle-valued scalar $\phi : M^{(3)} \to S^1$ with Lagrangian

$$\mathcal{L} = \frac{\Lambda}{4\pi} \int d\phi \wedge \ast d\phi$$
“Big Theory”

\[
\sum_{\mathcal{L}} \int \mathcal{D}\phi \mathcal{D}A\mathcal{D}B \exp \left[ -\frac{\Lambda}{4\pi} \int D_B \phi \wedge *D_B \phi + \frac{i}{2\pi} \int F_A \wedge B \right]
\]
“Big Theory”

\[ \sum \mathcal{L} \int D\phi DAD\phi \exp \left[ -\frac{\Lambda}{4\pi} \int D_B \phi \wedge *D_B \phi + \frac{i}{2\pi} \int F_A \wedge B \right] \]

- Integrate out \( A \) first, recover original theory

- Set \( \phi \) to zero using gauge invariance. Integrate out \( B \) by completing the square.

"Dual Theory"
“Big Theory”

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- Set $\phi$ to zero using gauge invariance. Integrate out $B$ by completing the square.

"Dual Theory"

$$\mathcal{L} = \sum_{\mathcal{L}} \int \mathcal{D}A \exp \left[ -\frac{1}{4\pi\Lambda} F_A \wedge *F_A \right]$$
Masses in 3D

\[ \mathcal{N} = 4 \text{ SUSY} \rightarrow \text{No superpotential} \]
Masses in 3D

- $\mathcal{N} = 4$ SUSY $\rightarrow$ No superpotential
- 3D CS interaction
Masses in 3D

- $\mathcal{N} = 4$ SUSY $\rightarrow$ No superpotential
- 3D CS interaction
- FI D-terms
Potential energy

\[ V = \frac{1}{4e^2} \sum_{i<j} \text{Tr}[\phi_i, \phi_j]^2 \]
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\( \phi_i \) pairwise commute
Potential energy

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- Vacua: \( V = 0 \)
- \( \phi_i \) pairwise commute
- \( G \rightarrow U(1)^r \)
Example

\[ \phi_i = \begin{pmatrix} a_i & 0 \\ 0 & -a_i \end{pmatrix} \]

- Moduli space of vacua: \((\mathbb{R}^3 \times S^1)/\mathbb{Z}_2\)
- Classical Metric:

\[ ds^2 = \frac{1}{e^2} \sum d\phi_i^2 + e^2 d\sigma^2 \]
Quantum Corrections

- Region at infinity in $\mathbb{R}^3$ asymptotically $S^2$ with radius $|\phi|$
- $S^1$ with fixed circumference $e$
- Possibly non-trivial fibration $S^1 \to S^2$.

$$ ds^2 = \frac{1}{e^2} \sum d\phi^2 + e^2 \left( d\sigma - sB_i(\phi)d\phi^i \right)^2 $$

- Can be generated at 1-loop.
Low energy effective theory on $\mathbb{R}^3 \times S^1$.

The low energy effective theory on $\mathbb{R}^3 \times S^1_R$ should interpolate between $\mathcal{N} = 4$ gauge theory in 3D as $R \to 0$ and $\mathcal{N} = 2$ gauge theory in 4D as $R \to \infty$.

- Massless scalars from holonomy

$$\tilde{\phi}^I = \oint_{S^1} A_4^I dx^4$$

- “Magnetic Wilson lines”

$$\tilde{\phi}^I = \oint_{S^1} (A_D,4)_I dx^4$$

obtained from dualizing the 3D gauge field.
$R \rightarrow \infty$ Limit

Want to study gauge theory at energy scale $\mu$ where

- $\mu \ll \Lambda$
- $\mu \ll 1/R$

The $R \rightarrow \infty$ limit lets us read off spectrum of 4D gauge theory.
3D Moduli space is a $2r$ dimensional torus fibration over the 4D vector multiplet moduli space.

$$\mathcal{J} \rightarrow \mathcal{M}_v$$

where $\mathcal{J}$ is parametrized by the electric and “magnetic” Wilson lines $(\tilde{\phi}^I, \phi_I)$. Denote the fiber over a point $u \in \mathcal{M}_v$ by $\tilde{\mathcal{J}}_u$. 

Richard Eager UCSB 3D $\mathcal{N} = 4$ Supersymmetric Gauge Theories and Hyperkähler Metrics
We take the $R \to \infty$ limit by truncating fields to be independent of $x^4$. This reduction of the 4D Lagrangian yields

$$\mathcal{L}^{(3)} = (\mathcal{S}_\tau) \left( -\frac{R}{2} \left| da \right|^2 - \frac{R}{2} F^{(3)} \wedge \ast F^{(3)} - \frac{1}{8\pi^2 R} d\tilde{\phi}^2 \right) + \left( \mathcal{R}_\tau \right) \left( \frac{1}{2\pi} d\phi \wedge F^{(3)} \right)$$  \hspace{1cm} (1)

Dualizing the 3D gauge field $A^I$ to a scalar yields

$$\mathcal{L}^{3\text{dual}} = -\frac{R}{2} (\mathcal{S}_\tau) \left| da \right|^2 - \frac{1}{8\pi^2 R} (\mathcal{S}_\tau)^{-1} \left| d\phi - \tau d\tilde{\phi} \right|^2$$

Semiflat metric

$$g^{sf} = R (\mathcal{S}_\tau) \left| da \right|^2 + \frac{1}{4\pi^2 R} (\mathcal{S}_\tau)^{-1} \left| dz \right|^2$$

where

$$dz_I = d\phi_I - \tau_{IJ} d\tilde{\phi}^J$$
Why semiflat?

Recall the semiflat metric
\[ g^{sf} = R(\mathcal{S} \tau)|da|^2 + \frac{1}{4\pi^2 R} (\mathcal{S} \tau)^{-1}|dz|^2 \]

here the torus fibers \( \mathcal{J}_u \) are flat. The fibers have volume
\[ \text{vol}(\mathcal{J}_u) \propto \left( \frac{1}{R} \right)^r \]

In the \( R \to \infty \) limit the torus is small.
What is new physics going from $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$?

Still have localized 4D instantons.

New kinda of instanton: A magnetic monopole or dyon that wraps around $S^1_R$.

For large $R$ the action is $I = 2\pi R M$. 
How do these new instantons affect the moduli space? $M = |a_D + na|$ is not holomorphic so it cannot correct the distinguished complex structure on $M$. However the monopoles do contribute to the metric on $M$. 
Definition

A \textbf{Kähler} manifold is a Riemannian manifold $X$ with an almost complex structure $J$ which is covariantly constant.
Definition

A *hyperkähler* manifold is a Riemannian manifold \((M, g)\) with an three covariantly constant almost complex structures \(I, J, K\) which satisfy

\[
I^2 = J^2 = K^2 = -1 \quad \text{and} \quad IJ = -K, \quad JK = -I, \quad KL = -J
\]

Examples

- **\(K3\) surfaces**
  \[
  \{(x, y, z, w) \in \mathbb{CP}^3 : x^4 + y^4 + z^4 + w^4 = 0\}
  \]

- [Moduli spaces from physics!](#)
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- Moduli spaces from physics!
Definition

The *twistor space* $Z$ of a hyperkähler manifold $M$ is the product

$$Z = M \times S^2$$

equipped with the almost complex structure

$$I = (aI + bJ + cK, I_0)$$
Topologically $Z$ is a trivial bundle.
- Topologically $Z$ is a trivial bundle.
- $Z$ is not trivial as a holomorphic bundle.
▶ Topologically $Z$ is a trivial bundle.
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Example

Holomorphic line bundles on an elliptic curve

$$Pic \, X \cong Jac \, X \times \mathbb{Z}$$
What are the (1,0) forms for complex structure $I$?

Suppose $\phi$ is a (1,0) form on $M$ in complex structure $I$.

\[ I\phi = i\phi \]

Define $\theta = \phi + \zeta K\phi$

Quick calculation:

\[ I\theta = i\theta. \]
Let $\phi_1, \ldots, \phi_n$ be a basis of $(1,0)$ forms for $(M, I)$. Then $\phi_i + \zeta K \phi_i$ and $d\zeta$ are a basis of $(1,0)$ forms for $(Z, I)$. 

The holomorphic (2,0) form

\[ \omega_+ = \omega_2 + i \omega_3 \]

can locally be written as

\[ \omega_+ = \sum \phi_i \wedge \phi_{n+i} \]
Define

\[ \frac{1}{2} \bar{\omega} = \sum \hat{\phi}_i \wedge \hat{\phi}_{n+i} \]

Calculation

\[ \bar{\omega} = \omega_+ + 2\zeta \omega_1 - \zeta^2 \omega_- \]
Theorem (HKLM)

Let $M^{4n}$ be a hyperkähler manifold and $Z$ its twistor space. Then

(i) $Z$ is a holomorphic fiber bundle $p: Z \rightarrow \mathbb{CP}^1$ over the projective line.

(ii) the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.

(iii) there exists a holomorphic section $\omega$ of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ defining a symplectic form on each fiber

(iv) $Z$ has a real structure $\tau$ compatible with (i),(ii),(iii) and inducing the antipodal map on $\mathbb{CP}^1$. 